A NEW CONTINUOUS DEMAND MODEL FOR MARKET LEVEL DATA*

Peter Davis† Ricardo Ribeiro‡

September 2007

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Abstract

This paper considers a new method of uncovering demand information from market level data on differentiated products. In particular, we propose a continuous-choice demand model with distinct advantages over the models currently in use and describe the econometric techniques for its estimation. The proposed model combines key properties of both the discrete- and continuous-choice traditions: *i*) it is flexible in the sense of Diewert (1974), *ii*) can deal with the entry and exit of products over time, and *iii*) incorporates a structural error term. Furthermore, it is relatively simple and fast to estimate which can prove a key advantage in competition policy issues where time and transparency are always crucial factors. Akin also to the continuous-choice tradition, the model encompasses a more general version (not consistent with an indirect utility function) that enables us to test the validity of symmetry properties and, for those cases it appears to be consistent with the data, also impose it *a priori*. In what concerns the estimation procedure in particular, we propose an analog to the algorithm derived in Berry (1994), Berry, Levinsohn and Pakes (1995). Along the way, we present an alternative procedure to BLP's contraction mapping for matching observed and predicted quantities.

*A previous version of parts of this paper was circulated under the title 'Demand Models for Market Level Data', Mimeo MIT (2001). Ricardo Ribeiro gratefully acknowledges financial support from the Fundação para a Ciência e a Tecnologia.

†Competition Commission, Victoria House, Southampton Row, London WC1B 4AD, UK. Email: peter.davis@cc.gsi.gov.uk.

‡STICERD, R5z15B, The London School of Economics and Political Science, Houghton Street, London WC2A 2AE, UK. Email: r.c.ribeiro@lse.ac.uk.

1 INTRODUCTION

This paper considers a new method of uncovering demand information from market level data on differentiated products. In particular, we propose a continuous-choice demand model with distinct advantages over the models currently in use and describe the econometric techniques for its estimation.

When products are differentiated, the number of parameters required to describe a demand system (without *a priori* restrictions on the substitution patterns) tends to be excessively large to estimate, given the number of observations in a typical dataset. As an illustration of the problem, note that even in the simplest and extremely restrictive of the demand specifications - the linear expenditure model - J products would yield at least J^2 parameters to be estimated, just to capture the substitution patterns. Although implied economic theory's restrictions (like the symmetry of the Slutsky matrix) could be imposed to increase the degrees of freedom on the estimation, they do not solve this dimensionality issue. And the use of a more flexible functional form would only naturally worsen the problem. Some structure must therefore be placed on the estimation procedure and the literature has, on this point, clearly evolved along two wide-ranging type of assumptions: the discrete- and the continuous-choice settings.

In a broad sense, the demand models under the first category assume that consumers are heterogeneous and purchase at most one unit of one of the available products. Furthermore, consumer preferences over products are typically mapped onto a space of characteristics (Lancaster, 1971), reducing therefore the number of parameters to be estimated: the parameter space is thereby defined by the number of characteristics rather than by the number of products. Within this set of assumptions, we can find the multinomial logit (McFadden, 1974), the nested multinomial logit (McFadden, 1978), the multinomial probit (Hausman and Wise, 1978), the mixed- or random-coefficients multinomial logit (McFadden, 1981) and the discrete choice analytically flexible (Davis, 2006) models. The most serious drawback of this branch of the literature relates to the typical trade-off between flexibility and computation requirements. On one hand, the standard and the nested multinomial logit models are fully analytical (and thereby relatively simple to estimate), but the nature of the implied substitution patterns tends to be model- instead of data-driven. On the other hand, though the probit and the mixed-coefficients logit multinomial models do provide increased flexibility by introducing unobserved consumer heterogeneity, they require the use of simulation techniques (which in turn increases substantially the computation requirements). The recent discrete choice analytically flexible model seems to present itself as an exception given it appears to combine the good properties from the two groups.

The assumption that consumers purchase at most one unit of one of the available products may, for some settings, seem somehow unrealistic. Yogurts, soft drinks and wine are just some down-to-earth examples of cases where many consumers typically buy more than one product in each of their shopping journeys. The discreteness assumption can, nevertheless, for some of those cases, be justified by an appropriate definition of the choice period. However, other times it just can not and the choice of continuous quantities must be modelled.

The continuous-choice literature typically assumes a representative agent that might consume all products and make use of functional forms that imply flexible substitution patterns. Within this set of assumptions, we can find the translog model of Christensen et al. (1975), the almost ideal demand system (AIDS) due to Deaton and Muellbauer (1980) and the distance metric model from Pinkse et al. (2002). Consumer preferences can be defined directly over products (as in the translog or in the AIDS cases) or mapped onto a space of characteristics in a way akin to the discrete-choice literature (as in the distance metric model). This set of models present though a serious limitation as, in opposition to the discrete-choice case, they can not be used to uncover demand information from markets with significant entry and exit of products.

The above problem has obviously been addressed before in the literature but never in a way that, to the best of our knowledge, we could categorize as adequate. The typical solutions are largely limited to either consider substitution patterns between broad aggregates of products as, for example, in Christensen et al. (1975), Deaton and Muelbauer (1980), and Hausman et al. (1994), or to estimate the demand system using data only from time periods when all products are present in the market as, for example, in Hausman (1994), Ellison et al. (1997), and Pinkse and Slade (2004).

In this paper, we follow the continuous-choice literature and develop a representative consumer flexible demand model. Our starting point is the specification of an indirect utility function from which, via Roy's identity, a continuous-choice demand system is derived. The demand function implied by the model is fully analytical and therefore avoids the burden of simulation. The model is flexible in the sense of Diewert (1974) as the implied own- and cross-price elasticities are capable of capturing the true substitution patterns in the data. In addition, the model can accommodate the use of data on the entry and exit of products. The importance of this last property is twofold as *i*) not being able to cope with entry and exit patterns limits the application of the above models and *ii*) the ability to deal with discrepancies in the set of choices available to consumers provides pseudo-price variation which might be instrumental in evaluating the degree of substitution between products.

Akin also to the continuous-choice tradition, the model encompasses a more general version (not consistent with an indirect utility function) that enables us to test the validity of symmetry properties and, for those cases it appears to be consistent with the data, also impose it *a priori*. In what concerns the estimation procedure in particular, we propose an analog to the algorithm derived in Berry (1994), Berry, Levinsohn and Pakes (1995) (henceforth BLP). Following this line of the literature, the error term is structurally embedded in the model and thereby circumvents the critique provided by Brown and Walker (1989) related to the addition of add-hoc errors and their induced correlations. Along the way, we present an alternative procedure to BLP's contraction mapping for matching observed and predicted expenditure shares.

We believe that our proposed new continuous-choice model combines key properties of both the discrete- and continuous-choice traditions: *i*) it is flexible in the sense of Diewert (1974), *ii*) can deal with the entry and exit of products over time, and *iii*) incorporates a structural error term. Furthermore, it is relatively simple and fast to estimate which can prove a key advantage in competition policy issues where time and transparency are always crucial factors.

The paper proceeds in five sections. In section 2, we describe the new continuous model and establish its properties. In section 3, we discuss computation and estimation issues. Section 4 concludes.

2 THE DEMAND MODEL

Consider a choice framework with J inside options, $j = 1, \ldots, J$, and an outside option, $j = 0$, that aggregates all other products. Within this setup, consumers choose therefore between the set $\Im$ of those $J+1$ options. We shall follow the continuous choice literature and define the demand system by specifying a parametric model for the indirect utility function of the representative consumer. See Gorman (1953, 1961) or Blackorby et al. (1978) for conditions under which aggregation across consumers is consistent. Let $V(p, y; \theta_0, \Im)$ denote such a function,

$$V(p, y; \theta_0, \Im) = \sum_{i=0}^{J} \sum_{j=0}^{J} b_{ij} \left[r_i(y, p_i; \delta_i) + r_j(y, p_j; \delta_i) \right]^{\frac{1}{2}}, \tag{1}$$

where p denotes the $(J+1)$ vector of $p_i \in \Re^+$ prices, $y \in \Re^+$denotes the representative consumer's income, and $r_k(y, p_k; \delta_k)$ denotes a known (possibly parametric) function of income and product-specific price. Prices and income enter only through the $r_k(y, p_k; \delta_k)$ function, and therefore $V(p, y, \delta; \theta_0, \Im)$ can also be rewritten as $V(r; \theta_0, \Im)$. Lastly, θ_0 refers to the $\{b_{ij}, \delta_i\}$ parameters, which we assume to have support $\Theta_0 \subseteq R^{\theta_0}$. Following the demand literature, we normalize the price of the outside option to one.

In order for the above indirect utility function to be a member of the class of consistent indirect utility functions, conditional on $b_{ij} > 0$ for all i and j, $r_k(y, p_k; \delta_k)$ must be i) a continuous function at all positive (p_k, y), ii) non-increasing in p_k, non-decreasing in y, and homogeneous of degree zero in (p_k, y), and iii) a convex function of p_k with y normalized to one. Although many particular functional forms for $r_k(y, p_k; \delta_k)$ are possible, for concreteness, let us consider the following for reasons to be made precise below,

$$r_k(y, p_k; \delta_k) = \exp(\ln y - \ln p_k + \delta_k), \tag{2}$$

which, in fact, satisfies conditions $(i) - (iii)$ above.

At this point, the indirect utility function is completely deterministic. However, as Brown and Walker (1989) point out, the introduction of the random utility hypothesis "is appealing for several reasons. Primarily, it motivates the randomness existing in an applied demand model. While we prefer to retain the assumption that individuals follow rational utility maximizing behaviour, it is clear that some randomness exist (...). The random utility hypothesis resolves this conflict. Furthermore, the use of random utility models provides a structure for the stochastic specification" of the disturbance terms for the demand equations.

We introduce the random utility hypothesis in a way akin to Pinkse et al. (2002) and define the parameters δ_i on the characteristics space, $\delta_i(x_i, \xi_i; \theta_\beta)$, where x_k denotes the K-dimensional vector of characteristics associated with product k, observed by both the consumer and the econometrician, ξ_k denotes the value of product k's characteristics observed by the consumer but not by the econometrician, and finally θ_β refers to the taste parameters of interest.

The precise functional form for $\delta_i(x_i, \xi_i; \theta_\beta)$ is an issue that can be examined using conventional testing procedures. For concreteness, we will assume the following specification,

$$\delta_i(x_i, \xi_i; \theta_\beta) = \sum_{k=1}^{K} \beta_{ik} \ln(x_{ik}) + \ln(\xi_i), \tag{3}$$

which has the desirable property of being monotonic in the value of a given product's characteristics. The imposition of the constraint that $\beta_{ik} = \beta_{jk}$ for all $i, j \in \Im$ is, naturally, an option available to the researcher.

The presence of unobserved product characteristics allows for a product-level source of sampling error, giving an explicit structural interpretation to the error term. Furthermore, the approach of placing unobservables directly into the utility function ensures that the model is internally consistent and thereby avoids the fundamental critique of the ad-hoc approach to introducing unobservables provided by Brown and Walker (1989).

In what follows, for notational purposes, we will decompose $\theta_0 = (\theta_{nl}, \delta)$, where θ_{nl} refers to the non-linear $\{b_{ij}\}$ parameters, whereas δ refers to the $\{\delta_i\}$ linear parameters (in a way to make precise below), which will be a function of the θ_β taste parameters.

2.1 Demand Derivation

Standard duality results establish conditions on the function $V(r; \theta_{nl}, \delta(\theta_\beta), \Im)$ which ensure that specifying a parametric functional form for the indirect utility function, and then solving for the demand system using Roy's identity, is entirely equivalent to specifying the direct utility function and budget constraint. By taking this dual approach, the resulting parametric demand systems are assured to be consistent with utility maximization, at least for some subset of parameter values.

The demand system is obtained via Roy's identity on the function $V(r; \theta_{nl}, \delta(\theta_\beta), \Im)$. In particular, the expenditure share for product m is as follows,

$$w_m(r; \theta_{nl}, \delta(\theta_\beta), \Im) = \frac{p_m\left(-\frac{V_{p_m}}{V_y}\right)}{y} = 2r_m \frac{V_{r_m}}{V}, \tag{4}$$

where V_{p_m}, V_y and V_{r_r} denote the first derivative of the indirect utility function V with respect to p_m, y and r_m, respectively. For notational convenience, whenever no ambiguity arises, the explicit dependence of the different variables on $(r; \theta_{nl}, \delta(\theta_\beta), \Im)$ will be dropped. For completeness,

$$V_{r_m} = \frac{1}{2}\sum_{j=0}^{J}(b_{mj} + b_{jm})(r_m + r_j)^{-\frac{1}{2}}. \tag{5}$$

In addition to the classical properties, the above expenditure share function satisfies also a global regularity property as it can be specialized down in an entirely consistent fashion over arbitrary subsets of products.

In order to ensure that the expenditure share function presented is globally consistent, we assume that the case where a product is not present (in a given market or/and time period) is entirely equivalent to the situation where the product is present as long as specific conditions on prices are verified. In particular, that the price is (positive) infinity. This condition imply that $r_j(y, p_j; \delta_j) = 0$ for all $j \notin \Im'$, and therefore that the expenditure share function can be specialized over arbitrary subsets of products. As a result, not only the model can be estimated using datasets where significant product entry and exit occurs, but also provides an useful source of pseudo-price variation instrumental for the estimation of substitutability patterns. This is not true of the present generation of continuous choice models.

Definition 1 *An expenditure share function* $w_k\left(r^{\Im}; \theta_{nl}^{\Im}, \delta^{\Im}(\theta_\beta), \Im\right)$ *is globally consistent iff for any set of products* $\Im' = \{0, 1, 2, \ldots, N\}$ *for* $N < J$ *and* $\Im'' = \{0, 1, 2, \ldots, J\}$*, where products* $N+1$ *to* J *have zero observed quantities,*

$$w_k\left(r^{\Im''}; \theta_{nl}^{\Im''}, \delta^{\Im''}(\theta_\beta), \Im''\right) = 0, \quad \textit{for any } k \notin \Im'. \tag{6}$$

Surprisingly, this extremely mild and intuitive regularity condition is not satisfied by the vast majority of existing continuous choice which have many terms like $\alpha_j \ln p_j$ and $\beta_{jk} \ln p_j \ln p_k$.

In many policy applications, including merger simulation, the key object of interest is the matrix of own- and cross-price demand elasticities. The analytical expressions for the own- and cross-price expenditure share elasticities predicted by the model for any given products m and n, are the following,

$$\varepsilon^s_{mn}\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right)=\begin{cases}\left(\frac{-V_{r_m r_m}V+V^2_{r_m}-\frac{1}{r_m}V^2_{r_m}}{V^2}\right)\left(\frac{2r_m r_m}{w_m}\right) & \text{for } \ m=n\\ \\ \left(\frac{-V_{r_m r_n}V+V_{r_m}V_{r_n}}{V^2}\right)\left(\frac{2r_m r_n}{w_m}\right) & \text{for } \ m\neq n\end{cases}\tag{7}$$

where, for completeness,

$$\begin{aligned}V_{r_m r_m}&=-\tfrac{1}{4}\textstyle\sum_{j=0}^{J}\left(b_{mj}+b_{jm}\right)\left(r_m+r_j\right)^{-\frac{3}{2}}\left(1+I\left(j=m\right)\right)\\ V_{r_m r_n}&=-\tfrac{1}{4}\left(b_{mn}+b_{nm}\right)\left(r_m+r_n\right)^{-\frac{3}{2}}.\end{aligned}\tag{8}$$

From the own- and cross-price expenditure share elasticiticies, ε^s_{mn}, we can straightforwardly obtain the implied own- and cross-price demand elasticities, ε^d_{mn}, from the following one for one relationship,

$$\varepsilon^d_{mn}\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right)=\begin{cases}\varepsilon^s_{mm}\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right)-1 & \text{for } \ m=n\\ \\ \varepsilon^s_{mn}\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right) & \text{for } \ m\neq n\end{cases}\tag{9}$$

2.2 Symmetry

The model described above is observationally equivalent to a symmetric model with $b_{ij}=b_{ji}=\frac{(b_{ij}+b_{ji})}{2}$, the reason being that both the demand and the elasticities functions do not depend on the $\{b_{ij}\}$ parameters in itself but only on their sum. This property of the model rends a great advantage in terms of the estimation procedure, as the number of parameters to be estimated decrease substantially. Furthermore, the model predicts symmetry in the cross-price effects, a restriction which is not, in general, expected to hold for market level data. For these reasons, it would be of interest to estimate a model which did not impose such symmetry restrictions. In order to accomplish that, we can estimate the model using the more general demand function,

$$w_m\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right)=2r_m\frac{N_{r_m}}{N},\tag{10}$$

where N_{r_r} and N denote,

$$N_{r_m} = \frac{1}{2}\sum_{j=0}^{J} b_{mj}\left(r_m + r_j\right)^{-\frac{1}{2}}$$
$$N = \sum_{i=0}^{J}\sum_{j=0}^{J} b_{ij}\left(r_i + r_j\right)^{\frac{1}{2}}. \tag{11}$$

When symmetry is not imposed, the model will obviously not be consistent with consumer utility maximization, but in turn it will allow us to test the validity of the symmetry constraint and, for those cases it appears to be consistent with the data, also impose it *a priori*.

2.3 Flexibility

An algebraic functional form for a complete system of consumer expenditure share functions $w_i\left(r;\theta_{nl},\delta\left(\theta_\beta\right),\Im\right)$ is said to be flexible if, at any given set of non-negative prices and income, the parameters can be chosen so that the complete system of consumer expenditure share functions, their own- and cross- price demand and income elasticities are capable of assuming arbitrary values at the given set of prices and income (subject only to the requirements of theoretical consistency). See Diewert (1974) and Lau (1986).

First, we will show that it would always be possible to solve for the vector of deltas associated with each option that makes predicted and actual quantities equal for all products. This result provides the first step in establishing flexibility results about the demand system since it ensures the model can always match the vector of observed quantities, one requirement for a model to be a Diewert (1974) flexible functional form. We then proceed to provide a result establishing the model's ability to also match own- and cross-price elasticities.

Lemma 1 *Let $\Im \equiv \{0,1,\ldots,J\}$ be the set of products and $\Im^+ \equiv \left\{j|q_j^* > 0, j \in \Im\right\}$ be the set of products with strictly positive observed quantities. Denote $w_j\left(r;\theta_{nl},\delta,\Im\right)$ and w_j^* as the predicted and observed expenditure shares. Let $w_j\left(r;\theta_{nl},\delta,\Im\right)$ be continuous and differentiable. Further suppose $w_j\left(r;\theta_{nl},\delta,\Im\right)$ has the following properties: (i) if $r_j = 0$ then $w_j\left(r;\theta_{nl},\delta,\Im\right) = 0$, (ii) $w_j\left(r;\theta_{nl},\delta,\Im\right)$ is homogeneous of degree zero in r, (iii) $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k} < 0$ for all $k \neq j$ with $k,j \in \Im$, and (iv) $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k} = \frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial r_k} r_k$ for all $k,j \in \Im$. Then there exists a finite vector of δ's that solve the $J+1$ vector of equations $w_j^* = w_j\left(r;\theta_{nl},\delta,\Im\right)$ for $j \in \Im$. If $w_j^* = 0$ then the solution sets $r_j = 0$ for $j \in \Im$. Moreover, the solution to the subset of equations $w_j^* = w_j\left(r;\theta_{nl},\delta,\Im^+\right)$ is unique.*

Proof. First notice that if $w_j^* = 0$ for any product j, property (i) on the $w_j(r;\theta_{nl},\delta,\Im)$ function ensures that $r_j = 0$ will solve the jth equation exactly. Having set the r's corresponding to products with zero expenditure shares, we can progress to consider the solution to the smaller set of equations $w_j^* = w_j(r;\theta_{nl},\delta,\Im^+)$ for $j \in \Im^+$, defined as in the lemma.

To prove the later result, we will work the following function defined in terms of δ : $g_j(r;\theta_{nl},\delta,\Im) = w_j(r;\theta_{nl},\delta,\Im) - w_j^*$ for $j \in \Im$. Next, recall that a suficient condition for uniqueness of a system of equations $g(r;\theta_{nl},\delta,\Im) = 0$ is that the Jacobian matrix of a function $D_\delta g(r;\theta_{nl},\delta,\Im)$ is positive definite. A symmetric matrix $D_\delta g(r;\theta_{nl},\delta,\Im)$ with a positive and dominant diagonal, is positive definite. Recall also that the matrix $D_\delta g(r;\theta_{nl},\delta,\Im)$ has a dominant diagonal if there is $(z_1,\ldots,z_{J+1}) \gg 0$ such that for every $j \in \Im$, $\left|z_j \frac{\partial g_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j}\right| > \sum_{k\neq j}\left|z_k \frac{\partial g_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k}\right|$. Note also that if $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j} > 0$ and $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k} < 0$ for all $k \neq j$ with $k,j \in \Im$, the dominant diagonal condition can be written as $z_j \frac{\partial g_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j} > -\sum_{k\neq j} z_k \frac{\partial g_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k}$.

We have now to establish that the dominant diagonal holds. Property (iii) insures $\frac{\partial g_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k} < 0$ for all $k \neq j$ with $k,j \in \Im$. Further, from the homogeneity property (ii), we must have $\sum_{k\in\Im} r_k \frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial r_k} = 0$ for each $j \in \Im$. The later equality can however, under property (iv) of the lemma, be rewritten as $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j} + \sum_{k\neq j\in\Im} \frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k} = 0$. Given property (iii), we have that $\frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j} > 0$ which insures the positive diagonal condition is satisfied. Furthermore, if $z_j > z_k$ for $(z_1,\ldots,z_{J+1}) \gg 0$ and for all $k \neq j$ with $k,j \in \Im$, we have $z_j \frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_j} > -\sum_{k\neq j\in\Im} z_k \frac{\partial w_j(r;\theta_{nl},\delta,\Im)}{\partial \delta_k}$, which is exactly the dominant diagonal condition. Thus there is a unique vector δ that solves the system of equations $w_j^* = w_j(r;\theta_{nl},\delta,\Im)$ for $j \in \Im^+$, that is, that equates actual and predicted quantities for products with strictly positive observed quantities. We have already noted that for those products with zero observed quantities, setting their r's to zero solves their equations. Thus, as the lemma claims, there is a solution to the full problem and a unique solution to the reduced problem of those products with strict positive quantities. ■

Having established lemma 1 above, in order to prove the model can always match the vector of observed quantities, we just have to show that our particular continuous-choice derived expenditure share function $w_j(r;\theta_{nl},\delta(\theta_\beta),\Im)$ does satisfy properties $(i)-(iv)$.

Corollary 1 *Let $w_j(r;\theta_{nl},\delta,\Im) = 2r_j\frac{V_{r_j}}{V}$, according to equation (5) for all $j \in \Im$. Let also $r_k(y,p_k;\delta_k)$ be defined as in equation (2). Then, if $b_{jk} > 0$ for all $j,k \in \Im$, there exists a finite vector of δ's that solve the $J+1$ vector of equations $w_j^* = w_j(r;\theta_{nl},\delta,\Im)$*

for $j \in \Im$. *If* $w_j^* = 0$ *then the solution sets* $r_j = 0$ *for* $j \in \Im$. *Moreover, the solution to the subset of equations* $w_j^* = w_j(r; \theta_{nl}, \delta, \Im^+)$ *is unique.*

Proof. In order to establish the above corollary, we just have to show that the expenditure share function $w_j(r; \theta_{nl}, \delta, \Im) = 2r_j \frac{V_{r_j}}{V}$ satisfies properties $(i) - (iv)$ of lemma 1, where $r_k(y, p_k; \delta_k)$ is as in equation (2).

First, we establish property (i). Notice that since $w_j(r; \theta_{nl}, \delta, \Im) = 2r_j \frac{V_{r_j}}{V}$, setting any $r_j = 0$ ensures that the predicted expenditure share for that product is $w_j(r; \theta_{nl}, \delta, \Im) = 0$.

Next, we establish that condition (ii) of the lemma holds, namely that $w_j(r; \theta_{nl}, \delta, \Im)$ is homogenous of degree zero in r. V_{r_j} is homogeneous of degree $-\frac{1}{2}$ in r, whereas V is homogenous of degree $\frac{1}{2}$. Given the analytical expression for the expenditure function, it is then immediate the homogeneity of degree zero in r.

Now let us establish property (iii): $\frac{\partial w_j(r; \theta_{nl}, \delta, \Im)}{\partial \delta_k} = 2r_j r_k \left(\frac{V_{r_j r_k} V - V_{r_j} V_{r_k}}{V^2} \right)$, where V_{r_j}, V_{r_j} and $V_{r_j r_k}$ are given as in equations (5) and (8). Conditional on $b_{jk} > 0$ for all $j, k \in \Im$, we have that $V_{r_j r_k} < 0$ and both $V_{r_j} > 0$ and $V_{r_k} > 0$, which yields $\frac{\partial w_j(r; \theta_{nl}, \delta, \Im)}{\partial \delta_k} < 0$.

Finally, we establish that property (iv) of the lemma holds. Given the analytical expression for $r_k(y, p_k; \delta_k)$ is as in equation (2), we have $\frac{\partial w_j(r; \theta_{nl}, \delta, \Im)}{\partial \delta_k} = \frac{\partial w_j(r; \theta_{nl}, \delta, \Im)}{\partial r_k} \frac{\partial r_k}{\partial \delta_k}$. Furthermore, $\frac{\partial r_k}{\partial \delta_k} = r_k$. Commbining the two results, yields in fact property (iv). ■

Given that the model is capable of matching observed with predicted quantities, we proceed to state the proposition which establishes the model's ability to also match own- and cross-price elasticities, which concludes the flexibility result.

Proposition 1 *There exists a vector of parameters* θ_β *and* θ_{nl} *such that the model can match any matrix of own- and cross-price elasticities.*

Proof. Omitted. ■

2.4 Identification

The $\{b_{ii}\}$ and $\{b_{ij}\}$ parameters define the degree of price substitution and are therefore identification requires variation in prices. The former will be identified by quantity effects due to variations in own-price, whereas the later will be identified by quantity effects due to variation in cross-prices. Given the normalization on the price of the outside option, we can not expect to be able to identify the $\{b_{i0}\}$ parameters that define the degree of substitution towards product i when the price of the outside option varies. We will, thereby, normalize b_{00}, which fixes the scale of the $\{b_{ij}\}$ parameters, and impose the $b_{i0} = b_{0i}$ symmetry for $i \neq 0$. The justification for the later assumption is that the $\{b_{0i}\}$ parameters define the degree of substitution towards the outside option when the price of inside product i varies. Given the typical price variation of an inside product, we can expect to identify the later parameters and, therefore, the imposition of symmetry seems a natural restriction.

3 Computation and Estimation

The data available to the researcher is crucial for the estimation procedure. Consider a panel with data on prices, income and observed quantities for a set of J products across time or from a number of markets. We proceed by describing the estimation algorithms for the cases where i) the number of products is relatively small so that the dimensionality problem does not constitute an issue and ii) the number of products yields a too great number of parameters to be estimated.

3.1 Small Number of Products

The estimation algorithm that we propose is based in Berry (1994) and BLP (1995), and encompasses four steps that we now describe.

Step One Set initial values for the vector of deltas δ_t and for the $\{b_{ij}\}$ parameters in θ_{nl}.

Step Two For a given θ_{nl}, solve for the δ_{jt}'s that ensure that the observed w_{jt}^{obs} and the predicted $w_{jt}\left(r_t; \theta_{nl}, \delta, \Im\right)$ expenditure shares are equated. The solution to this problem can again be found using BLP's contraction algorithm,

$$\delta_{jt}^{n} = \delta_{jt}^{n-1} + \ln\left[w_{jt}\left(r_t; \theta_{nl}, \delta^{n-1}, \Im\right)\right] - \ln\left(w_{jt}^{obs}\right) \tag{12}$$

where δ_{it} is solved for recursively. The initial guess for δ_{jt}^{n-1} is used to compute the predicted expenditure shares $w_{jt}\left(r_t; \theta_{nl}, \delta^{n-1}, \Im\right)$ and by (12) gives rise to a new computed δ_{jt}^{n}. The process is then repeated until $w_{jt}\left(r_t; \theta_{nl}, \delta, \Im\right)$, the predicted expenditure shares, equate the observed ones, and consequently convergence is achieved for δ_{jt}.

Alternatively, the solution to the problem of equating oberved and predicted expenditure shares can also be found as the unique solution to the following optimization problem for each time period or market,

$$\max_{\delta_{jt}} \ln V\left(r_t;\theta_{nl},\delta,\Im\right)-\frac{1}{2}\sum_{i=1}^{J}\delta_{it}w_{it}^{obs} \tag{13}$$

Proof. The first-order condition to the above optimization problem with respect to a given δ_{mt} is simply $\frac{V_{r_{mt}}r_{mt}}{V}-\frac{1}{2}w_{mt}^{obs}=0$, which in fact yields $w_{mt}^{prd}=2r_{mt}\frac{V_{r_{mt}}}{V}=w_{mt}^{obs}$. ∎

Let the solution vector of the δ_{jt}'s that ensure that the observed w_{jt}^{obs} and the predicted $w_{jt}\left(r_t;\theta_{nl},\delta,\Im\right)$ expenditure shares are equated, be denoted by $\delta_t\left(r_t,w_t^{obs};\theta_{nl},\Im\right)$.

Step Three Run a Berry (1994) style regression, again for a given θ_{nl}, on the relationship

$$\delta_{jt}\left(r_t,w_t^{obs};\theta_{nl},\Im\right)=\sum_{i=1}^{K}\beta_{jk}\ln\left(x_{jkt}\right)+\ln\left(\xi_{jt}\right) \tag{14}$$

and obtain estimates for the parameters θ_β and for the unobserved characteristics ξ_{jt}. Please note that the later estimates will be a function of both the θ_β and θ_{nl} parameters.

Step Four Estimate the $\{b_{ij}\}$ parameters in θ_{nl} by a Generalized Method of Moments procedure. The approach relies on an identifying restriction on the distribution of the true unobserved characteristics and is based on the sample analogue to the population condition.

The standard identifying restriction states that, at the true values of the parameters, $\theta_0^*=\left(\theta_{nl}^*,\delta^*\right)'$, the true unobserved characteristics are mean independent of a set of instruments $Z_{jt}=\left[z_{jt}^1,\ldots,z_{jt}^M\right]$,

$$E\left[\xi_{jt}\left(\theta_0^*\right)|Z_{jt}\right]=0 \tag{15}$$

Please note that other identifying restrictions would also enable the estimation of the model. In particular, given the typical panel structure of the data, an alternative assumption could incorporate the likelihood of the econometric error and the set of instruments to be more similar for a given product across time, than for those of different products. Please see Berry, Levinsohn, and Pakes (1995) and Davis (2006) for a more detailed analysis on this subject.

The above population moment conditions can be used, akin to Hansen (1982), to render a method of moments estimator of θ_0^* by interacting the estimated unobserved characteristics with the set of instruments, and then search for the value of the θ_0 parameters that set the sample analogues of the moment conditions as closed as possible to zero. Let $G_n(\theta_0)$ denote the sample analogues of the moment conditions,

$$G_n(\theta_0) = \frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{J}\tilde{\xi}_{jt}(\theta_0)\tilde{Z}'_{jt} \tag{16}$$

where for notational purposes $\tilde{\xi}_{jt}(\theta_0) = \xi_{jt}(\theta_0)\chi_{it}$, $\tilde{Z}_{jt} = \left[z_{jt}^1\chi_{it}, \ldots, z_{jt}^M\chi_{it}\right]$, and $\chi_{it} =$ 1 if product j is sold in period t and zero otherwise. χ_{it} provides, thereby, a missing value indicator used to compute $n = \sum_{t=1}^{T}\sum_{j=1}^{J}\chi_{it}$.

Formally, the method of moments estimator for $\hat{\theta}_0$ is the argument that minimizes the weighted norm criterion of $G_n(\theta_0)$, for some weighting matrix A_n with rank at least equal to the dimension of θ_0,

$$\hat{\theta}_0 = \arg\min_{\theta_b} \quad \|G_n(\theta_0)\|_{A_n} = G_n(\theta_0)' A_n G_n(\theta_0) \tag{17}$$

The strong non-linearity of the objective function requires a minimization routine. The non-linear search over θ_0 can be simplified by making use of the fact that the first order conditions for a minimum of $\|G_n(\theta_0)\|_{A_n}$ are linear for the subset $\delta(\theta_\beta)$ of the parameters of estimation in $\theta_0 = (\theta_{nl}, \delta)$. In particular, it is possible, given the standard instrumental variables results, to express θ_β as function of θ_{nl}, and limit the non-linear search over θ_{nl},

$$\hat{\theta}_\beta = \left(X'ZA_n^{-1}Z'X\right)^{-1} X'ZA_n^{-1}Z'\delta(\theta_{nl}) \tag{18}$$

where X denotes the matrix of observed characteristics.

Hansen (1982) establish the formal conditions under which $\hat{\theta}_0$, the method of moments estimator, is consistent and asymptotically normal with bounded variance, consistently estimated as follows,

$$\sqrt{n}\left(\hat{\theta}_0-\theta_0^*\right)\sim N\left[0,\left(\widehat{\Gamma}'A_n\widehat{\Gamma}\right)^{-1}\widehat{\Gamma}'A_n\hat{\Phi}A_n\widehat{\Gamma}\left(\widehat{\Gamma}'A_n\widehat{\Gamma}\right)^{-1}\right] \tag{19}$$

where $\widehat{\Gamma}$ denotes a consistent estimator of the gradient of the objective function,

$$\widehat{\Gamma}'=\widehat{\Gamma_n\left(\hat{\theta}_0\right)}=\frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{J}\left[\frac{\partial\tilde{\xi}_{jt}\left(\hat{\theta}_0\right)\tilde{Z}'_{jt}}{\partial\theta'_0}\right] \tag{20}$$

and $\hat{\Phi}$ denotes a consistent estimator of the variance-covariance matrix of the moment conditions,

$$\hat{\Phi}=Var\left[\widehat{G_n\left(\hat{\theta}_0\right)}\right]=\frac{1}{n}\sum_{t=1}^{T}\sum_{j=1}^{J_t}\tilde{Z}_{jt}\tilde{\xi}_{jt}\left(\hat{\theta}_0\right)'\tilde{\xi}_{jt}\left(\hat{\theta}_0\right)\tilde{Z}'_{jt} \tag{21}$$

In what the weighting matrix is concerned, the optimal weighting matrix is proportional to Φ^{-1}, giving less weigh to those moments with a higher variance.

3.2 Large Number of Products

If the number of products J is large, then the model would yield a too great number of parameters to be estimated. In that case, some aditional structure must be imposed in order to reduce the number of parameters. Here, we follow once again Pinkse et al. (2002) by mapping the $\{b_{ij}\}$ parameters to be parametric functions of 'distance metrics' on the characteristics space, which reduces the number of parameters to be estimated whenever the number of product characteristics is smaller than the number of products. One possible specification for the mapping could be the following where the parameters b_{ij} are defined as,

$$b_{ij}=\begin{cases}-\frac{1}{d_{ij}(x_{it},x_{jt};\alpha_1)} & \text{if } i\neq j\\ \\ \exp\left(x'_{it}\alpha_2\right) & \text{if } i=j\end{cases} \tag{22}$$

where $d_{ij}\left(x_{it},x_{jt};\alpha_1\right)=\sum_{l=1}^{L}\alpha_{1l}\left|x_{lit}-x_{ljt}\right|$ or $d_{ij}\left(x_{it},x_{jt};\alpha_1\right)=\sqrt{\sum_{l=1}^{L}\alpha_{1l}\left(x_{lit}-x_{ljt}\right)^2}$ measure the distance between products i and j in the characteristics space given their observed characteristics at date t. Independently, however, of the specification chosen, the $\{b_{ij}\}$ parameters in θ_{nl} are thereby mapped as functions of a set of characteristics and the vectors α_1 and α_2. As a result, the estimation procedure is thereby identical to the one outlined above, with the sole exception that instead of searching for the parameters b_{ij} that minimize the weighted norm criterion of $G_n(\theta_0)$, we now search for the vector of parameters α_1 and α_2.

The fundamental drawback of this approach is the fact that, although allowing the estimated b_{ij} to be data-driven, it imposes symmetry in the parameters and as result in the estimated cross-price effects. However, the precise mapping is, again, a functional form issue that can be examined using conventional testing procedures.

4 CONCLUDING REMARKS

In this paper, we consider a new method of uncovering demand information from market level data on differentiated products. We follow the continuous-choice literature and develop a representative consumer flexible demand model, which can accommodate the use of data on the entry and exit of products and incorporates a structural error term.

Furthermore, it is relatively simple and fast to estimate which can prove a key advantage in competition policy issues, where time and transparency are typically crucial factors. Akin also to the continuous-choice tradition, the model encompasses a more general version (not consistent with an indirect utility function) that enables us to test the validity of symmetry properties and, for those cases it appears to be consistent with the data, also impose it *a priori*. In what concerns the estimation procedure in particular, we propose an analog to the algorithm derived in Berry (1994) and BLP (1995). Along the way, we present an alternative procedure to BLP's contraction mapping for matching observed and predicted expenditure shares.

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