Risk premia in general equilibrium

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Abstract

This paper shows that non-linearities and non-normalities are important to generate empirically observed stylized facts of the risk premium. These key features can explain the equity premium puzzle and the time-varying behavior of the risk premium. We employ explicit solutions of dynamic stochastic general equilibrium (DSGE) models. It is shown that non-linearities in a prototype DSGE model can generate time-varying risk premia, while non-normalities can account for the observed risk-premium puzzle by drawing from the Barro-Rietz 'rare disaster hypothesis'.

JEL classification: E21, G11, O41

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1 Introduction

> "... the challenge now is to understand the economic forces that determine the stochastic discount factor, or put another way, the rewards that investors demand for bearing particular risks." (Campbell 2000, p.1516)

> "A major advantage of the continuous-time model over its discrete time analog is that one need only consider two types of stochastic processes: functions of Brownian motions and Poisson processes." (Merton 1971, p.412)

This paper shows that non-linearities and non-normalities are important to generate key features of the risk premium. We employ explicit solutions of dynamic stochastic general equilibrium (DSGE) models. Our macro-finance model is specified in terms of underlying preferences and technology parameters, such that the asset-pricing kernel is consistent with

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the macroeconomic dynamics (Diebold et al. 2005, p.420). It is shown that non-linearities can generate time-varying risk premia, and non-normalities are important to explain the observed risk-premium puzzle. For this purpose we readopt formulating models in continuous-time, which has closed-form solutions for a broad class of interesting models and/or parameter sets (Merton 1975, Eaton 1981, Cox et al. 1985).[1] This helps to provide further insights on the determinants of the risk premium in DSGE models.

Recent research on DSGE models has emphasized the importance of non-linearities and non-normalities in explaining the business cycle fluctuations for the US economy (Fernández-Villaverde and Rubio-Ramírez 2007, Justiniano and Primiceri 2008). However, the problem with discrete-time models is that they are hard to solve, and the literature uses approximation schemes to circumvent this problem. It has become very successful in characterizing certain aspects of dynamic properties, and in providing adequate answers to questions such as local existence and stability (Schmitt-Grohé and Uribe 2004). Most approximation schemes will fail, however, when it comes to the effects of uncertainty.

This paper contributes to the literature on the determinants of the risk premium, that is the rewards that investors demand for bearing particular risks (Campbell 2000). There has been a long discussion since Rietz (1988) proposed the 'rare disasters hypothesis' as a solution to the risk-premium puzzle (Mehra and Prescott 1985, 1988). Barro (2006, 2009) shows that disasters have been sufficiently frequent and large enough to account for the observed equity risk premia. Gabaix (2008) introduces variable disaster intensity in an endowment economy and shows that the rational, representative-agent framework is a workable paradigm in the macro-finance literature. We show that using a continuous-time formulation we can easily enrich the endowment economy by including non-linearities to the model. It clarifies the relationship between the equity premium and the implicit risk premium.

The remainder of the paper is organized as follows. In the Section 2 we solve in closed form a continuous-time version of Lucas' (1978) fruit-tree model with exogenous, stochastic production and obtain the risk-premium. Section 3 studies the effects of non-linearities on the risk premium in Merton's (1975) neoclassical growth model. We conclude in Section 4.

2 Lucas fruit-tree model in continuous-time

2.1 Lucas fruit-tree model in continuous-time (two assets)

Consider a fruit-tree economy (one risky asset or equity), and a riskless asset without default risk (government bond).

[1]Recent contributions of continuous-time DSGE models include e.g. Corsetti (1997), Wälde (1999, 2002), Steger (2005), and Turnovsky and Smith (2006).

2.1.1 Description of the economy

Technology. Consider a one-good pure exchange economy (Lucas 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time, $Y_t = A_t$ where A_t is the stochastic technology. Output is perishable. The law motion of A_t will be taken to follow a Markov process,

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t, \tag{1}$$

where B_t is a standard Brownian motion.

Suppose ownership of fruit-trees with productivity A_t is determined at each instant in a competitive stock market, and the production unit has outstanding one perfectly divisible equity share. A share entitles its owner to all of the unit's instantaneous output in t. Shares are traded at a competitively determined price, p_t. Suppose that for the risky asset,

$$dp_t = \mu p_t dt + \sigma p_t dB_t, \tag{2}$$

and for a riskless asset

$$dp_0(t) = p_0(t) r dt. \tag{3}$$

Because prices fully reflect all available information, the parameters will be determined in general equilibrium. The objective is to relate exogenously determined productivity changes to the market determined movements in asset prices. In fact, the evolution of prices ensures that assets are priced such that individuals are indifferent between holding more assets and consuming. Given initial wealth, we are looking for the optimal consumption path.

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes discounted expected life-time utility

$$U_0 \equiv E \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0.$$

Assuming no dividend payments, the budget constraint reads

$$dW_t = \left((\mu - r) w_t W_t + r W_t - C_t \right) dt + w_t \sigma W_t dB_t, \tag{4}$$

where W_t is real financial wealth and w_t denote a consumer's share holdings.

Equilibrium properties. In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and all shares will be held by capital owners.

2.1.2 The short-cut approach

Suppose that the only asset is the *market portfolio* is

$$dp_M(t) = \mu_M p_M(t)dt + \sigma_M p_M(t)dB_t. \tag{5}$$

Consider the portfolio choice as an independent decision of the consumption problem. The consumer obtains income and has to finance its consumption stream from wealth,

$$dW_t = (\mu_M W_t - C_t)\, dt + \sigma_M W_t dB_t. \tag{6}$$

One can think of the original problem with the budget constraint (4) as having been reduced to a simple Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of a (composite) asset (Merton 1973).

Define the *value function* as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t)dt, \quad s.t. \quad (6), \quad W_0 > 0. \tag{7}$$

The Bellman equation becomes when choosing the control $C_s \in \mathbb{R}_+$ at time s

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + (\mu_M W_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \right\}.$$

Because its a necessary condition, the *first-order conditions* are

$$u'(C_s) - V_W \quad = \quad 0 \quad \Rightarrow \quad V_W = u'(C_s) \tag{8}$$

for any interior solution at any time $s = t \in [0, \infty)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) &= \left((\rho - \mu_M)u'(C_t) - \sigma_M^2 W_t V_{WW}\right) dt + \sigma_M W_t V_{WW} dB_t \\ &= \left((\rho - \mu_M)u'(C_t) - \sigma_M^2 W_t u''(C_t) C_W\right) dt + \sigma_M W_t u''(C_t) C_W dB_t, \end{aligned} \tag{9}$$

which implicitly determines the optimal consumption path. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$).

To shed some light on the effects of uncertainty, we use the Euler equation and obtain the (implicit) risk premium as

$$\begin{aligned} \frac{du'(C_t)}{u'(C_t)} &= \left(\rho - \mu_M - \sigma_M^2 W_t \frac{u''(C_t)}{u'(C_t)} C_W\right) dt + \sigma_M W_t \frac{u''(C_t)}{u'(C_t)} C_W dB_t \\ \Rightarrow \quad \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right] &= \rho - \mu_M + E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t\right] \sigma_M^2 \end{aligned}$$

which may be written as

$$\mu_M - E\left[-\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2\right] = \rho - \frac{1}{dt}E\left[\frac{du'(C_t)}{u'(C_t)}\right]. \tag{10}$$

We may define the left-hand side as the *certainty equivalent rate of return on saving*, that is the expected rate of return on saving less the expected *implicit risk premium* (Steger 2005). The latter gives the minimum difference an individual requires to accept an uncertain rate of return, between its expected value and the certain rate of return that it is indifferent to. On the right-hand side, we have the expected cost of forgone consumption, i.e. the rate of time preference, and the expected rate of change of marginal utility .

2.1.3 A more comprehensive approach

Define the *value function* as

$$V(W_0) \equiv \max_{\{(w_t,C_t)\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t) dt, \quad s.t. \quad (4), \quad W_0 > 0. \tag{11}$$

The Bellman equation becomes when choosing the control $(w_s, C_s) \in \mathbb{R} \times \mathbb{R}_+$ at time s

$$\rho V(W_s) = \max_{(w_s,C_s)} \left\{u(C_s) + (w_s(\mu - r)W_s + rW_s - C_s)\, V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW}\right\}.$$

Because its a necessary condition, the *first-order conditions* are

$$u'(C_s) - V_W = 0 \Rightarrow V_W = u'(C_s) \tag{12}$$

$$(\mu - r)W_s V_W + w_s \sigma^2 W_s^2 V_{WW} = 0 \Rightarrow w_s = -\frac{V_W}{V_{WW} W_s}\frac{\mu - r}{\sigma^2} \tag{13}$$

for any interior solution at any time $s = t \in [0, \infty)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) &= \left((\rho - ((\mu - r)w_t + r))u'(C_t) - w_t^2 \sigma^2 W_t V_{WW}\right) dt + w_t \sigma W_t V_{WW} dB_t \\ &= ((\rho - ((\mu - r)w_t + r))u'(C_t) + (\mu - r)w_t u'(C_t))\, dt - u'(C_t)(\mu - r)/\sigma dB_t \\ &= (\rho - r)u'(C_t)dt - \pi u'(C_t) dB_t, \end{aligned} \tag{14}$$

where we defined $\pi \equiv (\mu - r)/\sigma$ as the market price of risk. Comparing to the Euler equation in (9), we notice that because the household optimally can choose its portfolio risk, there is no *implicit* risk premium. We show below that now the risk premia is available *explicitly*.

Given the demand function for the risky asset (13), we follow Merton (1973) and obtain an equilibrium relation for the price on the market portfolio

$$\begin{aligned} dp_M &= ((\mu - r)w_t + r)p_M(t)dt + w_t \sigma p_M dB_t \\ &\equiv \mu_M p_M(t)dt + \sigma_M p_M dB_t \end{aligned}$$

defining the instantaneous expected rate of return $\mu_M \equiv (\mu - r)w_t + r$, and the instantaneous variance of returns $\sigma_M^2 \equiv w_t^2\sigma^2$. Using the optimal portfolio weights (13) and (12)

$$\mu_M - r = -\frac{V_{WW}W_t}{V_W}\sigma_M^2 = -\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2, \tag{15}$$

which is the risk premium of the expected market portfolio rate of return over the riskless rate (henceforth *equity premium*). Comparing the result to the implicit risk premium in (10), we find that both approaches indeed give the reward that investors demand and consumers implicitly would be willing to pay for bearing/avoiding the systematic market risk.

2.1.4 General equilibrium prices

This section shows that general equilibrium conditions pin down the prices in the economy. For illustration, with constant relative risk aversion (CRRA) the Euler equation (14) implies

$$dC_t = \frac{\mu_M - \rho - \frac{1}{2}(1-\theta)\theta\sigma_M^2}{\theta}C_t dt + \sigma_M^2 C_t dB_t.$$

Because output is perishable, $Y_t = C_t = A_t$. Hence, we can determine the riskless rate and the market price of risk from

$$dA_t = \frac{\mu_M - \rho - \frac{1}{2}(1-\theta)\theta\sigma_M^2}{\theta}A_t dt + \sigma_M^2 A_t dB_t \quad \Leftrightarrow \quad dA_t = \bar{\mu}A_t dt + \bar{\sigma}A_t dB_t.$$

In general equilibrium it pins down

$$\begin{aligned} \theta\bar{\mu} &= \mu_M - \rho - \tfrac{1}{2}(1-\theta)\theta\sigma_M^2 = (\mu - r)w + r - \rho - \tfrac{1}{2}(1-\theta)\theta w^2\sigma^2 \\ \Leftrightarrow r &= \rho + \theta\bar{\mu} - \frac{(\mu-r)^2}{\theta\sigma^2} + \tfrac{1}{2}(1-\theta)\frac{(\mu-r)^2}{\theta\sigma^2} = \rho + \theta\bar{\mu} - \tfrac{1}{2}(1+\theta)\frac{(\mu-r)^2}{\theta\sigma^2}, \end{aligned}$$

where we inserted optimal portfolio weights from (13), and the Sharpe ratio is

$$\bar{\sigma} = \sigma_M \Leftrightarrow \frac{\mu - r}{\sigma} = \theta\bar{\sigma}.$$

Thus we may write

$$r = \rho + \theta\bar{\mu} - \tfrac{1}{2}(1+\theta)\theta\bar{\sigma}^2$$

as the general equilibrium riskless rate (see also Wang 1996, Basak 2002). Observe that there is only a unique Sharpe ratio, but no unique μ and σ. We may employ identifying technical restrictions in (81) to further restrict the parameter space.

2.2 Lucas fruit-tree model in continuous-time (multiple assets)

This section shows that the analysis can be extended to the multiple asset case. Consider an economy with many fruit-trees (multiple risky asset or equity), and a riskless asset without default risk. Suppose that the number of trees equals the number of consumers.

2.2.1 Description of the economy

Technology. Consider a pure exchange economy (Lucas 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time. The consumption good is produced on n distinct productive units. Let $Y_i(t) = A_i(t)$ denote the output of unit i in period t, $i = 1, ..., n$, and let $A_t = [A_1, ..., A_n]^\top$ be the vector of production-specific stochastic technology. Output is perishable. The law of motion of A_t will be taken to follow a Markov process,

$$dA_t = \text{diag}(A_t)\bar{\mu}dt + \text{diag}(A_t)\bar{\gamma}dB_t, \tag{16}$$

where $\bar{\mu} \equiv [\bar{\mu}_1, ..., \bar{\mu}_n]^\top$, and $\bar{\gamma} \equiv [\bar{\gamma}_1, ..., \bar{\gamma}_n]^\top$ is a $n \times n$ matrix, $\bar{\Omega} \equiv \bar{\gamma}\bar{\gamma}^\top$ the positive definite, non-singular, instantaneous conditional covariance matrix, and $B_t \equiv [B_1(t), ..., B_n(t)]^\top$ is a n-dimensional (uncorrelated) standard Brownian motion. Observe that $\text{diag}(A_t)$ denotes the $n \times n$ matrix with the vector A_t along the main diagonal and zeros off the diagonal.

Suppose ownership of fruit-trees with productivity $A_i(t)$ is determined at each instant in a competitive stock market, and each production unit has outstanding one perfectly divisible equity share. A share entitles its owner to all of the unit's instantaneous output in t. Shares are traded at a competitively determined price, $p_t \equiv [p_1, ..., p_n]^\top$. Suppose for the risky asset,

$$dp_t = \text{diag}(p_t)\mu dt + \text{diag}(p_t)\gamma dB_t, \tag{17}$$

$\mu \equiv [\mu_i, ..., \mu_n]^\top$ is the vector of instantaneous conditional expected percentage price changes for the n risky assets, whereas $\gamma \equiv [\gamma_1, ..., \gamma_n]^\top$ is a $n \times n$ matrix, $\Omega \equiv \gamma\gamma^\top$ the positive definite, non-singular, instantaneous conditional covariance matrix. For a riskless asset

$$dp_0(t) = p_0(t)rdt.$$

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes discounted expected life-time utility

$$U_0 \equiv E\int_0^\infty e^{-\rho t}u(C_t)dt, \quad u' > 0, \; u'' < 0.$$

As shown in the appendix, the budget constraint reads

$$dW_t = \left(w_t^\top(\mu - \hat{r})W_t + rW_t - C_t\right)dt + w_t^\top\gamma W_t dB_t, \tag{18}$$

where W_t is real financial wealth and $w_t \equiv [w_1, ..., w_n]^\top$ is a consumer's share holdings.

Equilibrium properties. In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. Output will be consumed, $\sum_{i=1}^n Y_i(t) = C_t$, and all shares will be held by capital owners.

2.2.2 The short-cut approach

Suppose that the only asset is the *market portfolio* is

$$dp_M(t) = \mu_M p_M(t)dt + \sigma_M^\top p_M(t)dB_t. \tag{19}$$

Consider the portfolio choice as an independent decision of the consumption problem. The consumer obtains income and has to finance its consumption stream from wealth,

$$dW_t = (\mu_M W_t - C_t)\, dt + \sigma_M^\top W_t dB_t, \tag{20}$$

where

$$\mu_M \equiv w_t^\top(\mu - \hat{r}) + r, \quad \sigma_M \equiv (w_t^\top \gamma)^\top, \quad \sigma_M^2 \equiv \sigma_M^\top \sigma_M = w_t^\top \Omega w_t. \tag{21}$$

As shown in the two-assets case, the short cut approach simply separates the consumption decision from the optimal portfolio selection decision. We will study the effects as a special case of the more comprehensive approach below.

2.2.3 A more comprehensive approach

Define the *value function* for the representative agent as

$$V(W_0) \equiv \max_{\{(w_t,C_t)\}_{t=0}^\infty} E_0 \int_0^\infty e^{-\rho t} u(C_t)dt, \quad s.t. \quad (18), \quad W_0 > 0.$$

The Bellman equation becomes when choosing the control $(w_s, C_s) \in \mathbb{R}^n \times \mathbb{R}_+$ at time s

$$\rho V(W_s) = \max_{(w_s,C_s)} \left\{ u(C_s) + \left(w_s^\top(\mu - \hat{r})W_s + rW_s - C_s \right) V_W + \tfrac{1}{2} w_s^\top \Omega w_s W_s^2 V_{WW} \right\}.$$

Because its a necessary condition, the $n+1$ *first-order conditions* are

$$u'(C_s) - V_W = 0 \quad \Rightarrow \quad V_W = u'(C_s) \tag{22}$$

$$(\mu - \hat{r})W_s V_W + \Omega w_s W_s^2 V_{WW} = 0 \quad \Rightarrow \quad w_s = -\frac{V_W}{V_{WW} W_s} \Omega^{-1}(\mu - \hat{r}) \tag{23}$$

for any interior solution at any time $s = t \in [0, \infty)$.

It can be shown that the *Euler equation* for consumption is (cf. appendix)

$$du'(C_t) = \left((\rho - (w_t^\top(\mu - \hat{r}) + r))u'(C_t) - w_t^\top \Omega w_t W_t V_{WW} \right) dt + w_t^\top \gamma W_t V_{WW} dB_t, \tag{24}$$

which implicitly determines the optimal consumption path. Before we proceed inserting the optimal portfolio weights, we may obtain the implicit risk premia from the Euler equation as we would have obtained from the short-cut approach. Using the definitions in (21),

$$\begin{aligned} du'(C_t) &= \left((\rho-\mu_M)u'(C_t)-\sigma_M^2 W_t V_{WW}\right)dt+\sigma_M^\top W_t V_{WW} dB_t \\ \Leftrightarrow \frac{du'(C_t)}{u'(C_t)} &= \left(\rho-\mu_M-\sigma_M^2 W_t \frac{u''(C_t)C_W}{u'(C_t)}\right)dt+\sigma_M^\top W_t \frac{u''(C_t)C_W}{u'(C_t)} dB_t \\ \Rightarrow \quad \frac{1}{dt}E\left[\frac{du'(C_t)}{u'(C_t)}\right] &= \rho-\mu_M+E\left[-\frac{u''(C_t)}{u'(C_t)}C_W W_t\right]\sigma_M^2 \end{aligned}$$

which may be written as, similar to the two-assets case in (10),

$$\mu_M - E\left[-\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2\right] = \rho - \frac{1}{dt}E\left[\frac{du'(C_t)}{u'(C_t)}\right]. \tag{25}$$

In that the left-hand side is the *certainty equivalent rate of return on saving*, and the term in brackets defines the *implicit risk premium*.

Using the first-order condition for optimal portfolio weights (23),

$$\begin{aligned} du'(C_t) &= \left((\rho-(w_t^\top(\mu-\hat{r})+r))u'(C_t)+w_t^\top(\mu-\hat{r})u'(C_t)\right)dt-(\mu-\hat{r})^\top\Omega^{-1}\gamma u'(C_t)dB_t \\ &= (\rho-r)u'(C_t)dt-(\mu-\hat{r})^\top\Omega^{-1}\gamma u'(C_t)dB_t \\ &= (\rho-r)u'(C_t)dt-\pi^\top u'(C_t)dB_t, \end{aligned}$$

where $\pi \equiv \gamma^{-1}(\mu-\hat{r})$ denotes the *market price of risk* vector.

2.2.4 The security market line

Given demand for risky assets and market-clearing, an equilibrium relation between expected return on any asset and the expected return on the market can be derived (Merton 1973). Let $M_t = n_M(t)p_M(t) \equiv mW_t$ be the market value, where $p_M(t)$ is the price per 'share' of the market and $n_M(t)$ is the number of 'shares' of m investors,

$$\begin{aligned} dM_t &= n_M(t)dp_M(t)+p_M(t)dn_M(t) = n_M(t)dp_M+p_0(t)dn_0(t)+p_t dn_t \\ &= n_M(t)dp_M(t)-mC_t dt. \end{aligned}$$

Using the budget constraint (18), we obtain

$$\begin{aligned} n_M(t)dp_M(t)-mC_t dt &= \left(w_t^\top(\mu-\hat{r})mW_t+rmW_t-mC_t\right)dt+w_t^\top\gamma mW_t dB_t \\ \Leftrightarrow \; dp_M(t) &= \left(w_t^\top(\mu-\hat{r})+r\right)p_M(t)dt+w_t^\top\gamma p_M(t)dB_t \end{aligned} \tag{26}$$

where we substituted $mW_t = n_M(t)p_M(t)$ and collected terms. Whenever log-normality of prices is assumed, we can work, without loss of generality, with just two assets, one riskless and the other risky with its price log-normally distributed (Merton 1973).

The instantaneous expected rate of return μ_M, its variance, σ_M^2, and the covariance with the return on the ith asset, σ_{iM}, can be determined as

$$\begin{aligned}
\mu_M &\equiv w_t^\top(\mu - \hat{r}) + r \\
\sigma_M^2 &\equiv \frac{dp_M}{p_M}\frac{dp_M}{p_M} = w_t^\top \Omega w_t \quad \text{and} \quad \sigma_{iM} \equiv \frac{dp_M}{p_M}\frac{dp_i}{p_i} = w_t^\top \gamma\gamma_i = w_t^\top \Omega e_i.
\end{aligned}$$

Using the optimal portfolio weights for risky assets (23),

$$\mu - \hat{r} = -\frac{V_{WW}W_t}{V_W}\Omega w_t$$

and pre-multiplying by $w_t^\top$ (equivalent to multiplying by w_i and summing up), we have that

$$w_t^\top(\mu_t - \hat{r}) = -\frac{V_{WW}W_t}{V_W} w_t^\top \Omega w_t \quad \Leftrightarrow \quad \mu_M - r = -\frac{V_{WW}W_t}{V_W}\sigma_M^2.$$

Using the first-order condition for consumption (22), we obtain

$$\mu_M - r = -\frac{V_{WW}W_t}{V_W}\sigma_M^2 = -\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 \tag{27}$$

which is the *equity premium*. Comparing the result to the implicit risk premium in (25), or to the two-assets case, we find that the different approaches indeed give the same result.

Similarly, when pre-multiplying by $e_i^\top$ where $e_i \equiv [0, ..., 1, ..., 0]^\top$ such that $e_i^\top \mu = \mu_i$, we obtain from (23)

$$\mu_i - r = -\frac{V_{WW}W_t}{V_W}\sigma_{iM}.$$

This result implies, together with the equity premium, the well known *security market line*

$$\mu_i - r = \frac{\sigma_{iM}}{\sigma_M^2}(\mu_M - r), \tag{28}$$

which is the continuous-time analog of the equilibrium return relation of the classical capital asset pricing model (CAPM), as introduced by Merton (1973).

2.2.5 Explicit solutions

As shown in Merton (1971), for the class of hyperbolic absolute risk aversion (HARA) utility functions one can obtain explicit solutions where consumption is a linear function of wealth. For illustration, we present the results for CRRA and CARA preferences.

Proposition 2.1 (CRRA preferences) *If utility exhibits constant relative risk aversion, i.e. $-u''(C_t)C_t/u'(C_t) = \theta$, then optimal consumption is proportional to wealth and optimal portfolio weights are constant, where*

$$C_t = C(W_t) = \left(\rho - (1-\theta)r - \tfrac{1}{2}\pi^\top\pi/\theta\right)/\theta W_t, \quad w = \Omega^{-1}(\mu - \hat{r})/\theta. \tag{29}$$

Proof. see Appendix 5.2.3 ■

Corollary 2.2 *Use the policy function to obtain the risk premium (27) as*

$$\mu_M - r = \theta\sigma_M^2. \tag{30}$$

Proposition 2.3 (CARA preferences) *If utility exhibits constant absolute risk aversion, i.e.* $-u''(C_t)/u'(C_t) = \eta$*, then optimal consumption is linear in wealth and optimal portfolio weights are time-varying, where*

$$C_t = C(W_t) = \left(\rho - r + \tfrac{1}{2}\pi^\top\pi\right)/(\eta r) + rW_t, \quad w_t = w(W_t) = \Omega^{-1}(\mu - \hat{r})/(\eta r W_t). \tag{31}$$

Proof. see Appendix 5.2.4 ■

Corollary 2.4 *Use the policy function to obtain the risk premium (27) as*

$$\mu_M - r = \eta r W_t \sigma_M^2. \tag{32}$$

Though both solutions give linear policy functions for consumption, for CARA preferences consumption is not proportion to wealth (i.e. the marginal propensity to consume does not equal the average propensity). The nice result that CARA preferences imply a time-varying risk premium has the following caveat. The proportion of wealth invested in the risky asset is negative related to individual wealth, that means a wealthy person invests virtually all in the riskless asset. This result seems questionable from an empirical point of view.

As expected, we obtain the standard result that the risk premium is determined by the investors risk aversion parameter and the variance of the market portfolio. Given our priors about risk aversion and empirical estimates of the variance of consumption, (30) leads us to the risk-premium puzzle (Mehra and Prescott 1985). One explanation is the Barro-Rietz 'rare-disaster hypothesis', where the risk premium is shown to depend on the possibility of rare events which (Rietz 1988, Barro 2006). Hence, we proceed our analysis as follows. First, we extend the analysis by allowing for rare disasters to account for the observed equity premium puzzle. Second, we show that introducing a neoclassical production economy with non-linearities can generate time-varying behavior of the risk premium.

2.3 Lucas fruit-tree model with rare disasters (two assets)

This section shows how an extension to the possibility of rare disasters can account for the observed equity premium puzzle, drawing from the Barro-Rietz 'rare disaster hypothesis'. Consider a fruit-tree economy (one risky asset or equity), and a riskless asset in normal times but with default risk (government bond) as in Barro (2006).

2.3.1 Description of the economy

Technology. Consider a pure exchange economy (Lucas 1978). Suppose production is entirely exogenous: no resources are utilized, and there is no possibility of affecting the output of any unit at any time, $Y_t = A_t$ where A_t is the stochastic technology. Output is perishable. The law motion of A_t will be taken to follow a Markov process,

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t + \bar{J}_t A_{t-} dN_t, \tag{33}$$

where B_t is a standard Brownian motion, and N_t is a standard Poisson process. The jump size is assumed to be proportional to its value an instant before the jump, A_{t-}, ensuring that A_t does not jump negative. For illustration, the independent random variable $\bar{J}_t$ has a degenerated distribution $\bar{J}_t \equiv \exp(\bar{\nu}) - 1$. This assumption is purely for reading convenience and extensions to other distributions of the jump size $\bar{J}_t$ pose no conceptional difficulties but are notationally more cumbersome with little associated gain.

Suppose ownership of fruit-trees with productivity A_t is determined at each instant in a competitive stock market, and the production unit has outstanding one perfectly divisible equity share. A share entitles its owner to all of the unit's instantaneous output in t. Shares are traded at a competitively determined price, p_t. Suppose that for the risky asset,

$$dp_t = \mu p_t dt + \sigma p_t dB_t + p_{t-} J_t dN_t \tag{34}$$

and for a government bill with default risk

$$dp_0(t) = p_0(t) r dt + p_0(t_-) D_t dN_t, \tag{35}$$

where D_t is a random variable denoting a random default risk in case of a disaster, where q is the probability of default. For illustration, we assume

$$D_t = \begin{cases} 0 & \text{with} \quad 1-q \\ \exp(\kappa) - 1 & \text{with} \quad q \end{cases},$$

which can be generalized without any difficulty.

Because prices fully reflect all available information, the parameters r, μ, σ, and J_t will be determined in general equilibrium. The objective is to relate exogenous productivity changes to the market determined movements in asset prices. In fact, the evolution of prices ensures that assets are priced such that individuals are indifferent between holding more assets and consuming. Given initial wealth, we are looking for the optimal consumption path.

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes discounted expected life-time utility

$$U_0 \equiv E \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0.$$

Assuming no dividend payments, the budget constraint reads

$$dW_t = ((\mu - r)w_t W_t + rW_t - C_t)\, dt + w_t \sigma W_t dB_t + ((J_t - D_t)w_{t-} + D_t)W_{t-}dN_t, \qquad (36)$$

where W_t is real financial wealth and w_t denote a consumer's share holdings.

Equilibrium properties. In this economy, it is easy to determine equilibrium quantities of consumption and asset holdings. The economy is closed and all output will be consumed, $C_t = Y_t$, and all shares will be held by capital owners.

2.3.2 The short-cut approach

Suppose that the only asset is the *market portfolio*,

$$dp_M(t) = \mu_M p_M(t)dt + \sigma_M p_M(t)dB_t - \zeta_M p_M(t_-)dN_t. \qquad (37)$$

Consider the portfolio choice as an independent decision of the consumption problem. The consumer obtains income and has to finance its consumption stream from wealth,

$$dW_t = (\mu_M W_t - C_t)\, dt + \sigma_M W_t dB_t - \zeta_M W_{t-} dN_t \qquad (38)$$

which assumes a constant investment opportunity set, in particular, D_t is constant.

One can think of the original problem with the budget constraint (36) as having been reduced to a simple Ramsey problem, in which we seek an optimal consumption rule given that income is generated by the uncertain yield of a (composite) asset (Merton 1973).

Define the *value function* as

$$V(W_0) \equiv \max_{\{C_t\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t) dt, \quad s.t. \quad (38), \quad W_0 > 0. \qquad (39)$$

The Bellman equation becomes when choosing the control $C_s \in \mathbb{R}_+$ at time s

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + (\mu_M W_s - C_s)\, V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} + (V((1-\zeta_M)W_s) - V(W_s))\lambda \right\}.$$

Because its a necessary condition, the *first-order conditions* is

$$u'(C_s) - V_W(W_s) \;=\; 0 \quad \Rightarrow \quad V_W(W_s) = u'(C_s) \qquad (40)$$

for any interior solution at any time $s = t \in [0, \infty)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) &= \left((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t V_{WW} - u'(C((1-\zeta_M)W_t))(1-\zeta_M)\lambda\right) dt \\ &\quad + \sigma_M W_t V_{WW} dB_t + (u'(C((1-\zeta_M)W_{t-})) - u'(C(W_{t-})))dN_t \\ &= \left((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t u''(C_t)C_W - u'(C((1-\zeta_M)W_t))(1-\zeta_M)\lambda\right) dt \\ &\quad + \sigma_M W_t u''(C_t) C_W dB_t + (u'(C((1-\zeta_M)W_{t-})) - u'(C(W_{t-})))dN_t, \end{aligned} \qquad (41)$$

which implicitly determines the optimal consumption path. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$).

To shed some light on the effects of uncertainty, we use the Euler equation (41) and obtain the (implicit) risk premium as

$$
\begin{aligned}
\frac{du'(C_t)}{u'(C_{t-})} &= \left(\rho - \mu_M + \lambda - \sigma_M^2 W_t \frac{u''(C_t)}{u'(C_t)} C_W - \frac{u'(C((1-\zeta_M)W_t))}{u'(C(W_t))}(1-\zeta_M)\lambda\right) dt \\
&\quad + \sigma_M W_t \frac{u''(C_t)}{u'(C_t)} C_W dB_t + \left(\frac{u'(C((1-\zeta_M)W_{t-}))}{u'(C(W_{t-}))} - 1\right) dN_t \\
\Rightarrow \quad \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right] &= \rho - \mu_M + E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 + \frac{u'(C((1-\zeta_M)W_t))}{u'(C(W_t))}\zeta_M \lambda\right],
\end{aligned}
$$

which may be written as

$$
\mu_M - E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma_M^2 + \frac{u'(C((1-\zeta_M)W_t))}{u'(C(W_t))}\zeta_M \lambda\right] = \rho - \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right]. \tag{42}
$$

The full expected rate of return on equity is $\mu_M - \zeta_M \lambda$. We may define the left-hand side as the *certainty equivalent rate of return on saving*, that is the expected rate of return on saving, conditioned on no disasters, less the expected *implicit risk premium*. The latter gives the minimum difference an individual requires to accept an uncertain rate of return, between its expected value (conditioned on no disasters) and the certain rate of return that it is indifferent to. On the right-hand side, we have the expected cost of forgone consumption, i.e. the rate of time preference, and the expected rate of change of marginal utility.

2.3.3 A more comprehensive approach

Define the *value function* as

$$
V(W_0) \equiv \max_{\{(w_t, C_t)\}_{t=0}^{\infty}} E_0 \int_0^{\infty} e^{-\rho t} u(C_t) dt, \quad s.t. \quad (36), \quad W_0 > 0. \tag{43}
$$

The Bellman equation becomes when choosing the control $(w_s, C_s) \in \mathbb{R} \times \mathbb{R}_+$ at time s

$$
\begin{aligned}
\rho V(W_s) &= \max_{(w_s, C_s)} \big\{ u(C_s) + ((\mu - r) w_s W_s + r W_s - C_s) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} \\
&\quad + \big(V((e^{\kappa} + (e^{\nu_1} - e^{\kappa}) w_s) W_s) q \\
&\quad + V((1 + (e^{\nu_2} - 1) w_s) W_s)(1-q) - V(W_s) \big) \lambda \big\}.
\end{aligned}
$$

Because its a necessary condition, the *first-order conditions* are

$$
u'(C_s) - V_W = 0 \quad \Rightarrow \quad V_W = u'(C_s) \tag{44}
$$

$$
\begin{aligned}
0 &= (\mu - r)W_s V_W + w_s \sigma^2 W_s^2 V_{WW} + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)(e^{\nu_1} - e^\kappa)W_s q\lambda \\
&\quad + V_W((1 + (e^{\nu_2} - 1)w_s)W_s)(1-q)(e^{\nu_2} - 1)W_s\lambda \\
\Rightarrow \quad w_s &= -\frac{V_W(W_s)}{V_{WW}(W_s)W_s}\frac{\mu - r}{\sigma^2} - \frac{V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)}{V_{WW}(W_s)W_s}\frac{e^{\nu_1} - e^\kappa}{\sigma^2}q\lambda \\
&\quad - \frac{V_W((1 + (e^{\nu_2} - 1)w_s)W_s)}{V_{WW}(W_s)W_s}\frac{e^{\nu_2} - 1}{\sigma^2}(1-q)\lambda \qquad (45)
\end{aligned}
$$

for any interior solution at any time $s = t \in [0, \infty)$. In that, an analytical solution for the optimal shares is no longer available except for specific parametric restrictions, e.g., for the case where $\nu_1 \equiv \kappa$ and $q \equiv 1$, i.e., the bond default is equal to the size of the disaster. Then, the optimal share of wealth allocated to the risky asset will not be affected by rare events.

It can be shown that the *Euler equation* for consumption is (cf. appendix)

$$
\begin{aligned}
du'(C_t) &= \big((\rho - ((\mu - r)w_t + r) + \lambda)u'(C_t) - w_t^2\sigma^2 W_t V_{WW} \\
&\quad - u'(C((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t))(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\
&\quad - u'(C((1 + (e^{\nu_2} - 1)w_t)W_t))(1 + (e^{\nu_2} - 1)w_t)(1-q)\lambda\big)dt + w_t\sigma W_t V_{WW} dB_t \\
&\quad + (u'(C((1 + (J_t - D_t)w_{t-} + D_t)W_{t-})) - u'(C(W_{t-})))dN_t. \qquad (46)
\end{aligned}
$$

Before we proceed, we show that the (implicit) risk premium from the Euler equation is similar to the short-cut approach in the case of a constant investment opportunity set.

Case 2.5 (Constant investment opportunties) *Define*

$$
\mu_M \equiv (\mu - r)w_t + r, \quad \sigma_M \equiv w_t\sigma, \quad \zeta_M \equiv 1 - e^\kappa - (e^{\nu_1} - e^\kappa)w_t,
$$

and we obtain optimal portfolio shares,

$$
w_t = -\frac{V_W(W_t)}{V_{WW}(W_t)W_t}\frac{\mu - r}{\sigma^2} - \frac{V_W((1 - \zeta_M)W_t)}{V_{WW}(W_t)W_t}\frac{e^{\nu_1} - e^\kappa}{\sigma^2}\lambda. \qquad (47)
$$

The Euler equation (46) reads for $q = 1$,

$$
\begin{aligned}
du'(C_t) &= \big((\rho - \mu_M + \lambda)u'(C_t) - \sigma_M^2 W_t V_{WW} - u'(C((1 - \zeta_M)W_t))(1 - \zeta_M)\lambda\big)dt \\
&\quad + \sigma_M W_t V_{WW} dB_t + (u'(C((1 - \zeta_M)W_{t-})) - u'(C(W_{t-})))dN_t
\end{aligned}
$$

which coincides with (41), and the implicit risk premium is

$$
RP \equiv E\left[-\frac{u''(C_t)}{u'(C_t)}C_W W_t \sigma_M^2 + \frac{u'(C((1 - \zeta_M)W_t))}{u'(C(W_t))}\zeta_M\lambda\right], \qquad (48)
$$

It is interpreted as the premium individuals are willing to pay to hedge the market risks over a certainty equivalent return on saving.

2.3.4 Towards a security market line

Given demand for the risky asset and market-clearing, one usually obtains the equilibrium relation between expected return on any asset and the expected return on the market. Let $M_t = n_M(t)p_M(t) \equiv mW_t$ be the market value, where $p_M(t)$ is the price per 'share' of the market and $n_M(t)$ is the number of 'shares' of m investors (Merton 1973),

$$\begin{aligned} dM_t &= n_M(t)dp_M(t) + p_M(t)dn_M(t) = n_M(t)dp_M + p_0(t)dn_0(t) + p_t dn_t \\ &= n_M(t)dp_M(t) - mC_t dt. \end{aligned}$$

Using the budget constraint (36), we obtain

$$\begin{aligned} n_M(t)dp_M(t) - mC_t dt &= ((\mu - r)w_t mW_t + rmW_t - mC_t)\,dt + w_t\sigma mW_t dB_t \\ &\quad +((J_t - D_t)w_{t-} + D_t)mW_{t-}dN_t \\ \Leftrightarrow\ dp_M(t) &= ((\mu - r)w_t + r)\,p_M(t)dt + w_t\sigma p_M(t)dB_t \\ &\quad +((J_t - D_t)w_{t-} + D_t)p_M(t_-)dN_t, \end{aligned} \tag{49}$$

where we substituted $mW_t = n_M(t)p_M(t)$ and collected terms.

Conditioning on no disasters, we may define the instantaneous expected percentage change $\mu_M \equiv (\mu - r)w_t + r$, and its variance, $\sigma_M^2 \equiv w^2\sigma^2$ as in Merton (1973). Similarly, we define the random variable $\zeta_M(t) \equiv ((J_t - D_t)w_t + D_t)$. The full expected percentage change on equity, which includes the jump-risk and the default possibility, is lower,

$$\begin{aligned} E\left[\frac{dp_M(t)}{p_M(t_-)}\right] &= \mu_M + E\left[(J_t - D_t)w_t + D_t\right]\lambda \\ &= \mu_M - (1 - (e^{\nu_1} - e^{\kappa})w_t - e^{\kappa})\,q\lambda - (1 - e^{\nu_2})w_t(1-q)\lambda. \end{aligned} \tag{50}$$

Similarly, we obtain expected percentage change on the risky asset, and on government bills,

$$E\left[\frac{dp_t}{p_{t-}}\right] = \mu - (1 - e^{\nu_1})q\lambda - (1 - e^{\nu_2})(1-q)\lambda, \quad E\left[\frac{dp_0(t)}{p_0(t_-)}\right] = r - (1 - e^{\kappa})q\lambda. \tag{51}$$

Given the demand for risky assets, we obtain the equity premium. Using the first-order condition for consumption, optimal portfolio weights in (45) may be written as

$$\begin{aligned} w_t &= -\frac{u'(C(W_t))}{u''(C_t)C_W W_t}\frac{\mu - r}{\sigma^2} - \frac{u'(C((e^{\kappa} + (e^{\nu_1} - e^{\kappa})w_t)W_t))}{u''(C(W_t))C_W W_t}\frac{e^{\nu_1} - e^{\kappa}}{\sigma^2}q\lambda \\ &\quad - \frac{u'(C((1 + (e^{\nu_2} - 1)w_t)W_t))}{u''(C(W_t))C_W W_t}\frac{e^{\nu_2} - 1}{\sigma^2}(1-q)\lambda \\ \Leftrightarrow\ \mu_M - r &= -\frac{u''(C_t)C_W W_t}{u'(C(W_t))}\sigma_M^2 - \frac{u'(C((1 - \zeta_M)W_t))}{u'(C(W_t))}(e^{\nu_1} - e^{\kappa})q\lambda w_t \\ &\quad - \frac{u'(C((1 + (e^{\nu_2} - 1)w_t)W_t))}{u'(C(W_t))}(e^{\nu_2} - 1)(1-q)\lambda w_t, \end{aligned} \tag{52}$$

where $\mu_M \equiv (\mu - r)w_t + r$, $\sigma_M \equiv w_t \sigma$, and $\zeta_M \equiv 1 - e^{\kappa} - (e^{\nu_1} - e^{\kappa})w_t$. Hence, for $\nu_1 = \kappa$ and $q = 1$, the (traditional) equity premium will be unaffected by the jump risk. In the present analysis, the implicit risk premium, however, will still be affected. The reason is that implicit risk premium is obtained from the certainty equivalent return to saving, but the government bill has a risk of default, thus is inherently risky. This premium on the willingness to pay for avoiding the default risk is reflected in the implicit risk premium (48).

2.3.5 Explicit solutions

Similar to Merton (1971), for the class of constant relative risk aversion (CRRA) utility one obtains an explicit solution where consumption is a linear function of wealth.

Proposition 2.6 (CRRA preferences) *If utility exhibits constant relative risk aversion, i.e.* $-u''(C_t)C_t/u'(C_t) = \theta$, *then optimal consumption is proportional to wealth and optimal portfolio weights are constant,* $C_t = C(W_t) = bW_t$, *where*

$$b \equiv \left(\rho + \lambda - (1-\theta)\mu_M - \left((1-\zeta_M)^{1-\theta} q + (1 + (e^{\nu_2} - 1)w)^{1-\theta}(1-q)\right)\lambda + (1-\theta)\theta \tfrac{1}{2}\sigma_M^2\right)/\theta, \quad (53)$$

where $\zeta_M \equiv 1 - e^{\kappa} - (e^{\nu_1} - e^{\kappa})w$. *Optimal portfolio weights are (implicitly) given by*

$$w = \frac{\mu - r}{\theta\sigma^2} + (1-\zeta_M)^{-\theta} \frac{e^{\nu_1} - e^{\kappa}}{\theta\sigma^2} q\lambda + (1 + (e^{\nu_2} - 1)w)^{-\theta} \frac{e^{\nu_2} - 1}{\theta\sigma^2}(1-q)\lambda. \quad (54)$$

Proof. see Appendix 5.1.4 ■

Corollary 2.7 (Default risk) *Consider, for illustration, the case of a constant investment opportunity set. Use the policy function to obtain the risk premium (48) as*

$$RP = \theta\sigma_M^2 + (1-\zeta_M)^{-\theta}\zeta_M\lambda. \quad (55)$$

The conditional equity premium (conditioned on no disasters) from (52) would be

$$\begin{aligned} \mu_M - r &= \theta\sigma_M^2 - (1-\zeta_M)^{-\theta}(e^{\nu_1} - e^{\kappa})q\lambda w - (1 + (e^{\nu_2} - 1)w)^{-\theta}(e^{\nu_2} - 1)(1-q)\lambda w \\ &= \theta\sigma_M^2 + (1-\zeta_M)^{-\theta}\zeta_M\lambda q - (1-\zeta_M)^{-\theta}(1 - e^{\kappa})q\lambda \\ &\quad + (1 + (e^{\nu_2} - 1)w)^{-\theta}(1 - e^{\nu_2})(1-q)\lambda w, \end{aligned} \quad (56)$$

whereas in the present analysis,

$$\begin{aligned} EP &\equiv \mu_M - \left((1 - (e^{\nu_1} - e^{\kappa})w - e^{\kappa})\, q + (1 - e^{\nu_2})(1-q)w\right)\lambda - (r - (1 - e^{\kappa})q)\lambda \\ &= \mu_M - r + (e^{\nu_1} - e^{\kappa})wq\lambda - (1 - e^{\nu_2})(1-q)w\lambda \end{aligned} \quad (57)$$

is the full equity premium, which includes the jump-risk and the default possibility, i.e., the expected rate of return on the market portfolio net of the expected rate of return on bills.

Recall that the implicit risk premium (RP) was obtained under the assumption of constant investment opportunities, e.g., for $q = 1$. In this case, RP will be higher than the (traditional) equity premium by $(1 - \zeta_M)^{-\theta}(1 - e^{\kappa})\lambda$. This term is interpreted as the market price of the default risk. If there was no default, the implicit risk premium again has the usual interpretation of the equity premium.

2.3.6 General equilibrium prices

This section shows that general equilibrium conditions pin down the prices in the economy. For illustration, with constant relative risk aversion (CRRA) the Euler equation (46) implies

$$\begin{aligned} dC_t &= \frac{\mu_M - \rho - \lambda - \frac{1}{2}(1-\theta)\theta\sigma_M^2 + (1-\zeta_M)^{1-\theta}q\lambda + (1+(e^{\nu_2}-1)w_t)^{1-\theta}(1-q)\lambda}{\theta} C_t dt \\ &+ \sigma_M C_t dB_t + ((J_t - D_t)w_{t-} + D_t)C(W_{t-})dN_t \end{aligned} \tag{58}$$

where we employed the inverse function $C = g(u'(C))$ which has

$$g'(u'(C)) = 1/u''(C), \quad g''(u'(C)) = -u'''(C)/(u''(C))^3.$$

Because output is perishable, using the market clearing condition $Y_t = C_t = A_t$

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t + \bar{J}_t A_{t-} dN_t, \tag{59}$$

the risk free rate and the both market prices of risk are (implicitly) pinned down in general equilibrium. In particular, we obtain J_t as a function of random variables $\bar{J}_t$ and D_t, that is the investment opportunity set is inherently stochastic, $\bar{J}_t = (J_t - D_t)w + D_t$.

In general equilibrium it pins down

$$\theta\bar{\mu} = \mu_M - \rho - \tfrac{1}{2}(1-\theta)\theta\sigma_M^2 - \lambda + (1-\zeta_M)^{1-\theta}q\lambda + (1+(e^{\nu_2}-1)w_t)^{1-\theta}(1-q)\lambda$$

and the variance of the market portfolio conditioned on no disasters is $\bar{\sigma} = \sigma_M$. Conditioned on no default of government bills,

$$e^{\bar{\nu}} - 1 = (e^{\nu_2} - 1)w,$$

whereas conditioned on default of government bills,

$$e^{\bar{\nu}} - 1 = (e^{\nu_1} - e^{\kappa})w + e^{\kappa} - 1 \Leftrightarrow 1 - e^{\bar{\nu}} = \zeta_M.$$

In that, the price of the risky assets may jump even for $\bar{\nu} = 0$ because of general equilibrium effects of the government bill default, and for $\kappa = 0$ we obtain that $\nu_1 = \nu_2$.

3 A prototype production economy

3.1 A model of growth under uncertainty (one asset)

This section illustrates that non-linearities in a prototype neoclassical DSGE model can generate a time-varying risk premium. We use a version of Merton's (1975) asymptotic theory of growth under uncertainty (see also Eaton 1981, Cox et al. 1985).

3.1.1 Description of the economy

Technology. At any time, the economy has some amounts of capital, labor, and knowledge, and these are combined to produce output. The production function is a constant return to scale technology $Y_t = A_t F(K_t, L)$, where K_t is the aggregate capital stock, L is the constant population size, and A_t is the stock of knowledge or total factor productivity (TFP), which in turn is driven by a standard Brownian motion B_t

$$dA_t = \bar{\mu} A_t dt + \bar{\sigma} A_t dB_t. \tag{60}$$

A_t has a log-normal distribution with $E_0(\ln A_t) = \ln A_0 + (\bar{\mu} - \frac{1}{2}\bar{\sigma}^2)t$, and $Var_0(\ln A_t) = \bar{\sigma}^2 t$.

The capital stock increases if gross investment exceeds stochastic capital depreciation,

$$dK_t = (I_t - \delta K_t)dt + \sigma K_t dZ_t, \tag{61}$$

where Z_t is a standard Brownian motion (uncorrelated with B_t). Unlike in Merton's (1975) model, the assumption of stochastic depreciation introduces instantaneous riskiness, which makes physical capital indeed a risky asset (for similar examples see Turnovsky 2000).

Preferences. Consider an economy with a single consumer, interpreted as a representative "stand in" for a large number of identical consumers. The consumer maximizes expected life-time utility

$$U_0 \equiv E_0 \int_0^\infty e^{-\rho t} u(C_t) dt, \quad u' > 0, \; u'' < 0 \tag{62}$$

subject to

$$dW_t = ((r_t - \delta)W_t + w_t^L - C_t)dt + \sigma W_t dZ_t. \tag{63}$$

$W_t \equiv K_t / L$ denotes individual wealth, r_t is the rental rate of capital, and w_t^L is labor income. The paths of factor rewards are taken as given by the representative consumer.

Equilibrium properties. In equilibrium, factors of production are rewarded with value marginal products, $r_t = Y_K$ and $w_t^L = Y_L$. The goods market clearing condition demands

$$Y_t = C_t + I_t. \tag{64}$$

Solving the model requires the aggregate capital accumulation constraint (61), the goods market equilibrium (64), equilibrium factor rewards of perfectly competitive firms, and the first-order condition for consumption. It is a system of differential equations determining, given initial conditions, the paths of K_t, Y_t, r_t, w_t^L and C_t, respectively.

3.1.2 The short-cut approach

Define the value of the optimal program as

$$V(W_0, A_0) = \max_{\{C_t\}_{t=0}^{\infty}} U_0 \quad s.t. \quad (63) \quad \text{and} \quad (60) \tag{65}$$

denoting the present value of expected utility along the optimal program. It can be shown that the first-order condition for the problem is (cf. appendix)

$$u'(C_t) = V_W(W_t, A_t), \tag{66}$$

for any $t \in [0, \infty)$, making consumption a function of the state variables $C_t = C(W_t, A_t)$.

It can be shown that the *Euler equation* is (cf. appendix)

$$\begin{aligned} du'(C_t) &= (\rho - (r_t - \delta))u'(C_t)dt - \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} dB_t + V_{WW} W_t \sigma dZ_t \qquad (67) \\ &= (\rho - (r_t - \delta))u'(C_t)dt - \sigma^2 u''(C_t) C_W W_t dt + u''(C_t)\left(C_A A_t \bar{\sigma} dB_t + C_W W_t \sigma dZ_t\right), \end{aligned}$$

which implicitly determines the optimal consumption path. Using the inverse function, we are able to determine the path for consumption ($u'' \neq 0$).

To shed some light on the effects of uncertainty, we use the Euler equation and obtain the (implicit) risk premium

$$\begin{aligned} \frac{du'(C_t)}{u'(C_t)} &= \left(\rho - (r_t - \delta) - \frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma^2\right) dt + \frac{u''(C_t)}{u'(C_t)} \left(C_A A_t \bar{\sigma} dB_t + C_W W_t \sigma dZ_t\right) \\ \Rightarrow \quad \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right] &= \rho - E(r_t) + \delta + E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t\right] \sigma^2, \end{aligned}$$

which may be written as

$$E(r_t) - E\left[-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma^2\right] = \rho + \delta - \frac{1}{dt} E\left[\frac{du'(C_t)}{u'(C_t)}\right], \tag{68}$$

where the left-hand side is the *certainty equivalent rate of return on saving*, and the term in brackets defines the *implicit risk premium.* Hence, the implicit risk premium indeed refers to the rewards that investors demand for bearing systematic market risk, while it does not account for the risk of a stochastically changing investment opportunity set.

3.1.3 Explicit solutions

A convenient way is to describe the behavior of the economy is in terms of the evolution of C_t, A_t and W_t. Similar to the endowment economy there are explicit solutions available, due to the non-linearities only for specific parameter restrictions. Below we use two known restrictions where the *policy function* $C_t = C(A_t, W_t)$ is available, and all economic variables can be solved for in closed form.

Proposition 3.1 (linear-policy-function) *If the production function is Cobb-Douglas,* $Y_t = A_t K_t^\alpha L^{1-\alpha}$*, utility exhibits constant relative risk aversion, i.e.* $-u''(C_t)C_t/u'(C_t) = \theta$*, and* $\alpha = \theta$*, then optimal consumption is linear in wealth.*

$$\alpha = \sigma \quad \Rightarrow \quad C_t = C(W_t) = \phi W_t \quad \text{where} \quad \phi \equiv (\rho + (1-\theta)\delta)/\theta + \tfrac{1}{2}(1-\theta)\sigma^2 \tag{69}$$

Proof. see Appendix 5.3.2 ■

Corollary 3.2 *Use the policy function to obtain the (implicit) risk premium (68) as*

$$-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma^2 = \theta\sigma^2. \tag{70}$$

Proposition 3.3 (constant-saving-function) *If the production function is Cobb-Douglas,* $Y_t = A_t K_t^\alpha L^{1-\alpha}$*, utility exhibits constant relative risk aversion, i.e.* $-u''(C_t)C_t/u'(C_t) = \theta$*, and the subjective discount factor is* $\rho = (\alpha\theta - 1)\delta - \theta\bar{\mu} + \frac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right)$*, then optimal consumption is proportional to income (i.e. non-linear in wealth).*

$$\rho = \bar{\rho} \quad \Rightarrow \quad C_t = C(W_t, A_t) = (1-s)A_t W_t^\alpha, \quad \theta > 1, \quad \text{where} \quad s \equiv 1/\theta \tag{71}$$

Proof. see Appendix 5.3.3 ■

Corollary 3.4 *Use the policy function to obtain the (implicit) risk premium (68) as*

$$-\frac{u''(C_t)}{u'(C_t)} C_W W_t \sigma^2 = \alpha\theta\sigma^2. \tag{72}$$

As shown, the (implicit) risk premium for $u''(C_t)C_t/u'(C_t) = \theta$ depends on the curvature of the policy function. Moreover, any policy function where optimal consumption is a power function of wealth, $C_W(A_t, W_t)W_t = aC_t(A_t, W_t)$ where $a \in \mathbb{R}$, implies a constant (implicit) risk premium. Because these solutions are obtained only for specific parameter restrictions, we conclude that for the general case the (implicit) risk premium will be time-varying.

4 Conclusion

This paper shows that non-linearities and non-normalities are important to generate key features of the risk premium. For this purpose, we employ explicit solutions of DSGE models to shed light on the determinants of the risk premium in general equilibrium.

We derive closed-form solutions for a continuous-time version of Lucas' fruit-tree model, and for the case of non-linearities using a stochastic neoclassical growth model in order to study the risk premium. The main result is that the (implicit) risk premium in addition to the standard parameters for the risk aversion and the level of uncertainty, in general depends on non-linearities, e.g. the curvature of the policy function. Moreover we find that in most DSGE models the (implicit) risk premium should be time-varying.

From a theoretical point of view, this paper shows that formulating the DSGE model in continuous-time gives closed-form solution for a large class of interesting macro-finance models (in the tradition of Merton 1975, Eaton 1981, Cox et al. 1985). It thus circumvents the problem induced by approximation schemes which is especially important when analyzing the effects of uncertainty (Schmitt-Grohé and Uribe 2004).

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5 Appendix

5.1 Lucas fruit-tree model in continuous-time (two assets)

5.1.1 Deriving the budget constraint

Consider a risky asset and a government bill with default risk. Suppose the price of the risky asset follows

$$dp_t = \mu p_t dt + \sigma p_t dB_t + J_t p_{t-} dN_t$$

where μ denotes the instantaneous conditional expected percentage change in the price of asset i, σ^2 its instantaneous conditional variance, B_t is a standard Brownian motion, and J_t is a random variable representing the sensitivity of the asset price with respect to a jump of the Poisson process N_t at arrival rate λ. A government bill (riskless in normal times) obeys

$$dp_0(t) = p_0(t) r dt + D_t dN_t$$

where D_t is a random variable denoting a random default risk during a contraction.

Consider a portfolio strategy which holds n_t units of the risky asset and $n_0(t)$ units of the riskless asset with default risk, such that

$$W_t = n_0(t) p_0(t) + p_t n_t$$

denotes the portfolio value. Using Itô's formula, it follows

$$\begin{aligned} dW_t &= p_0(t) dn_0(t) + n_0(t) p_0(t) r dt + p_t dn_t + n_t p_t \mu dt + n_t p_t \sigma dB_t \\ &\quad + (n_t p_{t-} J_t + n_0(t) p_0(t_-) D_t)\, dN_t \\ &= p_0(t) dn_0(t) + n_0(t) p_0(t) r dt + p_t dn_t + w_t \mu W_t dt + w_t \sigma W dB_t \\ &\quad + (w_{t-} J_t + (1 - w_{t-}) D_t)\, W_{t-} dN_t \end{aligned} \tag{73}$$

where $w_t W_t \equiv n_t p_t$ denotes the amount invested in the risky asset. Since investors use their savings to accumulate assets,

$$p_0(t) dn_0(t) + p_t dn_t = (\pi_0(t) n_0(t) + \pi_t n_t - C_t)\, dt$$

where π_t denotes per unit dividend payments on asset the risky asset. Hence,

$$\begin{aligned} dW_t &= (\pi_0(t) n_0(t) + \pi_t n_t - C_t)\, dt + r W_t dt + (\mu - r) w_t W_t dt + \sigma w_t W_t dB_t \\ &\quad + ((J_t - D_t) w_{t-} + D_t)\, W_{t-} dN_t \end{aligned}$$

If there are no dividend payments, $\pi_t = 0$, the budget constraint reads

$$\begin{aligned} dW_t &= ((\mu - r) w_t W_t + r W_t - C_t)\, dt + \sigma w_t W_t dB_t \\ &\quad + ((J_t - D_t) w_{t-} + D_t)\, W_{t-} dN_t \end{aligned} \tag{74}$$

5.1.2 The short-cut approach

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s) \right\} \tag{75}$$

Using Itô's formula (see e.g. Protter 2004, Sennewald 2007),

$$\begin{aligned} dV(W_s) &= \left((\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \right) dt + \sigma_M W_s V_W dB_t + (V(W_s) - V(W_{s-}))dN_t \\ &= \left((\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} \right) dt + \sigma_M W_s V_W dB_t \\ &\quad + (V((1-\zeta_M)W_{s-}) - V(W_{s-}))dN_t \end{aligned}$$

where σ_M^2 is the instantaneous variance of the risky asset's return from the Brownian motion increments. If we take the expectation of the integral form, and use the property of stochastic integrals, we may write

$$E_s dV(W_s) = \left((\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} + (V((1-\zeta_M)W_s) - V(W_s))\lambda \right) dt$$

Inserting into (75) gives the Bellman equation

$$\rho V(W_s) = \max_{C_s} \left\{ u(C_s) + (\mu_M W_s - C_s) V_W + \tfrac{1}{2}\sigma_M^2 W_s^2 V_{WW} + (V((1-\zeta_M)W_s) - V(W_s))\lambda \right\}$$

The first-order condition (40) makes consumption a function of the state variable. Using the maximized Bellman equation for all $s = t \in [0, \infty)$,

$$\rho V(W_t) = u(C(W_t)) + (\mu_M W_t - C(W_t))V_W + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WW} + (V((1-\zeta_M)W_t) - V(W_t))\lambda$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W &= (\mu_M V_W + (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + (V_W((1-\zeta_M)W_t)(1+\zeta_M) - V_W(W_t))\lambda \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - \mu_M + \lambda)V_W &= (\mu_M W_t - C(W_t))V_{WW} + \sigma_M^2 W_t V_{WW} + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} \\ &\quad + V_W((1-\zeta_M)W_t)(1+\zeta_M)\lambda \end{aligned} \tag{76}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W(W_t) &= (\mu_M W_t - C_t) V_{WW} dt + \tfrac{1}{2}\sigma_M^2 W_t^2 V_{WWW} dt + \sigma_M W_t V_{WW} dB_t \\ &\quad + (V_W((1-\zeta_M)W_{t-}) - V_W(W_{t-}))dN_t \\ &= \left((\rho - \mu_M + \lambda)V_W - \sigma_M^2 W_t V_{WW} - V_W((1+\zeta_M)W_t)(1+\zeta_M)\lambda \right) dt \\ &\quad + \sigma_M W_t V_{WW} dB_t + (V_W((1-\zeta_M)W_{t-}) - V_W(W_{t-}))dN_t \end{aligned}$$

where we inserted the costate from (76). As a final step we insert the first-order condition and obtain the Euler equation (41).

5.1.3 A more comprehensive approach

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s) = \max_{(w_s, C_s)} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s) \right\} \tag{77}$$

Using Itô's formula,

$$\begin{aligned} dV(W_s) &= \left(\left((w_s(\mu - r) + r)W_s - C_s \right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} \right) dt + w_s \sigma W_s V_W dB_t \\ &\quad + (V(W_s) - V(W_{s-}))dN_t \\ &= \left(\left((w_s(\mu - r) + r)W_s - C_s \right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} \right) dt + w_s \sigma W_s V_W dB_s \\ &\quad + (V((1 + (J_s - D_s)w_{t_-} + D_s)W_{s_-}) - V(W_{s-}))dN_s \end{aligned}$$

where σ^2 is the instantaneous variance of the risky asset's return from the Brownian motion increments. If we take the expectation of the integral form, and use the property of stochastic integrals, we may write

$$\begin{aligned} E_s dV(W_s) &= \big(\left((w_s(\mu - r) + r)W_s - C_s \right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} \\ &\quad + (E[V((1 + (J_s - D_s)w_s + D_s)W_s)] - V(W_s))\lambda \big) dt \\ &= \big(\left((w_s(\mu - r) + r)W_s - C_s \right) V_W + \tfrac{1}{2} w_s^2 \sigma^2 W_s^2 V_{WW} + (V((e^\kappa + (e^{\nu_1} - e^\kappa)w_s)W_s)q \\ &\quad + V((1 + (e^{\nu_2} - 1)w_s)W_s)(1 - q) - V(W_s))\lambda \big) dt \end{aligned}$$

The first-order conditions (44) and (45) make the controls a function of the state variable. Using the maximized Bellman equation,

$$\begin{aligned} \rho V(W_t) &= u(C(W_t)) + ((\mu - r)w(W_t)W_t + rW_t - C(W_t)) V_W + \tfrac{1}{2} w(W_t)^2 \sigma^2 W_t^2 V_{WW} \\ &\quad + \big(V((e^\kappa + (e^{\nu_1} - e^\kappa)w(W_t))W_t)q + V((1 + (e^{\nu_2} - 1)w(W_t))W_t)(1 - q) \\ &\quad - V(W_t) \big) \lambda \end{aligned} \tag{78}$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W &= ((\mu - r)w(W_t) + r)V_W + ((\mu - r)w(W_t)W_t + rW_t - C(W_t))V_{WW} \\ &\quad + w(W_t)^2 \sigma^2 W_t V_{WW} + \tfrac{1}{2} w(W_t)^2 \sigma^2 W_t^2 V_{WWW} \\ &\quad + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\ &\quad + V_W((1 + (e^{\nu_2} - 1)w_t)W_t)(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda - V_W(W_t)\lambda \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - ((\mu - r)w_t + r) + \lambda)V_W &= ((\mu - r)w_t W_t + rW_t - C_t)V_{WW} \\ &\quad + w_t^2 \sigma^2 W_t V_{WW} + \tfrac{1}{2} w_t^2 \sigma^2 W_t^2 V_{WWW} \\ &\quad + V_W((e^\kappa + (e^{\nu_1} - e^\kappa)w_t)W_t)(e^\kappa + (e^{\nu_1} - e^\kappa)w_t)q\lambda \\ &\quad + V_W((1 + (e^{\nu_2} - 1)w_t)W_t)(1 + (e^{\nu_2} - 1)w_t)(1 - q)\lambda \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W(W_t) &= ((\mu-r)w_tW_t + rW_t - C_t)\,V_{WW}dt + \tfrac{1}{2}w_t^2\sigma^2W_t^2V_{WWW}dt + w_t\sigma W_tV_{WW}dB_t \\ &\quad +(V_W((1+(J_t-D_t)w_{t\_}+D_t)W_{t-}) - V_W(W_{t-}))dN_t \\ &= \Big((\rho-((\mu-r)w_t+r)+\lambda)V_W - w_t^2\sigma^2W_tV_{WW} \\ &\quad -V_W((e^\kappa+(e^{\nu_1}-e^\kappa)w_t)W_t)(e^\kappa+(e^{\nu_1}-e^\kappa)w_t)q\lambda \\ &\quad -V_W((1+(e^{\nu_2}-1)w_t)W_t)(1+(e^{\nu_2}-1)w_t)(1-q)\lambda\Big)dt \\ &\quad +w_t\sigma W_tV_{WW}dB_t + (V_W((1+(J_t-D_t)w_{t\_}+D_t)W_{t-}) - V_W(W_{t-}))dN_t \end{aligned}$$

where we inserted the costate from above. As a final step, we insert the first-order condition for consumption to obtain the Euler equation (46).

5.1.4 Proof of Proposition 2.6

For constant relative risk aversion, θ, the utility function reads

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad \theta > 0 \tag{79}$$

From (78) we have the maximized Bellman equation where we use functional equations from first-order conditions (44) and (45),

$$\begin{aligned} C(W_t) &= V_W^{-\frac{1}{\theta}} \\ w(W_t) &= \frac{C(W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1}C_WW_t}\frac{\mu-r}{\sigma^2} + \frac{C((e^\kappa+(e^{\nu_1}-e^\kappa)w_t)W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1}C_WW_t}\frac{e^{\nu_1}-e^\kappa}{\sigma^2}q\lambda \\ &\quad + \frac{C((1+(e^{\nu_2}-1)w_t)W_t)^{-\theta}}{\theta C(W_t)^{-\theta-1}C_WW_t}\frac{e^{\nu_2}-1}{\sigma^2}(1-q)\lambda \end{aligned}$$

We may use an *educated guess*,

$$\bar{V} = \mathbb{C}_0\frac{W_t^{1-\theta}}{1-\theta} \tag{80}$$

and $\bar{V}_W = \mathbb{C}_0W_t^{-\theta}$, and $\bar{V}_{WW} = -\theta\mathbb{C}_0W_t^{-\theta-1}$ to solve the resulting equation. Note that optimal consumption is linear in wealth, $C(W_t) = \mathbb{C}_0^{-1/\theta}W_t$, which implies that the optimal portfolio weight is constant and implicitly given by

$$w = \frac{\mu-r}{\theta\sigma^2} + (e^\kappa+(e^{\nu_1}-e^\kappa)w)^{-\theta}\frac{e^{\nu_1}-e^\kappa}{\theta\sigma^2}q\lambda + (1+(e^{\nu_2}-1)w)^{-\theta}\frac{e^{\nu_2}-1}{\theta\sigma^2}(1-q)\lambda$$

Using the result that $w(W_t) = w$ is constant, and inserting the candidate policy function for consumption into the maximized Bellman equation (78), we arrive at

$$\begin{aligned} \rho\mathbb{C}_0\frac{W_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_0^{-\frac{1-\theta}{\theta}}W_t^{1-\theta}}{1-\theta} + ((\mu-r)wW_t + rW_t - \mathbb{C}_0^{-\frac{1}{\theta}}W_t)\mathbb{C}_0W_t^{-\theta} - \theta\tfrac{1}{2}w^2\sigma^2\mathbb{C}_0W_t^{1-\theta} \\ &\quad +\Big((e^\kappa+(e^{\nu_1}-e^\kappa)w)^{1-\theta}q + (1+(e^{\nu_2}-1)w)^{1-\theta}(1-q) - 1\Big)\mathbb{C}_0\frac{W_t^{1-\theta}}{1-\theta}\lambda \end{aligned}$$

Defining $\mu_M \equiv (\mu - r)w + r$, $\sigma_M \equiv w\sigma$, $\zeta_M \equiv 1 - e^{\kappa} - (e^{\nu_1} - e^{\kappa})w_t$, and collecting terms,

$$\begin{aligned}
\rho &= \mathbb{C}_0^{-\frac{1}{\theta}} + (1-\theta)((\mu - r)w + r - \mathbb{C}_0^{-\frac{1}{\theta}}) - (1-\theta)\theta\tfrac{1}{2}w^2\sigma^2 \\
&\quad + \big((e^{\kappa} + (e^{\nu_1} - e^{\kappa})w)^{1-\theta}q + (1 + (e^{\nu_2} - 1)w)^{1-\theta}(1-q) - 1\big)\lambda \\
\Rightarrow \quad \mathbb{C}_0 &= \left(\frac{\rho + \lambda - (1-\theta)\mu_M - \big((1-\zeta_M)^{1-\theta}q + (1 + (e^{\nu_2} - 1)w)^{1-\theta}(1-q)\big)\lambda}{\theta} + (1-\theta)\tfrac{1}{2}\sigma_M^2\right)^{-\theta}
\end{aligned}$$

This proofs that the guess (80) indeed is a solution, and by inserting the guess together with the constant we obtain the policy functions for the portfolio weights and consumption.

5.1.5 Identifying restrictions in general equilibrium

The asset market 'solves' a problem at each instant of time, such that equilibrium prices should be expressible as some fixed function of the state of the economy, $p_t = F(A_t)$ which is assumed to be C^2 (following the idea of the Black-Scholes formula). Observe that by stating (34), we implicitly impose a restriction on the parameter space. The reason is that

$$\begin{aligned}
p_t &= F_A(dA_t + \bar{J}_t A_{t-} dN_t) + \tfrac{1}{2}F_{AA}\bar{\sigma}^2 A_t^2 dt + (F(A_t) - F(A_{t-}))dN_t \\
&= \bar{\mu}F_A A_t dt + \tfrac{1}{2}F_{AA}\bar{\sigma}^2 A_t^2 dt + \bar{\sigma}A_t F_A dB_t + (F((1+\bar{J}_t)A_{t-}) - F(A_{t-}))dN_t
\end{aligned}$$

implies

$$\begin{aligned}
\mu F &= F_A \bar{\mu} A_t + \tfrac{1}{2}F_{AA}\bar{\sigma}^2 A_t^2 \\
\sigma F &= F_A \bar{\sigma} A_t \quad \Rightarrow \quad F_{AA}\bar{\sigma}A_t = (\sigma - \bar{\sigma})F_A \\
(1 + J_t)F(A_{t-}) &= F((1+\bar{J}_t)A_{t-})
\end{aligned}$$

It does not help to pin down and/or it does not put a restriction of J_t as a function of $\bar{J}_t$. However, from the first two equations

$$\mu = \left(\bar{\mu}/\bar{\sigma} + \tfrac{1}{2}(\sigma - \bar{\sigma})\right)\sigma. \tag{81}$$

5.2 Lucas fruit-tree model in continuous-time (multiple assets)

5.2.1 Deriving the budget constraint (matrix notation)

Consider $n+1$ assets. Suppose that the price of an asset i follows

$$dp_i(t) = \mu_i p_i(t)dt + p_i(t)\gamma_i^{\top} dB_t$$

μ_i denotes the instantaneous conditional expected percentage change in the price of asset i, $\gamma_i \equiv [\gamma_{i1}, ..., \gamma_{in}]^{\top}$ a $n\times 1$ vector, $\gamma \equiv [\gamma_1, ..., \gamma_n]^{\top}$ a $n\times n$ matrix, $\Omega \equiv \gamma\gamma^{\top}$ the positive definite, non-singular, instantaneous conditional covariance matrix, and $B_t \equiv [B_1(t), ..., B_n(t)]^\top$ is a n-dimensional (uncorrelated) standard Brownian motion. Let $\mu \equiv [\mu_i, ..., \mu_n]^\top$ be the vector of instantaneous conditional expected percentage price changes for the n risky assets,

$$dp_t = \text{diag}(p_t)\mu dt + \text{diag}(p_t)\gamma dB_t \tag{82}$$

denotes a geometric Brownian motion. Observe that $\text{diag}(p_t)$ denotes the $n \times n$ matrix with the vector p_t along the main diagonal and zeros off the diagonal. Consider a portfolio strategy which holds $n_i(t)$ units of the asset $i \neq 0$ and $n_0(t)$ units of the riskless asset, where

$$W_t = n_0(t)p_0(t) + p_t^\top n_t$$

$p_t \equiv [p_1(t), ..., p_n(t)]^\top$, $n_t \equiv [n_1(t), ..., n_n(t)]^\top$ denotes the portfolio value, which obeys

$$\begin{aligned} dW_t &= p_0(t)dn_0(t) + n_0(t)p_0(t)rdt + p_t^\top dn_t + n_t^\top \text{diag}(p_t)\mu dt + n_t^\top \text{diag}(p_t)\gamma dB_t \\ &= p_0(t)dn_0(t) + n_0(t)p_0(t)rdt + p_t^\top dn_t + w_t^\top \mu W_t dt + w_t^\top \gamma W dB_t \end{aligned} \tag{83}$$

where $w_t W_t \equiv (n_t^\top \text{diag}(p_t))^\top = [n_1(t)p_1(t), ..., n_n(t)p_n(t)]^\top$ and $w_0(t) = 1 - w_t^\top \hat{1}$ denotes the amount invested in the risky asset. Since investors use their savings to accumulate assets,

$$p_0(t)dn_0(t) + p_t^\top dn_t = \left(\pi_0(t)n_0(t) + \pi_t^\top n_t - C_t\right) dt$$

where $\pi_t \equiv [\pi_1, ..., \pi_n]^\top$, and $\pi_i(t)$ denotes per unit dividend payments on asset i. Then

$$\begin{aligned} dW_t &= \left(\pi_0(t)n_0(t) + \pi_t^\top n_t - C_t\right) dt + n_0(t)p_0(t)rdt + n_t^\top \text{diag}(p_t)\mu dt + n_t^\top \text{diag}(p_t)\gamma dB_t \\ &= \left(\pi_0(t)n_0(t) + \pi_t^\top n_t - C_t\right) dt + rW_t dt + w_t^\top(\mu - \hat{r})W_t dt + w_t^\top \gamma W_t dB_t \end{aligned}$$

If there are no dividend payments, $\pi_t = 0$, the budget constraint reads

$$dW_t = \left(w_t^\top(\mu - \hat{r})W_t + rW_t - C_t\right) dt + w_t^\top \gamma W_t dB_t \tag{84}$$

where $\hat{r} \equiv [r, ..., r]^\top = [1, ..., 1]^\top r \equiv \hat{1}^\top r$.

5.2.2 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s) = \max_{(w_s, C_s)} \left\{ u(C_s) + \frac{1}{dt} E_s dV(W_s) \right\} \tag{85}$$

Using Itô's formula,

$$\begin{aligned} dV(W_s) &= \left(\left((w_s^\top(\mu - \hat{r}) + r)W_s - C_s\right) V_W + \frac{1}{2}(w_s^\top \gamma)(w_s^\top \gamma)^\top W_s^2 V_{WW} \right) dt + w_s^\top \gamma W_s V_W dB_t \\ &= \left(\left((w_s^\top(\mu - \hat{r}) + r)W_s - C_s\right) V_W + \frac{1}{2} w_s^\top \Omega w_s W_s^2 V_{WW} \right) dt + w_s^\top \gamma W_s V_W dB_t \end{aligned}$$

where $\Omega \equiv \gamma\gamma^\top = [\gamma_{ij}]$ is the $n \times n$ covariance matrix of the risky assets, which is symmetric and positive definite. If we take the expectation of the integral form, and use the property of stochastic integrals (assuming that the integrals exist), we may write

$$E_s dV(W_s) \;=\; \left(\left((w_s^\top (\mu - \hat{r}) + r) W_s - C_s \right) V_W + \frac{1}{2} w_s^\top \Omega w_s W_s^2 V_{WW} \right) dt$$

The first-order conditions (22) and (23) make the controls a function of the state variable. Using the maximized Bellman equation,

$$\rho V(W_t) = u(C(W_t)) + (w^\top (\mu - \hat{r})(W_t) W_t + r W_t - C(W_t)) V_W + \frac{1}{2} w(W_t)^\top \Omega w(W_t) W_t^2 V_{WW}$$

Use the envelope theorem to compute the costate

$$\begin{aligned} \rho V_W \;&=\; (w(W_t)^\top (\mu - \hat{r}) + r) V_W + (w^\top (\mu - \hat{r})(W_t) W_t + r W_t - C(W_t)) V_{WW} \\ &\quad + w^\top (W_t) \Omega w(W_t) W_t V_{WW} + \frac{1}{2} w^\top (W_t) \Omega w(W_t) W_t^2 V_{WWW} \end{aligned}$$

Collecting terms, we obtain

$$\begin{aligned} (\rho - (w^\top (\mu - \hat{r})(W_t) + r)) V_W \;&=\; (w^\top (\mu - \hat{r})(W_t) W_t + r W_t - C(W_t)) V_{WW} \\ &\quad + w^\top (W_t) \Omega w(W_t) W_t V_{WW} \\ &\quad + \frac{1}{2} w^\top (W_t) \Omega w(W_t) W_t^2 V_{WWW} \end{aligned} \tag{86}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W(W_t) \;&=\; \left(w_t^\top (\mu - \hat{r}) W_t + r W_t - C_t \right) V_{WW} dt + \frac{1}{2} w_t \Omega w W_t^2 V_{WWW} dt \\ &\quad + w_t^\top \gamma W_t V_{WW} dB_t \\ &=\; \left((\rho - (w_t^\top (\mu - \hat{r}) + r)) V_W - w_t^\top \Omega w_t W_t V_{WW} \right) dt + w_t^\top \gamma W_t V_{WW} dB_t \end{aligned}$$

where we inserted the costate from (86). As a final step, we insert the first-order conditions to obtain the Euler equation (24).

5.2.3 Proof of Proposition 2.1

For constant relative risk aversion, θ, the utility function reads

$$u(C_t) = \frac{C_t^{1-\theta}}{1-\theta}, \quad \theta > 0 \tag{87}$$

the maximized Bellman equation reads,

$$\rho V(W_t) = \frac{C_t^{1-\theta}}{1-\theta} + \left(w_t^\top (\mu - \hat{r}) W_t + r W_t - C_t \right) V_W + \frac{1}{2} w_t^\top \Omega w_t W_t^2 V_{WW}$$

where we insert functional equations from first-order conditions (22) and (23)

$$C(W_t) = V_W^{-\frac{1}{\theta}}, \quad \text{and} \quad w(W_t) = -\frac{V_W}{V_{WW}W_t}\Omega^{-1}(\mu - \hat{r})$$

We may use an *educated guess*,

$$\bar{V} = \mathbb{C}_0 \frac{W_t^{1-\theta}}{1-\theta} \tag{88}$$

and $\bar{V}_W = \mathbb{C}_0 W_t^{-\theta}$, and $\bar{V}_{WW} = -\theta\mathbb{C}_0 W_t^{-\theta-1}$ to solve the resulting equation,

$$\rho\mathbb{C}_0\frac{W_t^{1-\theta}}{1-\theta} = \frac{\mathbb{C}_0^{-\frac{1-\theta}{\theta}}W_t^{1-\theta}}{1-\theta} + \left(w_t^\top(\mu-\hat{r})W_t + rW_t - \mathbb{C}_0^{-\frac{1}{\theta}}W_t\right)\mathbb{C}_0 W_t^{-\theta} - \tfrac{1}{2}w_t^\top\Omega w_t\theta\mathbb{C}_0 W_t^{1-\theta}$$

we insert optimal portfolio weights w_t and solve for $\mathbb{C}_0$,

$$\begin{aligned}\frac{\rho}{1-\theta} &= \frac{\mathbb{C}_0^{-\frac{1}{\theta}} - (1-\theta)\mathbb{C}_0^{-\frac{1}{\theta}}}{1-\theta} + \frac{(\mu-\hat{r})^\top\Omega^{-1}(\mu-\hat{r})}{\theta} + r - \frac{\frac{1}{2}(\mu-\hat{r})^\top\Omega^{-1}(\mu-\hat{r})}{\theta} \\ &\Leftrightarrow\ \mathbb{C}_0 = \left(\frac{\rho-(1-\theta)r}{\theta} - (1-\theta)\frac{\frac{1}{2}(\mu-\hat{r})^\top\Omega^{-1}(\mu-\hat{r})}{\theta^2}\right)^{-\theta}\end{aligned}$$

This proofs that the guess (88) indeed is a solution, and by inserting the guess together with the constant we obtain the policy functions for the portfolio weights and consumption.

5.2.4 Proof of Proposition 2.3

For constant absolute risk aversion, η, the utility function reads

$$u(C_t) = -\frac{\exp(-\sigma C_t)}{\eta}, \quad \eta > 0 \tag{89}$$

the maximized Bellman equation reads,

$$\rho V(W_t) = -\frac{1}{\eta}\exp(-\eta C_t) + \left(w_t^\top(\mu-\hat{r})W_t + rW_t - C_t\right)V_W + \frac{1}{2}w_t^\top\Omega w_t W_t^2 V_{WW}$$

where we insert functional equations from first-order conditions (22) and (23)

$$C(W_t) = -\frac{1}{\eta}\ln V_W, \quad \text{and} \quad w(W_t) = -\frac{V_W}{V_{WW}W_t}\Omega^{-1}(\mu-\hat{r})$$

We may use an *educated guess*,

$$\bar{V} = -\frac{\mathbb{C}_0}{\mathbb{C}_1}\exp\left(-\mathbb{C}_1 W_t\right) \tag{90}$$

and $\bar{V}_W = \mathbb{C}_0 \exp(-\mathbb{C}_1 W_t)$, and $\bar{V}_{WW} = -\mathbb{C}_0\mathbb{C}_1 \exp(-\mathbb{C}_1 W_t)$ to solve the resulting equation.

$$\begin{aligned}
\rho V(W_t) &= -\frac{1}{\eta}V_W + \left(w_t^\top(\mu - \hat{r})W_t + rW_t + \frac{1}{\eta}\ln V_W\right)V_W + \frac{1}{2}w_t^\top \Omega w_t W_t^2 V_{WW} \\
\Leftrightarrow \; -\rho\frac{\mathbb{C}_0}{\mathbb{C}_1}\exp(-\mathbb{C}_1 W_t) &= -\frac{1}{\eta}\mathbb{C}_0\exp(-\mathbb{C}_1 W_t) \\
&\quad + \left(w_t^\top(\mu - \hat{r})W_t + rW_t + \frac{1}{\eta}\ln \mathbb{C}_0 - \frac{1}{\eta}\mathbb{C}_1 W_t\right)\mathbb{C}_0\exp(-\mathbb{C}_1 W_t) \\
&\quad - \frac{1}{2}w_t^\top \Omega w_t W_t^2 \mathbb{C}_0\mathbb{C}_1\exp(-\mathbb{C}_1 W_t)
\end{aligned}$$

Hence, requiring that $\mathbb{C}_1 = \eta r$ we obtain,

$$-\frac{\rho}{\mathbb{C}_1} = -\frac{1}{\eta} + \frac{1}{2}\frac{1}{\mathbb{C}_1}(\mu - \hat{r})^\top \Omega^{-1}(\mu - \hat{r}) + \frac{1}{\eta}\ln \mathbb{C}_0$$

Collecting terms, the second constant is pinned down by

$$\ln \mathbb{C}_0 = \frac{r - \rho - \frac{1}{2}(\mu - \hat{r})^\top \Omega^{-1}(\mu - \hat{r})}{r}$$

This proofs that the guess (90) indeed is a solution, and by inserting the guess together with the constants we obtain the policy functions for the portfolio weights and consumption.

5.3 The prototype production economy

5.3.1 The Bellman equation and the Euler equation

As a necessary condition for optimality the Bellman's principle gives at time s

$$\rho V(W_s, A_s) = \max_{C_s}\left\{u(C_s) + \frac{1}{dt}E_s dV(W_s, A_s)\right\}$$

Using Itô's formula yields

$$\begin{aligned}
dV &= V_W dW_s + V_A dA_s + \frac{1}{2}V_{AA}\bar{\sigma}^2 A_s^2 dt \\
&= ((r_s - \delta)W_s + w_s^L - C_s)V_W dt + V_W \sigma W_s dZ_s + V_A \bar{\mu} A_s dt + V_A \bar{\sigma} A_s dB_s \\
&\quad + \frac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right)dt
\end{aligned}$$

Using the property of stochastic integrals, we may write

$$\rho V(W_s, A_s) = \max_{C_s}\left\{u(c_s) + ((r_s - \delta)W_s + w_s^L - C_s)V_W + V_A\bar{\mu}A_s + \frac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_s^2 + V_{WW}\sigma^2 W_s^2\right)\right\}$$

for any $s \in [0, \infty)$. Because it is a necessary condition for optimality, we obtain the first-order condition (66) which makes optimal consumption a function of the state variables.

For the *evolution of the costate* we use the maximized Bellman equation

$$\rho V(W_t, A_t) = u(C(W_t, A_t)) + ((r_t - \delta)W_t + w_t^L - C(W_t, A_t))V_W + V_A \bar{\mu} A_t + \frac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) \tag{91}$$

where $r_t = r(W_t, A_t)$ and $w_t^L = w(W_t, A_t)$ follow from the firm's optimization problem, and the envelope theorem (also for the factor rewards) to compute the costate,

$$\begin{aligned} \rho V_W &= \bar{\mu} A_t V_{AW} + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} + (r_t - \delta)V_W + \frac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &\quad + V_{WW}\sigma^2 W_t \end{aligned}$$

Collecting terms we obtain

$$\begin{aligned} (\rho - (r_t - \delta))V_W &= V_{AW}\bar{\mu} A_t + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} + \frac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) \\ &\quad + \sigma^2 V_{WW} W_t \end{aligned}$$

Using Itô's formula, the costate obeys

$$\begin{aligned} dV_W &= V_{AW}\bar{\mu} A_t dt + V_{AW}\bar{\sigma} A_t dB_t + \frac{1}{2}\left(V_{WAA}\bar{\sigma}^2 A_t^2 + V_{WWW}\sigma^2 W_t^2\right) dt \\ &\quad + ((r_t - \delta)W_t + w_t^L - C_t)V_{WW} dt + V_{WW}\sigma W_t dZ_t \end{aligned}$$

where inserting yields

$$dV_W = (\rho - (r_t - \delta))V_W dt - \sigma^2 V_{WW} W_t dt + V_{AW} A_t \bar{\sigma} dB_t + V_{WW} W_t \sigma dZ_t$$

which describes the evolution of the costate variable. As a final step, we insert the first-order condition (66) to obtain the Euler equation (67).

5.3.2 Proof of Proposition 3.1

The idea of this proof is to show that together with an educated guess of the value function, both the maximized Bellman equation (91) and first order condition (66) are fulfilled. We may guess that the value function reads

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + f(A_t) \tag{92}$$

From (66), optimal consumption is a constant fraction of wealth,

$$C_t^{-\theta} = \mathbb{C}_1 W_t^{-\theta} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\theta} W_t$$

Now use the maximized Bellman equation (91), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L_t^{-\alpha}$, together with the transformation $K_t \equiv LW_t$, and insert the solution candidate,

$$\begin{aligned}
\rho V(W_t, A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + ((r_t - \delta)W_t + w_t^L - C(W_t, A_t))V_W + V_A \bar{\mu} A_t \\
&\quad + \frac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right) \\
\Leftrightarrow \ \rho \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (\alpha A_t W_t^{\alpha-1} W_t - \delta W_t + (1-\alpha) A_t W_t^{\alpha} - \mathbb{C}_1^{-1/\theta} W_t)\mathbb{C}_1 W_t^{-\theta} \\
&\quad - \frac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta}\sigma^2 - g(A_t)
\end{aligned}$$

where we defined $g(A_t) \equiv \rho f(A_t) - f_A \bar{\mu} A_t - \frac{1}{2} f_{AA} \bar{\sigma}^2 A_t^2$. When imposing the condition $\alpha = \theta$ and $g(A_t) = \mathbb{C}_1 A_t$ it can be simplified to

$$\begin{aligned}
\rho \frac{\mathbb{C}_1 W_t^{1-\theta}}{1-\theta} + g(A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{1-\theta}}{1-\theta} + (A_t W_t^{\alpha-\theta} - \delta W_t^{1-\theta} - \mathbb{C}_1^{-1/\theta} W_t^{1-\theta})\mathbb{C}_1 - \frac{1}{2}\theta \mathbb{C}_1 W_t^{1-\theta}\sigma^2 \\
\Leftrightarrow \ \rho W_t^{1-\theta} &= \theta \mathbb{C}_1^{-1/\theta} W_t^{1-\theta} - (1-\theta)\delta W_t^{1-\theta} - \frac{1}{2}\theta(1-\theta) W_t^{1-\theta}\sigma^2
\end{aligned}$$

which implies that

$$\mathbb{C}_1^{-1/\theta} = \frac{\rho + (1-\theta)\delta + \frac{1}{2}\theta(1-\theta)\sigma^2}{\theta}$$

This proofs that the guess (92) indeed is a solution, and by inserting the guess together with the constant we obtain the optimal policy function for consumption.

5.3.3 Proof of Proposition 3.3

The idea of this proof follows Section 5.3.2. An educated guess of the value function is

$$V(W_t, A_t) = \frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta} A_t^{-\theta} \tag{93}$$

From (66), optimal consumption is a constant fraction of income,

$$C_t^{-\theta} = \mathbb{C}_1 W_t^{-\alpha\theta} A_t^{-\theta} \quad \Leftrightarrow \quad C_t = \mathbb{C}_1^{-1/\theta} W_t^{\alpha} A_t$$

Now use the maximized Bellman equation (91), the property of the Cobb-Douglas technology, $F_K = \alpha A_t K_t^{\alpha-1} L^{1-\alpha}$ and $F_L = (1-\alpha) A_t K_t^{\alpha} L^{-\alpha}$, together with the transformation $K_t \equiv LW_t$, and insert the solution candidate,

$$\begin{aligned}
\rho V(W_t, A_t) &= \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{\alpha-\alpha\theta} A_t^{1-\theta}}{1-\theta} + ((r_t - \delta)W_t + w_t^L - C(W_t, A_t))V_W + V_A \bar{\mu} A_t \\
&\quad + \frac{1}{2}\left(V_{AA}\bar{\sigma}^2 A_t^2 + V_{WW}\sigma^2 W_t^2\right)
\end{aligned}$$

which is equivalent to

$$
\begin{aligned}
\rho\frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta}A_t^{-\theta} \quad = \quad & \frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}} W_t^{\alpha-\alpha\theta}A_t^{1-\theta}}{1-\theta} - \theta\frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta}\bar{\mu}A_t^{-\theta} \\
& + \left(\alpha A_t W_t^{\alpha} - \delta W_t + (1-\alpha)A_t W_t^{\alpha} - \mathbb{C}_1^{-1/\theta}W_t^{\alpha}A_t\right)\mathbb{C}_1 W_t^{-\alpha\theta}A_t^{-\theta} \\
& + \frac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right)\frac{\mathbb{C}_1 W_t^{1-\alpha\theta}}{1-\alpha\theta}A_t^{-\theta}
\end{aligned}
$$

Collecting terms gives

$$
\begin{aligned}
\rho = (1-\alpha\theta)&\frac{\mathbb{C}_1^{-\frac{1-\theta}{\theta}-1} W_t^{\alpha-1}A_t}{1-\theta} - \theta\bar{\mu} + (1-\alpha\theta)A_t W_t^{\alpha-1} - (1-\alpha\theta)\delta \\
& -(1-\alpha\theta)\mathbb{C}_1^{-1/\theta}W_t^{\alpha-1}A_t + \frac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right) \\
& \Leftrightarrow \ \rho + \theta\bar{\mu} - \frac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right) + (1-\alpha\theta)\delta = \\
& \qquad \left(\frac{\theta}{1-\theta}\mathbb{C}_1^{-1/\theta} + 1\right)(1-\alpha\theta)A_t W_t^{\alpha-1}
\end{aligned}
$$

which has a solution for $\mathbb{C}_1^{-1/\theta} = (\theta-1)/\theta$ and

$$
\rho = -\theta\bar{\mu} + \frac{1}{2}\left(\theta(1+\theta)\bar{\sigma}^2 - \alpha\theta(1-\alpha\theta)\sigma^2\right) - (1-\alpha\theta)\delta
$$

This proofs that the guess (93) indeed is a solution, and by inserting the guess together with the constant we obtain the optimal policy function for consumption.