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Term Structure of Interest Rates Implications of Habit Persistence

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ABSTRACT

This paper investigates the term structure implications of a simple structural model with non-separable preferences and habit formation. The distinguishing features of the model are that the drift of equilibrium spot interest rates is non-linear, interest rates depend on lagged values of monetary and consumption shocks, and the price of risk is *not* a constant multiple of interest rates volatility. We solve the model in closed-form and investigate its empirical properties. We find that habit persistence can help reproduce (i) the non-linearity of the spot rate process, (ii) the empirical Campbell and Shiller (1991) linear projection coefficients and the documented deviations from the expectations hypothesis, (iii) the extent of the persistence of conditional volatility of interest rates, (iv) the lead/lag relationship between interest rates and monetary aggregates, and (v) the dynamics of the inflation risk premium. We also describe the limitations of this particular form of habit persistence. Although the model improves some traditional models with respect to several dimensions, its ability to reproduce, at the same time, the equity risk premium and the conditional second moments of interest rates is still limited.

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In this paper, we investigate a structural model that links the fundamentals of a monetary economy with habit formation and the dynamics of the yield curve. First, we provide testable restrictions on how the dynamics of the nominal yield curve depend on both the habit stock and monetary factors. Then, we use data on nominal bonds to study whether habit persistence can help explain some of the empirical regularities highlighted by the existing term structure literature.

Both the economic and the psychological literature stress the importance of interpersonal effects and time non-separabilities in consumption choices. Duesenberry (1949) and Veblen (1899) argue that

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consumers imitate each others' purchases to conform to the expectations of the people in their reference group. Thereafter, a large literature has used preferences assuming some of these features to address a variety of questions ranging from criminal behavior, business cycle, rational addiction, and asset pricing. Both the economic and the psychological literature stress the importance of interpersonal effects and time non-separabilities in consumption choices. Duesenberry (1949) and Veblen (1899) argue that consumers imitate each others' purchases to conform to the expectations of the people in their reference group. Thereafter, a large literature have used preferences assuming some of these features to address a variety of questions ranging from criminal behavior, business cycle, rational addiction, and asset pricing. With regards to the latter, preferences with habit persistence have been found useful to explain some asset pricing empirical regularities. Constantinides (1990), Stambaugh and Kandel (1991), Abel (1990), Campbell and Cochrane (1999) argue that habit formation can help to explain the large realized equity excess returns and that the assumption of time-separability plays an important role in the difficulty of traditional asset pricing models to reproduce the empirical regularities of equity returns. However, very little is known with regards to the implications of these models in terms of the cross-sectional and time-series properties of the term structure of interest rates and whether they can help to explain some of the empirical features found in the data.

Campbell and Cochrane (1999) argue that habit formation can help to explain the large realized equity excess returns and that the assumption of time-separability plays an important role in the difficulty of traditional asset pricing models to reproduce the empirical regularities of equity returns. However, very little is known with regards to the implications of these models in terms of the cross-sectional and time-series properties of the term structure of interest rates and whether they can help to explain some of the following empirical features found in the data.

Duffee (2002) finds that across the maturity spectrum, the unconditional mean excess return to bonds is small relative to the variation in conditional mean excess returns and the conditional volatility of yields is very persistent. Moreover, linear projections of bond yield changes on the slope of the yield curve give large and negative Campbell-Shiller slope coefficients. This result is very robust across different time periods and statistical methods. Dai and Singleton (2002) find that "key to matching the empirical findings in Fama and Bliss (1987) and Campbell and Shiller (1991) are parametrization of the market price of risk that let the risk factors affect the market price of risk directly, and not only through their factor volatilities". Cheridito, Filipovic, and Kimmel (2003) investigate a class of flexible arbitrage-free specifications of the market price of risk.

Second, there is mounting evidence against the Fisher neutrality assumption. Benninga and Protopapadakis (1983), Fama (1990), Evans (1998) and Boudoukh (1993) find that the inflation rate is negatively related to the real interest rate in terms of both realized changes and expected values. Fama (1976), Fama (1990), Fama and Gibbons (1982), and Marshall (1992) find that real returns on nominal bonds decline when inflation increases. Moreover, Chen, Roll, and Ross (1986) find that assets that are positively correlated with inflation earn a lower risk premium.[1]

[1] Fama (1981) finds evidence of inflation non-neutrality also in the stock market as stock real returns are negatively correlated with inflation. Moreover, in the medium and long term, the real gross domestic product is negatively affected

Third, both real and nominal interest rates appear to be correlated with past (detrended) levels of output and money (Fiorito and Kollintzas (1994), Chari, Christiano, and Eichenbaum (1995), King and Watson (1996)). When the role of money is assumed away, it is hard to explain the correlation between asset returns and money growth. Marshall (1992) shows, however, that when money is introduced into the model to facilitate transactions, the negative correlation between inflation and stock returns and the positive correlation between money growth and asset returns can become an equilibrium property. This article follows this path and investigates an economy in which money facilitates consumption transactions.

Fourth, Conley, Hansen, Luttmer, and Scheinkman (1997) argue that "although linear specifications are convenient for deriving and estimating explicit models of the term structure of interest rates, from the viewpoint of data description it is important to specify the short term rate drift and possibly the diffusion in more flexible ways."[2] They estimate the stationary density of the fed funds rate and find evidence of non-linearity in the short-term rate using semi-parametric methods. Similar results are also discussed by Ait-Sahalia (1996) and Ait-Sahalia (1999) using different econometric methods.

In this paper, we use these insights to develop and estimate a simple structural model in which some of these features arise in equilibrium. We explore a tractable monetary version of an exchange economy with external habit formation in which the term structure of interest rates has the following properties in equilibrium. First, the market price of risk is not a constant multiple of interest rates volatility. The term premium is state dependent so that the model can accommodate deviations from the expectations hypothesis. Second, the inflation risk premium is positive and time varying, so that the model can allow for deviations from the Fisher hypothesis. Third, the model induces a lead-lag relationship between both nominal interest rates and money, and between nominal interest rates and consumption. Fourth, yield to maturities are not affine in the state variables. In particular, the spot interest rate has a non-linear drift that captures some of the empirical properties described by Ait-Sahalia (1996) and Ait-Sahalia (1999) and Conley, Hansen, Luttmer, and Scheinkman (1997).

Our model builds on the work by Campbell and Cochrane (1999) who investigate an economy with non-separable preferences and constant interest rates in which the representative agent's current utility depends not only on his own current consumption, but also on the history of aggregate consumption.[3] This generates a wedge between relative risk aversion and the intertemporal marginal rate of substitution. Negative endowment shocks, pushing current consumption toward the habit stock, make investors more risk averse. Therefore, during recessions, asset prices must drop more than in a time-separable economy in order to reflect the higher state-dependent risk premium. An important advantage of the extrinsic model specification is the avoidance of the implied excess interest rate volatility and negative Arrow-Debreu state price density, which is sometimes present in internal habit models (see Chapman (1998) for a discussion). We extend Campbell and Cochrane (1999)

by an increase in inflation (Fama and Gibbons (1982), Boudoukh (1993), Harvey (1988)).

[2] Conley, Hansen, Luttmer, and Scheinkman (1997), page 526.

[3] In an earlier working paper draft of the published article, Campbell and Cochrane (1998) discuss the implications of an economy with stochastic interest rates.

and Menzly, Santos, and Veronesi (2004) in two important ways. First, we allow interest rates to be stochastic. Second, we consider a monetary economy that supports positive monetary holdings in equilibrium. This has some important implications. The Fisher relationship does not hold, so that assets that are positively correlated with inflation earn lower returns, and the documented link between money growth and nominal interest rates is an equilibrium feature of the model. Moreover, since preferences are non-separable, the model can account for a persistent correlation between money growth and interest rates. The monetary aspect of the economy requires, however, to solve for the inflation rate as an endogenous stochastic process. This is a non-trivial step that links inflation and nominal interest rates to the state variables governing also the real side of the economy. We derive closed-form solutions for both the real and nominal term structure of interest rates, the inflation risk premium and conditional yield volatilities. We find that the model-implied equilibrium process of interest rates is non-linear. We subsequently estimate the structural model using data on U.S. nominal Treasury bonds from January 1960 to December 2000. We address the following questions.

First, to what extent does a model with habit persistence link term structure dynamics with economic fundamentals for reasonable values of the structural parameters? We find, based on asymptotic GMM tests, that the joint moment restrictions of the model on both bond yields and macroeconomic time-series, such as inflation and monetary holdings, are not rejected. Moreover, when we investigate the cross-sectional implications of the model, we find that the implied median absolute errors for the one year yield to maturity is 14.6 basis points and 13.1 basis points for the five year bond. The model is also quite accurate in fitting, at the same time, the conditional volatility of yields. We run a regression of squares in yield changes onto the model-implied conditional second moment of yield changes. We cannot reject the null hypothesis that the intercept is zero at any maturity and that the slope coefficient is one for maturities between 3 months and 3 years. The null hypothesis is, however, rejected for maturities above or equal to five years.

Second, does habit persistence help explain the Campbell-Shiller expectations puzzle? We compute the model-implied slope coefficients of a regression of changes in yields onto the slope of the yield curve. These coefficients, known as the Campbell and Shiller (1991) coefficients, are considered important statistics describing the conditional second moment properties of a term structure model (see Dai and Singleton (2000) and Duffee (2002)). We find that the model-implied Campbell-Shiller slope coefficients are negative and increasing (in absolute value) with the horizon. The magnitude of these coefficients matches those found in the expectations hypothesis literature. At two and five-year horizons, the empirical Campbell-Shiller slope coefficients are -0.95 and -1.72, while the model-implied coefficients are -0.339 and -1.274. We find that the time variation in the habit stock plays a crucial role in explaining the time variation in the forward premium.

Third, how large is the inflation risk premium and is it time varying? Increasing empirical evidence shows that nominal interest rates are not consistent with the Fisher hypothesis, which assumes that nominal interest rates are equal to real interest rates plus the expected inflation. For instance, the spread between yields of nominal bonds and index-linked bonds is, on average, larger than realized inflation and its dynamics are only partially explained by changes in expected inflation. In our model the spread between nominal and real interest rates includes a state-dependent inflation risk premium. We find that the inflation risk premium accounts for about one fourth of the spread between nominal and real interest rates. The inflation risk premium is upward-sloping and time-varying. The average inflation risk premium is 44 basis points for an eight year horizon and ranges between 20 and 90 basis points.

Fourth, to what extent does the time variation in the inflation risk premium explain the rejection of the expectations hypothesis? We regress the forward premium onto the inflation risk premium and find that a large component of the time variation of the forward premium is due to the time variation in the inflation risk premium. The results hold for any horizon between 6 months and 10 years.

Fifth, we find that the model implies a nonlinear spot interest rate process which is similar to the one estimated using both the semi-parametric method of Conley, Hansen, Luttmer, and Scheinkman (1997) and the nonparametric method of Ait-Sahalia (1999). Moreover, our results show that habit persistence help explain the hump-shaped response of consumption to monetary shocks and the lead-lag correlation between real interest rates and output (see also Fuhrer (2000) and Boldrin, Christiano, and Fisher (2001) for similar results based on calibration methods).

Related Literature. The model in this paper links two separate streams of the literature: monetary models of the term structure of interest rates and business cycle models with habit formation. Important contributions to the monetary literature include Bakshi and Chen (1996a), and Marshall (1992). Marshall (1992) shows that when money is introduced in the model to facilitate consumption transactions, it is possible to reproduce the negative correlation between inflation and stock returns. Bakshi and Chen (1996a) and Buraschi and Jiltsov (2005) derive implications for the term structure of interest rates endogenizing inflation in a money-in-the-utility-function model. These models, however, assume time-separable preferences. We explore a setting in which this assumption is relaxed. Sundaresan (1989), Constantinides (1990), Abel (1990) and Detemple and Zapatero (1991) were among the first to relax the time-separability assumption and to study preferences with habit formation. Constantinides (1990) discusses a rational expectations model in which the habit is "intrinsic", i.e. rationally anticipated by the investor when making optimal consumption and investment decisions. He shows that habit persistence induces a more realistic average equity risk premium and it helps to reduce the implied risk-free rate (risk-free rate puzzle).[4] In his model, habit persistence allows stock prices to be rationally volatile even if the consumption process is smooth. Menzly, Santos, and Veronesi (2004) investigate the cross-sectional expected equity returns implications of a Campbell and Cochrane (1999) economy and propose a structural explanation for the observed predictability in stock returns. Other contributions in this literature include Dunn and Singleton (1986), Ferson and Constantinides (1991), Bakshi and Chen (1996b), Daniel and Marshall (1997), Dai (2002), Wachter (2005), Carroll, Overland, and Weil (2000), Heien and Duhram (1991).

The empirical evidence on habit formation is mixed. Heaton (1995) finds evidence in support

[4] Mehra and Prescott (1985), Hansen and Jagannathan (1991) show that the traditional consumption-based CAPM is not consistent with the observed equity premium for reasonable levels of the risk aversion coefficient. Additional studies show that, with respect to the arrival of new information, the time series of aggregate consumption is too smooth, the real interest rate is too low and the volatility of stock prices too high in order to be reconciled with models with traditional complete-markets time-separable preferences.

of habit persistence if consumption is allowed to be locally substitutable. He finds that the stochastic discount factor of a model with long-term habit persistence is consistent with the Hansen and Jagannathan (1991) bounds and can match the autocorrelation pattern of the monthly returns of Treasury Bills and stocks. The model, however, finds it difficult to fit both the volatility of stock returns and the equity premium at the same time. Ferson and Constantinides (1991) show, using aggregate consumption data, evidence of habit formation in stock returns using Euler restrictions at monthly, quarterly, and annual frequencies.

Some new direct studies of preferences with habit formation are based on longitudinal consumption data. Dynan (2000) tests and rejects habit formation in consumption using annual household food consumption (PSID) data. Ravina (2004) uses a data set consisting of individual specific U.S. credit card accounts and finds evidence of habit persistence in household consumption once individual heterogeneity and credit constraints are allowed.

Some important questions, however, are raised by Lettau and Uhlig (2000) who argue, by simulating a real business cycle model with production, that the consumption reaction to technology shocks is too small when utility includes a consumption habit and that the labor input can even become countercyclical. Chapman (1998) highlights some problems of early specifications of models with habit persistence. He shows an example of an endowment process that matches the unconditional moments of consumption growth and asset returns (i.e. resolves the equity premium puzzle) but implies negative marginal utility with probability one in the case of intrinsic habit models. He describes the conditions that need to be satisfied to ensure positive Arrow-Debreu state prices.

The paper is also related to Bekaert, Engstrom, and Grenadier (2004) who explore the role of preference shocks in a multifactor affine model to explain empirical regularities of both equity returns and interest rates and to Wachter (2005) who studies the link between short-term real rates and lagged average real consumption growth rates. She shows that habit persistence helps explain this relation. While she focuses on a real economy, this article provides closed-form solutions for the nominal term structure in a (continuous-time) monetary economy. This result is useful as most of the empirical evidence and available data are on nominal bonds.

Our paper proceeds as follows: Section 1 presents the model. Section 2 and 3 discuss its asset pricing implications. Section 4 describes the data-set. Section 5 presents the econometric methodology. Section 6 summarizes the empirical results. Section 7 discusses the non-linear properties of the model-implied spot interest rate. Section 8 tests the model's implications for the lead-lag relationship between interest rates, consumption and money. Section 9 shows the extent to which the model can explain the Campbell-Shiller expectations puzzle. Section 10 investigates the properties of the model-implied inflation risk premium and compares it to the empirical evidence. Section 11 studies the extent of the trade-off between explaining the equity premium and the dynamics of interest rates. Section 12 concludes. All proofs are in the Appendix.

I. The Model

We study a representative agent endowment economy with habit formation. Real monetary holdings are assumed to provide a transaction service by reducing the total amount of resources X_t needed to achieve a given level of net consumption C_t. Thus, in this economy, money is held because of its positive marginal productivity. Clearly, one may desire to work with a simpler real model, abstracting from the nominal side of the economy. However, a real model would not give implications for nominal interest rates. Since most of the available empirical evidence is on nominal Treasury bonds, this would be a shortcoming. Alternatively, one could specify an exogenous inflation process and impose inflation neutrality. The empirical literature, however, finds strong evidence of violations of the Fisher hypothesis (see Buraschi and Jiltsov (2005)). Finally, a vast monetary literature finds a correlation between lagged real monetary shocks and nominal interest rates. Thus, a monetary framework is necessary to take into account the structural interaction between the monetary and real growth rates of the economy and the different dynamics of the real and nominal yield curves.

Assumption 1

(a) (Preferences). The representative agent is affected by external habit formation H_t. The agent chooses his consumption plan and nominal monetary holdings to maximize his expected value of utility $u(X,H)$

$$E_0 \int_0^\infty e^{-\rho t} \log\left(X_t - H_t\right) \, dt, \qquad \rho > 0 \qquad [A1]$$

(b) (Transaction Costs). The consumption of X_t entails a proportional transaction cost $1-\psi$. Given a level of gross consumption C_t, the level of net consumption is $X_t = \psi C_t$, with $\psi(C_t, \frac{M_t}{P_t}) = \psi_0 \left(\frac{M_t/P_t}{C_t}\right)^\gamma$, $0 \leq \gamma \leq 1$ and ψ_0 being a normalization factor such that $\psi \leq 1$. M_t and P_t are the nominal amount of monetary holdings and the general price level respectively.

This assumption implies that money reduces the transaction costs of obtaining the desired level of consumption. Thus, although money does not yield any interest, money is demanded in equilibrium.[5] A similar approach has been used to investigate monetary equilibria in different settings by Marshall (1992) and Bekaert (1996). In the case of time-separable preferences, Feenstra (1986) shows that money-in-the-utility models are equivalent to assuming that money facilitates consumption transactions. Thus, this specification is also related to Bakshi and Chen (1996a) who consider an economy with money-in-the-utility function.[6]

The stochastic sequence of habits H_t is regarded as exogenous by each agent and specified implicitly as a function of the past aggregate consumption and monetary holdings. The existence of

[5] In the period 1905-2004, money velocity $\frac{C_t}{M_t/P_t}$ ranged between 1 and 2. Thus, a value of $0 < \psi_0 < 2^\gamma$ guarantees that the *realized* value of the transaction cost function is $0 < \psi < 1$.

[6] The previous specification can also be thought as one in which the agent consumes a composite good $X = C^{1-\gamma} C_*^\gamma$ with C_* being a cash-good subject to a cash-in-advance constraint $C_* = M/P$.

transaction costs coupled with habit persistence makes monetary shocks have a persistent effect on future asset prices and interest rates. The intuition is simple. With monetary transaction costs, current money shocks affect ψ and therefore current marginal utility and optimal current consumption. Because current consumption, however, affects the habit stock H_t, current money shocks affect future marginal utilities and therefore future asset prices. This link is very important in order to reproduce the persistence found in the data.

The equilibrium level of risk aversion is state-dependent: individuals become more risk-averse in bad times (when consumption is low relative to its past values) than in good times (when consumption is high relative to its historical levels). Thus, the habit H_t affects the way in which consumption shocks change the level of risk aversion in a state-dependent fashion. This property allows one to investigate a (potentially) more flexible term structure model, even when consumption growth is *i.i.d.*, without giving up tractability. We follow Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004) and model the habit H_t as an external process in terms of the process $S_t = \frac{X_t - H_t}{X_t}$, the surplus to consumption ratio, or equivalently, its inverse Y_t. Clearly, given an equilibrium process for X_t, there is a one-to-one mapping between Y_t and H_t, so that the choice of which one to model is immaterial. However, since the local curvature of the utility function $-X_t \frac{u_{xx}}{u_x}$ is equal to $\frac{1}{S_t}$, it is more convenient to model Y_t directly.

Assumption 2 (Habit Formation)

Let the net surplus-consumption ratio be $S_t = \frac{X_t - H_t}{X_t}$ *and let* $Y_t = \frac{1}{S_t}$. Y_t *follows a stochastic mean reverting process*

$$dY_t = k_y (\theta_y - Y_t)\, dt - (Y_t - \lambda)\, \sigma_y dW_t \qquad [A2]$$

initialized at $Y_0 > \lambda > 1$.

The Y_t process is bounded below by λ. Thus, when $\lambda \geq 1$ the specification ensures that the habit H_t is always positive. This condition also implies that $X_t > H_t$ so that the marginal utility is always finite and positive. Y_t is mean reverting to θ_y and its stochastic innovations are driven by *unexpected* innovations in net consumption X_t. Thus, the dynamics of Y_t are a function of both the gross-consumption endowment shocks and the liquidity shocks affecting their service flow. Assumption [*A*2] is originally made in Menzly, Santos, and Veronesi (2004) to study the cross-section of (real) expected equity returns. This diffusion process is not affine since the local variance is quadratic in the state variable. It is easy, however, to verify that the drift and diffusion coefficients satisfy global Lipschitz and growth conditions which imply the existence and uniqueness of a strong solution to [*A*2].[7] Of course, for the process to be stationary additional restrictions are required. The stationarity of the process depends on the boundary behavior of Y_t. The following Proposition discusses the

[7]For the statement of Ito's classical results of strong form existence and uniqueness using global conditions see Oksendal (1992), Theorem 5.5); for a statement using local Lipschitz see Karatzas and Shreve (1991), Theorem 2.5, p. 287; for even weaker conditions see the Yamada-Watanabe theorem (Karatzas and Shreve (1991), Theorem 2.13 p. 291).

conditions under which the process $[A2]$ admits a unique strong solution and a stationary density density p, such that when the diffusion is initialized with a density p, the diffusion is stationary with stationary density p (see Hansen and Scheinkman (1995), p. 774 and Karlin and Taylor (1981), p. 221).[8] The Proposition also provides the conditions under which the process is square integrable.

Proposition 1

A. (Stationarity). If the process $[A2]$ satisfies condition $[C1]$

$$\theta_y > \lambda > 1, \qquad k_y > 0, \qquad Y_0 > \lambda. \qquad [C1]$$

then Y_t does not explode and its unique strong solution is

$$Y(t) = \lambda + \omega_t^{-1}\left[(Y_0 - \lambda) + k(\theta - \lambda)\int_0^t \omega_s ds\right]$$
$$\omega_t = \exp\{(k + \frac{1}{2}\sigma_y^2)t + \sigma_y(W_t - W_0)\}$$

Moreover, the boundaries of $(\lambda, +\infty)$ are both entrance boundaries and the stationary density $p(y)$ of Y is Inverse-Gamma:

$$p(y) = \mathcal{N}\cdot(y-\lambda)^a \exp\left(\frac{b}{y-\lambda}\right)$$

with $\mathcal{N} = \frac{(-b)^{-1-a}}{\Gamma(-1-a)}$, $a = -2(1 + \frac{k_y}{\sigma_y^2})$ and $b = -2\frac{k_y(\theta_y - \lambda)}{\sigma_y^2}$.

B. (Square-integrability). If, in addition to $[C1]$, the following condition is also satisfied, then $E|Y|^2 < \infty$. The process is square-integrable.[9]

$$2k_y - \sigma_y^2 > 0 \qquad [C2]$$

The first part of Proposition 1 shows that the inverse of the surplus-consumption ratio $Y(t)$ is a weighted average of the lagged shocks W_s, for $0 \le s \le t$. The persistence of the shocks depends on $(2k + \sigma_y^2)$. When this is positive, the solution converges. Moreover, it shows that when the process is properly initialized, mean reversion (i.e. $k > 0$) is a sufficient condition for the two boundaries to be entrance boundaries.[10] The second part shows that in order for the process to have bounded unconditional second moments, mean-reversion is not sufficient and a stricter condition is required. A full characterization of the conditional moments are given in Proposition 3.

The parameter λ is the lower bound of Y_t, so that $(0, \frac{1}{\lambda})$ is the support of the surplus ratio S_t. Campbell and Cochrane (1999) restrict λ to yield a constant real interest rate. Since our focus is

[8] See Ait-Sahalia (1996), Appendix 1, Conley, Hansen, Luttmer, and Scheinkman (1997), Assumption 1 and 2, p. 532 and 533.

[9] In this case the process is also said stationary in a wide sense. For a definition of strict and wide sense stationarity see Liptser and Shiryaev (2001), page 23.

[10] For the sake of comparison, it is insightful to observe that in the case of square-root diffusions, Feller (1951) shows that zero is an entrance boundary for the process if $2k\theta > \sigma^2$.

to model the dynamics of the term structure of interest rates, we do not impose such a restriction. Instead, we require $\lambda \geq 1$ to avoid negative habit formation.

The growth rate of the aggregate consumption endowment process is i.i.d. and follows the process

$$\frac{dC_t}{C_t} = \mu_c dt + \sigma_c dW_t^c$$

The assumption of an i.i.d. consumption growth rate process is motivated by a large body of empirical evidence that argues that deviations from this assumption, which is consistent with the permanent income hypothesis (PIH), are small. At the aggregate level, the autocorrelation of U.S. consumption[11] at quarterly frequency is 0.22 in 1947-1996 and it becomes -0.117 in the 1891-1995 period.[12] This assumption is also consistent with the empirical literature on the PIH based on household data.[13] Thus, we follow Campbell and Cochrane (1999) and exogenously constrain the consumption growth process to be *i.i.d.* and investigate whether it is possible to generate the required autocorrelation in the stochastic discount factor (SDF) via a parsimonious habit specification.

Stock and Watson (1989) find that the $M1$ money supply process in the U.S. can be described as a stationary process around a positive deterministic time trend.[14] Thus, we consider a process for the money supply which is given by two components: a deterministic (exponential) rate μ_M and a stochastic deviation from this trend L_t, which can be thought of as the detrended inverse of the money supply:

$$d\ln M_t = \mu_M dt - d\ln L_t \qquad \mu_M > 0 \tag{1}$$

Assumption 3 (Money Supply)

Let L_t be the aggregate liquidity shock generated by n factors ℓ_{it} following the stochastic process

$$d\ell_{it} = k_{\ell_i} \left(\theta_{\ell_i} - \ell_{it}\right) dt + \sigma_{\ell_i} \ell_{it} dW_t^{\ell_i} \qquad \text{with} \qquad E(dW_t^{\ell_i} dW_t^c) = \rho_{\ell_i,c} dt \qquad [A3]$$

with $L_t = \sum_{i=1}^{n} \ell_{it}$ and $E(dW_t^{\ell_i} dW_t^{\ell_j}) = 0$.

This particular specification of the ℓ_{it} process is of special interest for two reasons. First, Nelson (1990) shows that this process is the continuous time limit of the Garch(1,1)-M studied by Engle and Bollerslev (1986). It has been extensively used to model empirically log-excess equity returns but never to investigate the implications of a term structure model. Second, it will be shown that the interest rate dynamics depend on the ℓ_t process and that the quadratic local variance of ℓ_t produces in equilibrium a non linear drift in interest rates. We investigate the extent to which this particular

[11] For the U.K. the autocorrelation is -0.017 (1970-1996), for Canada it is 0.113 (1970-1996).

[12] See "Asset Prices, Consumption, and the Business Cycle," in Handbook of Macroeconomics, edited by John B. Taylor and Michael Woodford.

[13] Attanasio and Weber (1995) use a time series of cross-sections from the Consumer Expenditure Survey (1980-1990) to test the PIH after controlling for several variables that are likely to affect family composition and labor supply behavior over the business cycle. They use a robust IV technique to control for the potential endogeneity of the explanatory variables. They find that the excess sensitivity to labor income disappears after controlling for these effects (see Table 3) and conclude that consumption growth display very modest autocorrelation.

[14] **?** also investigate a specification for the log of the money supply with a quadratic time trend. To simplify the derivation, we decided to consider the simpler case with a linear time trend.

model-implied non-linearity in interest rates is close to what found in the data by Conley, Hansen, Luttmer, and Scheinkman (1997) and Ait-Sahalia (1999).

Since ℓ_t is a shifted version of Y_t, with a boundary set at zero, the conditions for the existence of a stationary density and for the square-integrability are analogous to Conditions $[C1]$ and $[C2]$ with $\lambda = 0$. Thus, the process ℓ_t is stationary if $\theta_\ell > 0$, $k_\ell > 0$, $\ell_0 > 0$ and square-integrable if $2k_\ell - \sigma_\ell^2 > 0$.[15] The closed-form solution of the stationary density can be obtained from the result of Proposition 1a setting $\lambda = 0$.

Using Ito's rule, it follows that in a one monetary factor economy $\frac{dM_t}{M_t} = \left((\mu_M + k_\ell + \sigma_\ell^2) - \frac{k_\ell \theta_\ell}{\ell_t}\right) dt - \sigma_\ell dW_t^\ell$. The expected growth rate of the nominal money supply can be either positive or negative depending on the level of the state variable ℓ_t. The drift of the money growth rate is positive (negative) if ℓ_t is above (below) $\frac{k_\ell \theta_\ell}{\mu_M + \sigma_\ell^2 + k_\ell}$. The nominal stock of money supply is however always positive. For expositional simplicity and to streamline the notation, whenever clear from the context we drop the subscript i from ℓ_t.

Since the innovations of the inverse-surplus ratio $\sigma_y dW_t$ must be functions of the same Brownian innovations driving the state variables of the economy, namely $dW_t = \left[dW_t^c, dW_t^{\ell_i}\right]$, we simply define $\sigma_y dW_t \equiv \left[\sigma_{cy} dW_t^c - \sigma'_{\ell y} dW_t^\ell\right]$ with $\sigma_y^2 \equiv \sigma_{cy}^2 + \sum_i (\sigma_{\ell_i y}^2 \text{ - } 2\rho_{\ell_i,c}\sigma_{cy}\sigma_{\ell_i y})$. When $\sigma_{cy} > 0$, a negative consumption shock increases the inverse surplus to consumption level, thus inducing an increase in the investor's implied (local) risk aversion. The second component comes from liquidity shocks. Due to habit formation in consumption, the transaction service generated by liquidity shocks at time t affect the marginal utility of consumption in future time periods as well. If monetary holdings do not provide a transaction service, the dynamics of the habit stock are driven only by the past real consumption. We let the data illustrate the empirical magnitude of $\sigma_{\ell y}$.

In the monetary literature, it is common to interpret the long-term mean of the monetary policy as the targets of a Taylor rule in which the monetary authority adjusts the money supply according to the deviations of some observable economic aggregates from their target levels.[16] Since Taylor rules are usually specified in terms of one real factor (deviations from a real consumption growth target) and one nominal factor (deviations from an inflation target), we consider a two factor process for the money supply.

In addition to Condition 1 and 2 and $\rho > 0$, we require additional parameter restrictions to ensure that $Cov(dH_t, dC_t)/dt > 0$, so that positive innovations in consumption increase the habit. Let $c^* = \max_i \sigma_{\ell_i}\sigma_c\rho_{c,\ell_i}$, it is easy to show that a sufficient condition is that $[(1-S_t)\sigma_c^2\text{-}(1-S_t)\gamma c^*\text{-}S_t\sigma_y(1-\lambda S_t)\psi_c] \geq 0$ with $\psi_c = \sigma_{yc}\sigma_c - \sigma_{y\ell_i}\sigma_c\rho_{\ell_i,c}$. The lower bound of this conditions is reached at $S_* = v\frac{1}{\sigma_y}$ with $v = \frac{\sigma_y\psi_c + \sigma_c^2 - \gamma c^*}{2\lambda\psi_c}$. Substituting S_* back, we find that $Cov(dH_t, dC_t)/dt > 0$ if $\sigma_y \geq \frac{v\sigma_c^2 - v\gamma c^* + \lambda\psi_c v^2}{\sigma_c^2 - \gamma c^* - v\psi_c}$.[17]

[15] In the case of a Cox, Ingersoll, and Ross (1985) square-root processes, the equivalent condition for zero being non-attracting is $2k\theta - \sigma^2 > 0$.

[16] We refer to Buraschi and Jiltsov (2005) for a discussion of a monetary model of the term structure with an explicit two factor Taylor rule. They derive reduced form equations similar to $[A3]$.

[17] The condition requires $a\alpha^2 + b\sigma + c \geq 0$, with $\hat{a} = \frac{1}{4}\frac{\sigma_L\psi_L}{\lambda}$, $\hat{b} = (\frac{1}{2\lambda} - 1)\sigma_c\sigma_L\rho_{c,L}$, $\hat{c} = \sigma_L q_1(\sigma_c\rho_{c,L} - \lambda\psi_L)$ and

A. Aggregation in Heterogeneous-Agent Economies

Since the aggregation properties of economies with this type of habit preferences are not completely known, in this section we briefly investigate the link between the competitive equilibrium prevailing in a heterogeneous-agent economy and the representative agent equilibrium. We consider two heterogenous economies. In the first economy, all agents have identical logarithmic preferences (as in our model) but different endowments. In the second economy, agents have heterogeneous preferences.

Case 1. (Heterogeneous endowment) - When markets are dynamically complete it is possible to show that equilibrium asset prices in a heterogeneous-agent economy are the same as those prevailing in a representative agent economy in which the single agent's consumption is equal to the total available resources $X_t = \sum_i x_t^i$ and $H_t = \sum_i h_t^i = X_t - \sum_i s_t^i x_t^i$. When markets are dynamically complete, the stochastic discount factor $\xi_{t,T}^i = e^{-\delta(T-t)} \frac{u'(x_T^i - h_T^i)}{u'(x_t^i - h_t^i)}$ of each agent in the heterogeneous economy is unique and identical across agents, so that $\xi_{t,T}^i = \bar{\xi}_{t,T}$ $\forall i$. With logarithmic preferences, this implies that $\left[e_{t,T}^{\delta(T-t)} \bar{\xi}_{t,T}\right](x_T^i - h_T^i) = (x_t^i - h_t^i)$. Since the term in square bracket is agent independent, we can aggregate across agents and obtain that $\bar{\xi}_{t,T}$ is equal to the intertemporal marginal rate of substitution of an agent with logarithmic preferences who consume the aggregate endowment, i.e. $u(X_t, H_t) = \log(X_t - H_t)$. Since $s_t^i x_t^i = x_t^i - h_t^i$ and in equilibrium $x_t^i - h_t^i = x_t^j - h_t^j$, then $\sum_i s_t^i x_t^i = X_t - H_t$ so that the representative agent surplus consumption ratio is a weighted average of the individual surplus ratios, i.e. $S_t = \sum_i \omega_i s_t^i$ where the weights ω_i are equal to the individual consumption share $\omega_i = (x_t^i / X_t)$.

Case 2. (Heterogeneous preferences) - Since for analytical convenience in the rest of the article we assume the existence of a representative agent with logarithmic preferences, in what follows we study if a disaggregated economy with multiple agents and heterogeneous CRRA preferences admits a representative agent aggregation. The case of a representative agent with logarithmic preferences is a special case. Let us assume that each agent has preferences of the type $u_t = e^{-\delta t} \frac{1}{1-\gamma}(x_t(\gamma) - h_t(\gamma))^{1-\gamma}$ and $u_t = e^{-\delta t} \ln(x_t - h_t)$ for the agent with $\gamma = 1$.

In a heterogeneous economy, the social planner allocates the aggregate endowment X_t across agents to achieve Pareto efficiency. Let $\omega(\gamma)$ be the social weight attributed to agent of type γ. Since there is no intertemporal transfer of resources via a production technology, at each time t the social planner maximizes the following objective function

$$\sup_{x_t(X_t, h_t; \gamma)} \int \omega(\gamma) \frac{1}{1-\gamma} (x_t(\gamma) - h_t(\gamma))^{1-\gamma} d\gamma \qquad \forall t$$

subject to the resource constraint $\int x_t(\gamma) d\gamma \leq X_t$. Let z_t be the Lagrange multiplier associated to the resource constraint, the first order conditions are $\omega(\gamma)(x_t(\gamma) - h_t)^{-\gamma} = z_t$ and $\int x_t(\gamma) d\gamma = X_t$. Solving for x_t and substituting in the resource constraint we have $X_t - H_t = \int \omega(\gamma)^{1/\gamma} z_t^{-1/\gamma} d\gamma$. The Lagrange multiplier z_t is an implicit function of the aggregate surplus consumption $X_t - H_t$.

$q_1 = \frac{\sigma_c \rho_{c,L}}{2\lambda \psi_L}$, $\psi_L = \sigma_{yc} \sigma_L \rho_{c,L} - \sigma_{yL} \sigma_L$.

Let $z_t = \varphi(X_t - H_t)$ be the solution of the previous implicit function. Asset prices depends on the representative agent stochastic discount factor $\xi_{t,T}$, which is equal to $e^{-\delta(T-t)}(z_T/z_t)$. Substituting the solution for the Lagrange multiplier, we have $\xi_{t,T} = e^{-\delta(T-t)}\frac{\varphi(X_T-H_T)}{\varphi(X_t-H_t)}$.

Two things should be noticed. First, the representative agent stochastic discount factor is a deterministic function of the excess aggregate consumption $X_T - H_T$. The representative agent inherits the external habit preferences of the heterogenous agents as the argument of φ is the aggregate surplus consumption. The function φ, however, is not an arithmetic average of each agent's power functions.

Second, from the first order conditions it is possible to obtain that in equilibrium the consumption share of an agent of type γ is equal to $\frac{x_t(\gamma)}{X_t} = \frac{h_t(\gamma)}{X_t} + \omega(\gamma)^{1/\gamma} y_t(\gamma)$, where $y_t(\gamma) = \frac{z_t^{-1/\gamma}}{X_t}$ is the solution of $S_t = \int \omega(\gamma) y_t(\gamma) d\gamma$. Since, under Condition 1, S_t is stationary then each agent's consumption share $\frac{x_t(\gamma)}{X_t}$ is stationary. No single agent dominates the economy in the long-run. This contrasts with heterogenous-agent economies with standard CRRA agents in which the economy becomes eventually dominated by the least risk averse agent (Wang (1996)). The reason for the different asymptotic behavior is due to the effect of the habit on the cross-section of marginal utilities. In a traditional economy, as the economy grows, the agent with lower risk aversion experiences higher marginal utility of consumption. Thus, he is allocated a progressively higher share of total endowment. This does not occur in the habit economy since consumption growth has also the effect of increasing the habit stock which reduces the marginal utility of consumption. This reduction is larger the lower the risk aversion coefficient. Thus, habit persistence is an interesting and effective way to achieve stationarity in the distribution of consumption allocation in the disaggregated heterogeneous-agent economy.

In order to obtain closed-form solutions, our analysis will focus on economies that support $\varphi(X_T - H_T) = (X_T - H_T)^{-1}$, i.e. economies supporting a logarithmic representative agent.

Chetty and Szeidl (2003) argue that a non-psychological interpretation of external habit preferences can be given in the context of economies with forced consumption commitments. Suppose that an individual i has preferences defined over both c_t^i, a vector of consumption goods (or characteristics) of which some components are not directly observable, and h_t^i, a consumption good that can be adjusted only at discrete frequencies, such as real estate or monetary holdings. At any generic point of time t the individual has to commit to a stream of consumption goods h_s^i between t and the subsequent readjustment date. When preferences are time-separable, the aggregate behavior of this economy is observationally equivalent to a representative agent economy with habit persistence. To show the equivalence, let $u_t^i = e^{-\rho t}\left(\ln c_t^i + \beta^i \ln h_t^i\right)$. Since c_t^i can be freely adjusted at no cost if markets are complete, there exists a unique stochastic discount factor $\xi_{t,T}$ such that $\xi_{t,T} = e^{-\rho\tau}\frac{1/c_T^i}{1/c_t^i}$. Thus, using the previous aggregation argument and defining X_t to be the total observable consumption, $X_t = C_t + H_t$, with $\sum_i h_t^i = H_t$ and $\sum_i c_t^i = C_t$, then $X_t - H_t = (X_T - H_T)\left[e^{\rho\tau}\xi_{t,T}\right]$ so that

$$\xi_{t,T} = e^{-\rho\tau}\frac{X_t - H_t}{X_T - H_T}$$

The Euler condition that describes bond and asset prices in an heterogeneous competitive equilibrium is thus identical to the Euler equation of a representative agent economy with habit persistence:

$$B(t,T) = E_t \left[e^{-\rho\tau} \frac{X_t - H_t}{X_T - H_T} \right]$$

Since h_t^i can change only at discrete frequencies, the aggregate committed consumption H_t is a weighted average of lagged consumption innovations. Clearly, a key assumption for this equivalence to be useful empirically is that some of the individual components of c_t^i are not directly observable.

B. The Equilibrium Price Level

First, we solve for the general price level in equilibrium. Then, we use the result to solve for equilibrium asset prices. Let us define the real stock of money as $m = M/P$. First, notice that in equilibrium the following transversality condition needs to hold in order for an interior condition to exist:

Transversality Condition

$$\lim_{T\to\infty} e^{-\rho T} E_t \left[u_x(X_T, H_T) \frac{\partial X_T}{\partial C_T} \frac{1}{P_T} \right] = 0 \tag{2}$$

If (2) were not satisfied, a small reduction in consumption at time t would yield a large discounted marginal utility by using money as a store of value and delaying consumption up to time T.

To solve for the equilibrium price level, we follow Bakshi and Chen (1996a) and consider the marginal decision between consumption and monetary holdings. Since the inverse of the general price level $\frac{1}{P_s}$ is the relative price of the monetary holdings with respect to consumption, a marginal reduction in one unit of real monetary stock yields a marginal utility reduction of $u_x(X_t, H_t)\frac{\partial X_t}{\partial C_t}\frac{1}{P_t}$. The marginal revenue, over a dt interval of time with $s = t+dt$, is $E_t e^{-\rho dt} u_x(X_s, H_s)(\frac{\partial X_s}{\partial m_s}dt + \frac{\partial X_s}{\partial C_s})\frac{1}{P_s}$. The first term is the service flow of the monetary stock, the second term comes from the increase in consumption at the later period. Since in equilibrium the agent must be indifferent between reducing consumption or the money stock, the following Euler restriction must be satisfied for $\forall dt$:

$$u_x(X_t, H_t)\frac{\partial X_t}{\partial C_t}\frac{1}{P_t} = E_t e^{-\rho dt} \left[u_x(X_s, H_s)(\frac{\partial X_s}{\partial m_s}dt + \frac{\partial X_s}{\partial C_s})\frac{1}{P_s} \right] \tag{3}$$

Equation (3) can be solved forward. Using the the transversality condition and taking the continuous time limit, we obtain

$$\frac{1}{P_t} = E_t \left[\int_t^\infty e^{-\rho(s-t)} \frac{u_x(X_s, H_s)\frac{\partial X_s}{\partial m_s}}{u_x(X_t, H_t)\frac{\partial X_t}{\partial C_t}} \frac{1}{P_s} ds \right] \tag{4}$$

Money is a durable stock, thus its value reflects its service flow *in all future periods.*[18] Taking

[18] Bakshi and Chen (1996a) use this Euler condition to solve for a stochastic monetary equilibrium in an economy with time-separable preferences.

derivatives, we obtain $u_x(X_t, H_t)\frac{\partial X_s}{\partial C_s} = (1-\gamma)\frac{1}{S_t C_t}$ and $u_x(X_s, H_s)\frac{\partial X_s}{\partial m_s} = \gamma\frac{1}{S_s m_s}$. So that

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma} C_t S_t \int_t^{\infty} e^{-\rho(s-t)} E_t\left[\frac{P_s}{S_s M_s}\frac{1}{P_s}\right] ds \tag{5}$$

$$= \frac{\gamma}{1-\gamma} C_t S_t \int_t^{\infty} e^{-(\rho+\mu_M)(s-t)} \frac{1}{L_t M_t} E_t\left[Y_s L_s\right] ds \tag{6}$$

Since L is the inverse of the money supply, for simplicity one could initialize $L_0 M_0 = 1$, so that $\frac{1}{L_t M_t} = e^{-\mu_M t}$. To solve for the equilibrium price level, which allows one to solve for nominal interest rates, we need to solve for the conditional expected value of the product of two Garch-Ito processes. The following Lemma provides the general result for this class of processes.

Lemma 1 *Consider a linear system of two mean-reverting Garch-Ito processes* ξ_{1t} *and* ξ_{2t}

$$d\xi_{1t} = k_{\xi_1}\left(\theta_{\xi_1} - \xi_{1t}\right) dt + \left(\xi_{1t} - \lambda_{\xi_1}\right)\left[\upsilon dW_t^2 + \sigma_{\xi_1} dW_t^1\right]$$

$$d\xi_{2t} = k_{\xi_2}\left(\theta_{\xi_2} - \xi_{2t}\right) dt + \left(\xi_{2t} - \lambda_{\xi_2}\right)\sigma_{\xi_2} dW_t^2, \qquad E(dW_t^1 \cdot dW_t^2) = \rho dt$$

The conditional expectation of their product $q_t = \xi_{1t}\xi_{2t}$ *is non-linear in the state variables and equal to*

$$E_t\left[q_{t+\tau}\right] = A_0(\tau) + A_1(\tau)\xi_{1t} + A_2(\tau)\xi_{2t} + A_3(\tau) q_t \tag{7}$$

with $A_i(\tau)$ *being deterministic functions of the expectation horizon and fully characterized in the appendix.* ■

Clearly, this result is important for asset pricing applications since the price of any contingent claim is the conditional expected value of the product of the stochastic discount factor and the future cash flows. Moreover, it can be used in econometric applications to calculate conditional covariances of pairs of Garch-Ito processes. To solve for the equilibrium price level, let $\Theta_Y = [k_y, \theta_y, \sigma_y, \lambda]$ and $\Theta_{\ell_i} = [k_{\ell_i}, \theta_{\ell_i}, \sigma_{\ell_i}, 0]$ be the vectors of structural parameters for Y and ℓ_i, then let us use Lemma 1 to solve for $E_t\left(Y_s \ell_{i,s}\right)$. The equilibrium price level is:

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma}\frac{C_t}{Y_t}\frac{1}{L_t M_t}\int_t^{\infty} e^{-(\rho+\mu_M)(s-t)}\left[\sum_{i=1}^{2}\left(\begin{array}{c} A_{0i}(s-t;\Theta_Y,\Theta_{\ell_i}) + A_{\ell_i}(s-t;\Theta_Y,\Theta_{\ell_i})\ell_{it} + \\ A_Y(s-t;\Theta_Y,\Theta_{\ell_i})Y_t + A_{\ell_i Y}(s-t;\Theta_Y,\Theta_{\ell_i})\ell_{it}Y_t \end{array}\right)\right] ds$$

If Conditions $[C1]$ and $[C2]$ are satisfied and the parameters $A_i(s)$ are bounded, the integral converges and the economy admits a monetary equilibrium with a finite market clearing price level. Sufficient conditions are that $-k_y - k_{\ell_i} - (\sigma_{yc}\sigma_{\ell_i}\rho + \sigma_{y\ell_i}\sigma_{\ell_i}) < 0$, $\forall i$. We summarize the result as follows:

Proposition 2 *If, in addition to Conditions* $[C1]$ *and* $[C2]$, $-k_y - k_\ell - (\sigma_{yc}\sigma_\ell\rho + \sigma_{y\ell}\sigma_\ell) < 0$ *then the economy supports a monetary equilibrium whose price level is*

$$\frac{1}{P_t} = \frac{\gamma}{1-\gamma}\frac{C_t}{Y_t}\frac{1}{L_t M_t}\psi_t$$

$$\psi_t = \sum_{i=1}^{2}\Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i})Y_t + \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i})\ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i})Y_t\ell_{it}$$

where $\Gamma(\cdot)$ are deterministic functions of the structural parameters whose functional form is fully described in the appendix. ■

From the solution of the equilibrium price level we can determine the money velocity. This is defined as $v_t = \frac{P_t C_t}{M_t}$, thus $v_t = \frac{(1-\gamma)L_t Y_t}{\gamma \psi_t}$. Under Conditions $[C1]$ and $[C2]$ both ℓ_{it} and Y_t are strictly positive stationary processes, thus both the numerator and the denominator are stationary. Thus, it is possible to show that a sufficient condition for v_t to be stationary is that the denominator does not cross zero inside the support of Y_t and ℓ_t. A sufficient condition is that the coefficients Γ_0, Γ_1, Γ_2 and Γ_3 are non-negative.

II. The Term Structure of Interest Rates

A. Nominal Spot Interest Rate. The nominal spot rate can be obtained from the first order conditions observing that the agent in equilibrium must be indifferent between reducing current consumption to invest in a (nominal) riskless opportunity paying off $1 + R_t dt$ and reducing current consumption to increase the money stock. In the first case, the expected discounted increase in marginal utility is $e^{-\rho dt} E_t[u_x(X_s, H_s)\ \frac{\partial X_s}{\partial C_s}(1 + R_t dt)\ \frac{1}{P_s}]$. In the second case, the increase is $e^{-\rho dt} E_t[u_x(X_s, H_s)$ $(\frac{\partial X_s}{\partial m_s} dt + \frac{\partial X_s}{\partial C_s})\frac{1}{P_s}]$. In equilibrium, these two quantities must be equal. Simplifying and taking the continuous time limit, we obtain $R_t = \frac{u_x(X_t,H_t)\partial X_t/\partial m_t}{u_x(X_t,H_t)\partial X_t/\partial C_t}$ which yields the following relationship:

$$R_t = \frac{\gamma}{1-\gamma} \frac{C_t P_t}{M_t}$$

Substituting the equilibrium price process in Proposition 2, we find the following relationship between the inverse surplus ratio and the instantaneous nominal interest rate:

$$R_t = \frac{Y_t L_t}{\left[\sum_{i=1}^{2} \left(\Gamma_{0i}(\Theta_Y, \Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y, \Theta_{\ell_i}) Y_t + \Gamma_{\ell_i}(\Theta_Y, \Theta_{\ell_i}) \ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y, \Theta_{\ell_i}) Y_t \ell_{it}\right)\right]} \tag{8}$$

The nominal interest rate is non linear in the state variables. It depends on both the habit stock and the factors affecting the monetary aggregate. A higher expected growth in monetary holdings increases the interest rate. The higher the surplus ratio S_t (i.e. the higher the current level of consumption with respect to the habit), the higher the incentive to save for future consumption and the lower the nominal interest rate. Thus, the model implies that the lagged real money-adjusted consumption predicts both real and nominal interest rates. This empirical implication of the model is consistent with the finding of a lagged effect between changes in monetary holdings and interest rates, which has been the focus of substantial interest in the monetary literature.

B. Bond Prices. Given the equilibrium price process, we can solve for the term structure of nominal and real bond prices. Let $N(t, \tau)$ and $B(t, \tau)$ be time t prices of two pure discount bonds paying at time $t + \tau$ one unit of currency and one unit of the consumption good respectively. The price of the second bond is equal to the price of an index-linked bond. The price of the nominal bond must satisfy the following Euler condition:

$$N(t,\tau) = e^{-\rho\tau} E_t \left[\frac{u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t}} \frac{P_t}{P_{t+\tau}} \right]$$

The marginal cost of reducing consumption and purchasing a nominal bond is $N(t,\tau)\, u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t} \frac{1}{P_t}$. In equilibrium, it must be equal to the marginal utility of the future consumption that can be obtained from the bond investment $u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}} \frac{1}{P_{t+\tau}}$. The Euler restriction for the real bond is similar. However, since the payoff is indexed to the price level:

$$B(t,\tau) = e^{-\rho\tau} E_t \left[\frac{u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t}} \right]$$

The price of the two bonds are affected by the habit via the consumption surplus ratio as $\frac{u_x(X_{t+\tau}, H_{t+\tau}) \frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t}} = \frac{C_t S_t}{C_{t+\tau} S_{t+\tau}}$. Using the result of Proposition 2 for the price level, we obtain $C_{t+\tau} S_{t+\tau} P_{t+\tau} = \frac{\gamma}{1-\gamma} e^{\mu_M \tau} L_t M_t / \Psi_{t+\tau}$ with $\Psi_{t+\tau} = \sum_{i=1}^{2} (\Gamma_{Y,i}(\Theta_Y, \Theta_{\ell_i}) Y_{t+\tau} + \Gamma_{\ell_i}(\Theta_Y, \Theta_{\ell_i}) \ell_{it+\tau} + \Gamma_{\ell_i Y}(\Theta_Y, \Theta_{\ell_i}) \ell_{it+\tau} Y_{t+\tau} + \Gamma_{0i}(\Theta_Y, \Theta_{\ell_i}))$. Hence, applying Lemma 1, we obtain the following result:

Theorem 1 (The Nominal Term Structure) *The price of a nominal bond $N(t,\tau)$ at time t with time to maturity τ is given by*

$$N(t,\tau) = e^{-(\rho-\mu_M)\tau} \frac{\sum_{i=1}^{2} \left(\Lambda_{0i}(\tau;\Theta_Y,\Theta_{\ell_i}) + \Lambda_{Y,i}(\tau;\Theta_Y,\Theta_{\ell_i}) Y_t + \Lambda_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) \ell_{it} + \Lambda_{\ell_i Y}(\tau;\Theta_Y,\Theta_{\ell_i}) Y_t \ell_{it} \right)}{\sum_{i=1}^{2} \left(\Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i}) Y_t + \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i}) \ell_{it} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) Y_t \ell_{it} \right)}$$

where

$$\begin{aligned}
\Lambda_{Y,i}(\tau;\Theta_Y,\Theta_{\ell_i}) &= \Gamma_{Y,i}(\Theta_Y,\Theta_{\ell_i}) e^{-k_y\tau} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) \\
\Lambda_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) &= \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i}) e^{-k_{\ell i}\tau} + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_Y(\tau;\Theta_Y,\Theta_{\ell_i}) \\
\Lambda_{\ell_i Y}(\tau;\Theta_Y,\Theta_{\ell_i}) &= \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_{\ell_i}(\tau;\Theta_Y,\Theta_{\ell_i}) \\
\Lambda_{0i}(\tau;\Theta_Y,\Theta_{\ell_i}) &= \Gamma_{0i}(\Theta_Y,\Theta_{\ell_i}) + \Gamma_Y(\Theta_Y,\Theta_{\ell_i})\theta_y \left(1 - e^{-k_y\tau}\right) + \Gamma_{\ell_i}(\Theta_Y,\Theta_{\ell_i})\theta_{\ell i} \left(1 - e^{-k_{\ell i}\tau}\right) + \\
&\quad + \Gamma_{\ell_i Y}(\Theta_Y,\Theta_{\ell_i}) A_{0i}(\tau;\Theta_Y,\Theta_{\ell_i})
\end{aligned}$$

where $\Gamma_{\cdot}(\cdot)$, and $A_{\cdot i}(\tau;\Theta_Y,\Theta_{\ell_i})$ are deterministic functions of the structural parameters Θ_Y and Θ_{ℓ_i}. Their functional form is given in Proposition 2 and Lemma 1 respectively. ■

C. The Price of Risk. To gain further intuition on the properties of the term structure of interest rates, it is instructive to study the properties of the price of risk that are implied by the model. In the term structure literature, the issue of flexible specifications of the price of risk has received considerable attention. Let ξ_t be the stochastic discount factor, $\xi_t = e^{-\rho t} u'(c_t^*, M_t^*)$, so that the discounted value of any tradable asset is a martingale, $\xi_t B_t^\tau = E_t(\xi_s B_s^\tau)$. The diffusion process of the stochastic discount factor must be of the form

$$\frac{d\xi_t}{\xi_t} = -r_t dt - \Lambda_t dW_t$$

with $\Lambda_t dW_t$ being the price of risk. From the equilibrium solution of the structural model, we obtain

$$-\Lambda_t dW_t = -\left[\sigma_{cy}(1-\frac{\lambda}{Y_t}) + \sigma_c\right] dW_t^c - \sum_{i=1}^{2}\left[\sigma_{\ell_i y}(1-\frac{\lambda}{Y_t})\right] dW_t^{\ell_i} \tag{9}$$

so that the price of risk of unexpected consumption innovations is $\left[\sigma_{cy}(1-\frac{\lambda}{Y_t}) + \sigma_c\right]$ and the price of risk of the monetary risk factor is $\left[\sigma_{\ell_1 y}(1-\frac{\lambda}{Y_t})\right]$. Both prices of risk are state dependent. When the consumption level is close to the habit level H_t, Y_t is low and the local curvature of the utility function is high. The higher implied risk aversion generates a higher price of risk which affects expected returns independently of interest rate volatility. To see this, let us use Ito's rule to obtain the unexpected innovation in the nominal interest rate $dR_t - E_t dR_t$. We obtain:

$$R_t(1-\frac{\lambda}{Y_t})\sigma_y dW_t^y + R_t\sigma_L dW_t^L + R_t\Xi_t \;\left[(\Gamma_1+\Gamma_3 L_t)(Y_t-\lambda)\sigma_y - (\Gamma_2+\Gamma_3 Y_t)L_t\sigma_L dW_t^L\right] \tag{10}$$

$$\text{with} \qquad \Xi_t = \Gamma_1 Y_t + \Gamma_2 L_t + \Gamma_3 L_t Y_t + \Gamma_0,$$

A comparison of (9) and (10) show that the price of risk is not a constant multiple of interest rates volatility. Similar to Duffee (2002) and Dai and Singleton (2002), we find that such a property is important to explain, at the same time, the joint empirical properties of first and second conditional moments of the term structure and expected bond returns.

III. The Term Structure of Index-Linked Bonds

The term structure of real bonds prices can be obtained by solving $B(t,\tau) = e^{-\rho\tau} E_t\left[\frac{u_x(X_{t+\tau},H_{t+\tau})\frac{\partial X_{t+\tau}}{\partial C_{t+\tau}}}{u_x(X_t,H_t)\frac{\partial X_t}{\partial C_t}}\right] = e^{-\rho\tau} E_t\left[\frac{C_t}{Y_t}\frac{Y_{t+\tau}}{C_{t+\tau}}\right]$. Since $d(1/C_t) = -(\mu_c - \sigma_c^2)dt - \sigma_c dW_t^c$, it is straightforward to use Lemma 1 to solve for the conditional expectation $E_t\left[\frac{Y_{t+\tau}}{C_{t+\tau}}\right]$. Let $\xi_{1t} = \frac{1}{C_t}$ and $\xi_{2t} = Y_t$ and consistent with the notation of Lemma 1 let Θ_y and $\Theta_{1/c}$ be the four-dimensional vector of parameters of the two diffusion processes for dY_t and $d(1/C_t)$.[19] Since the inverse of the consumption is a geometric Brownian motion, then $\theta_{\xi_1} = \lambda_{\xi_1} = 0$ and so that we obtain that $A_0 = A_y = 0$ in Lemma 1. Thus, we obtain the following result:

Theorem 2 (The Term Structure of Index-Linked Bonds) *The price $B(t,\tau)$ of an index-linked bond with time to maturity τ is equal to*

$$B(t,\tau) = e^{-\rho\tau}\left[A_{Y/c}(\tau;\Theta_Y,\Theta_{1/c}) + A_{1/c}(\tau;\Theta_Y,\Theta_{1/c})\frac{1}{Y_t}\right]$$

[19] Thus, $\Theta_y = [k_y,\, \theta_y,\, \sigma_y,\, \lambda]$ and $\Theta_{1/c} \equiv [-(\mu_c - \sigma_c^2),\, 0,\, -\sigma_c,\, 0]$.

Index-linked bonds are affected by monetary innovations through the surplus ratio $1/Y_t$. The higher the current consumption relative to the current habit stock is, the higher $1/Y_t$ and therefore the higher the yield to maturity. Since both monetary and endowment shocks affect Y_t, innovations on both state variables affect the real term structure of interest rates.

Instantaneous Real Interest Rate. The instantaneous real interest rate can be obtained either by taking the limit for $\tau \to 0$ of the previous result, or even more simply from the drift of the stochastic discount factor. Letting $\pi_t = e^{-\rho t} u_x(X_t, H_t) \frac{\partial X_t}{\partial C_t}$ and using Ito's Lemma one obtains[20]

$$\frac{d\pi_t}{\pi_t} = \mu(\pi, t)dt + \sigma(\pi, t)' dW_t$$

with

$$\begin{aligned} \mu(\pi, t) &= -\rho - \mu_c + \sigma_c^2 + k_y \frac{(\theta_y - Y_t)}{Y_t} - \frac{(Y_t - \lambda)}{Y_t} \sigma_{cy} \sigma_c \\ -\sigma(\pi, t)' dW_t &= \left[\sigma_{cy} \frac{(Y_t - \lambda)}{Y_t} - \sigma_c \right] dW_t^c + \sigma_{\ell_i y} \frac{(Y_t - \lambda)}{Y_t} dW_t^{\ell_1} + \sigma_{\ell_2 y} \frac{(Y_t - \lambda)}{Y_t} dW_t^{\ell_2} \end{aligned}$$

The real interest rate is given by $r_t = -\mu(\pi, t)$. Thus,

$$r_t = \rho + \mu_c - \sigma_c^2 - k_y \frac{(\theta_y - Y_t)}{Y_t} + \frac{(Y_t - \lambda)}{Y_t} \sigma_{cy} \sigma_c \tag{11}$$

The first two terms stem from the intertemporal consumption smoothing motive, the third term is the precautionary motive and the last two terms are due to the presence of the habit. Given the focus of their paper, Campbell and Cochrane (1999) assume constant real interest rates by using a specific value of λ. In our model, this would be equivalent to assuming that $\lambda = -\frac{k\theta}{\sigma_{y_c} \sigma_c}$. Since the focus of our paper is the dynamics of the term structure of interest rates, we discuss the properties of the model without imposing this restriction.

IV. The Dataset

The empirical results are based on the sample period between January 1960 and December 2000. The data-set consists of three main components: interest rate data, price level data, and money supply data. Interest rate data from January 1960 to February 1991 is obtained from the McCulloch and Kwon data-set[21]. This database contains end-of-month zero-coupon yields and forward curves based on the McCulloch (1975) methodology from one month to 10 years. We extend this dataset using the daily GovPX data-set which provides end-of-day prices for all Treasury securities. The data are based on the transactions done by the primary dealers through five of the six inter-dealer brokers for all active and off-the-run U.S. Treasuries. We keep the methodology for the construction of the zero-coupon yield curve as close as possible to that of McCulloch (1990) and Kwon (1992).

[20] Observing that $d\pi = d\left(e^{-\rho t} \frac{Y_t}{C_t}\right)$ and using Ito's Lemma, we have: $\frac{d\pi}{\pi} = -\rho dt + \left(\frac{dY_t}{Y_t}\right) - \left(\frac{dC_t}{C_t}\right) + \left(\frac{dC_t}{C_t}\right)^2 - \frac{dY_t}{Y_t} \frac{dC_t}{C_t}$

[21] See McCulloch (1990) and Kwon (1992).

We select the last business day of each month and remove all callable bonds from consideration. The number of Treasury securities in the McCulloch data-set increases from slightly over 40 in the 1950s to over 200 in the late 1980s. The average number of Treasury securities in each cross-section of our implied spot curve is 134, ranging from 100 to 200.

Inflation data is based on the Consumer Price Index (CPI) for all urban consumers, which is available from January 1947. The money supply data used in this study is from the official H.6 release of the Federal Reserve Board of Governors. We choose the M2 money stock measure since it includes money market deposit accounts, which can be used for purchasing products and services. This is the closest representation of money in our model. For our purposes, M3 is too wide a measure because it includes instruments that pay significant interest rates which can not be classified as money in our framework. The quarterly per capita consumption series is from the Citibase data-set.

Summary statistics for our sample are given in Table 1. We find that the correlation between M2 growth and the yield on the 5 year zero coupon bond is 15%, supporting the evidence of an important link between monetary shocks and nominal interest rates. Moreover, the monthly correlation between M2 growth and inflation is 16.8%.

V. Econometric Methodology

This section uses the restrictions from Propositions 2 and 1 to estimate the structural model. The term structure is not affine as yields are non linear functions of a set of underlying factors that follow non Gaussian diffusions. Methods for estimation and inference that can be applied to continuous-time non linear Markov models when data is sampled discretely have been proposed, among others, by Lo (1988), Hansen and Scheinkman (1995), Conley, Hansen, Luttmer, and Scheinkman (1997), Ait-Sahalia (1996, 1999), Ait-Sahalia (2002) and Stanton (1997).

With discrete time sampling, Lo (1988) discusses the computation of the likelihood function solving numerically the Fokker-Planck partial differential equation. Since a solution has to be obtained for each maximum likelihood iteration, Ait-Sahalia (1999) proposes a method to approximate the correct transition function using Hermite polynomials in the context of a maximum likelihood estimation. Hansen and Scheinkman (1995) construct generalized method of moments estimators of the unknown parameter vector using the properties of both the original and reverse-time infinitesimal generator.

With respect to the literature, our econometric task is considerably simplified by the fact that we are in the privileged situation of knowing both the stationary density of the states $z(t)$ in closed-form and the conditional moments up to any order. Thus, we can obtain consistent estimators of the unknown parameter vector using GMM. Consistency is achieved for an increasing number of observations ($T \to \infty$), even when these are sampled discretely ($\Delta t > \delta > 0$).

The unobservable states are expressed as functions of the observable economic variables $X(t)$ using the model's solutions. The panel data $X(t)$ consists of both nominal bond yields and macro variables, including real consumption, inflation and monetary holdings. The vector of moment conditions used to estimate the structural parameters refer to the level of nominal bond yields, moments of the distribution of changes in bond yields, the inflation rate, and the growth in monetary holdings.

1. No-arbitrage Cross-sectional Restrictions

The model imposes tight no-arbitrage restrictions among bond with different maturities. When bond prices are observed with a measurement error these restrictions take the form of moment conditions. If the rank of the covariance matrix of the observation errors is lower than full rank by at least the number of state variables, then we can use two yields to maturities $y(t, \tau_A)$ to obtain the two (liquidity) states by inverting the yield equations. The third state variable Y_t is directly observable using the result in Proposition 1a. The remaining yields $y(t, \tau_B)$ must be equal, by no-arbitrage, to a function $\mathcal{F}$ of $y(t, \tau_A)$ plus an observation error.[22] Let

$$h_1(y_t^{\tau}; \theta) = y(t, \tau_B) - \mathcal{F}\left(y(t, \tau_A)\right)$$

The first set of cross-sectional moment conditions is:

$$E\left[h_1(y_t^{\tau}; \theta_0)\right] = 0$$

We select 5 bond yields to construct the moment condition for h_1.

2. Time Series Moment Restrictions

A. Moments from the Stationary Density.

In addition to cross-sectional restrictions on the term structure, we consider a sufficiently large set of moment restrictions generated by both the stationary and conditional density. Let $X(z, t)$ be an observable economic variable, function of the state z_t, such as bond yields, and let $\phi(X)$ be a smooth function of X. Since the stationary density $p(z; \theta)$ of the states is known in closed-form (Inverse Gamma), moments of X can easily be computed by integration $E\phi = \int \phi(X)p(z; \theta)dz$. The estimation of θ can be posed in a standard generalized-method-of-moments framework by setting

$$h_2(X_t; \theta) = \left[\phi(X_t) - E\phi\right] \tag{12}$$

The second set of moment conditions is:

$$E\left[h_2(X_t; \theta_0)\right] = 0$$

We choose ϕ and X so that h_2 is a vector of second moments of five bond yields and of the first two moments of M_t, P_t, and C_t. The total number of moment restrictions is therefore $5 + 6 = 11$.

[22] Clearly, the choice of which bonds are observed with no error is arbitrary (i.e. the generalized inverse of the covariance matrix is not unique). Thus, we decide to select the instruments that imply the lowest pricing errors for the remaining bonds. We find that this is achieved when we select the 3 month and 2 year yield to maturity. The empirical results are not, however, very sensitive to this choice as long as they sufficiently span the maturity structure.

Similarly, when the inverse function of the instrument is not unique, we choose the state vector that minimize the bond pricing errors.

B. Conditional Moment Restrictions.

Let $\phi(z)$ be a smooth function of the state vector $z_t = [Y_t, \ell_t]$. Consider a Taylor expansion of $\phi(z_{t+1})$ at z_t. Taking the conditional expected value:

$$E_t\phi(z_{t+1}) = \phi(z_t) + \sum_{i=1}^{J} \frac{1}{i!}\frac{\partial^i}{\partial z^i}\phi(z_t)E_t\left[z_{t+1} - z_t\right]^i + o\left(E\left[z_{t+1} - z_t\right]^i\right)$$

The moments $E_t\left[z_{t+1}\right]^i$ can be obtained using the results of the following Proposition, so that the conditional expectations of $E_t\phi(z_{t+1})$ can be obtained up to any desired degree of approximation.

Proposition 3 *Given a linear stochastic process z_t satisfying assumption [A2] with entrance boundary λ, the conditional second moment is equal to*[23]

$$E_t\left(z_T^2\right) = e^{-(2k-\sigma^2)T}z_0^2 + 2\left(k\theta - \lambda\sigma^2\right)\left[\frac{\theta}{(2k-\theta)} + \frac{e^{-kT}(z_0-\theta)}{(k-\theta)}\right]$$

All other moments can be obtained recursively by integrating the following differential equation

$$\frac{d}{dt}E_t\left(z_T^n\right) = E_t\left(z_T^{n-2}\right)\left[-nk + \frac{n(n-1)}{2}\sigma^2\right] + E_t\left(z_T^{n-1}\right)\left[nk\theta\right]$$

■

Note that in order for the process to admit a finite unconditional moment of order n, the parameters must satisfy the condition $\left[-2k + (n-1)\sigma^2\right] < 0$. This restriction becomes progressively tighter as the order n increases.

For the estimation, we choose ϕ to be $[y(t+1,\tau),\ y^2(t+1,\tau)]$, so that the parameters are estimated using the first two conditional moments of the observable process. Let

$$\begin{aligned} h_3\left(X_t,\theta\right) &= \left[y_{t+\Delta}^{\tau} - E\left(y_{t+\Delta}^{\tau}|X_t\right)\right]\otimes\xi\left(X_t\right) \\ h_4\left(X_t,\theta\right) &= \left[\left(y_{t+\Delta}^{\tau}\right)^2 - E\left(\left(y_{t+\Delta}^{\tau}\right)^2|X_t\right)\right]\otimes\xi\left(X_t\right) \end{aligned}$$

We select the lagged values of consumption and money growth to build a set of instruments $\xi(X_t)$. Thus, since the number of independent bond yields is five, the total number of restrictions from this set of moments is $5 \times 2 \times 2 = 20$.

The state variables.

An explicit characterization of $Y(t)$, the inverse surplus ratio, as a function of the accumulated consumption and monetary shocks is obtained in Proposition 1a as a solution of [A2]: $Y(t) = \lambda + \omega_t^{-1}\left[(Y_0 - \lambda) + k(\theta - \lambda)\int_0^t \omega_s ds\right]$ with $\omega_t = \exp\{(k+\frac{1}{2}\sigma_y^2)t + \sigma_y(W_t - W_0)\}$, where $\sigma_y dW_t = \sigma_y(\frac{dX}{X} - E\frac{dX}{X}) = \sigma_y((1-\gamma)\sigma_y(\frac{dC_t}{C_t} - \mu_c) + \gamma(\frac{dm_t}{m_t} - \mu_m))$. The other unobservable state variables are ℓ_{it}. They can be obtained by inverting two of the measurement equations to express ℓ_{it} as a function of the

[23] Clearly, the process ℓ_t is a special case of Y_t with boundary $\lambda = 0$.

remaining vector of observable economic variables and θ, $\ell_{it} = \mathcal{F}(X_t, \theta)$. This approach is also used by Chen and Scott (1993) and Duffie and Singleton (1997) for the swap curve. Once we substitute these restrictions on Y_t and ℓ_{it} back into $h_t(X_t, \theta)$, all moment conditions depend exclusively on observable economic variables and structural parameters.

Estimation.

We merge the no-arbitrage cross-sectional restrictions and the moment conditions from both the stationary and conditional distribution in a vector $h_t(X_t; \theta)$, with $h_t = [h_1,\, h_2,\, ...,\, h_4]'$, so that:

$$E\left[h_t\left(X_t; \theta\right)\right] = 0 \tag{13}$$

The total number of moment conditions is therefore $5 + 11 + 20 = 36$. Since the model has 19 parameters, the model is overidentified and the number of degrees of freedom is 17. One can obtain a consistent estimator of the vector of structural parameters by minimizing with respect to θ the quadratic criterion $J_T(X_t; \theta)$, based on the sample counterpart of the previous moments:

$$J_T(X_t; \theta) = \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} h_t\left(X_t; \theta\right)\right]' W_T^{-1} \left[\frac{1}{\sqrt{T}} \sum_{t=1}^{T} h_t\left(X_t; \theta\right)\right] \tag{14}$$

Under the null hypothesis that the model is correctly specified, $J_T(X_t; \theta) \to \chi^2_{(17)}$. This asymptotic distribution can be used to construct test statistics for the overidentifying restrictions of the model. It is important to notice that the condition to obtain consistent estimators for θ depend on the number of observation $T \to \infty$. They do not require that the sampling frequency converge to 0. This property differs from other estimation techniques for non-linear continuous-time models. The reason is simple. Even if the model is non-linear, we can identify the unconditional and conditional moment conditions even for a strictly positive sampling frequency $\Delta t > 0$. Thus, for our model consistency in the estimation does not require continuously observed sample processes. The weighting matrix W_T is the Newey-West heteroskedastic and autocorrelation consistent estimator of the covariance matrix, namely $W_T = \Gamma_0 + \sum_{i=1}^{q} \left(1 - \frac{i}{q+1}\right) \left(\Gamma_i + \Gamma_i'\right)$ with $\Gamma_i = \sum_{t=i+1}^{T} \left(h_t - h\right)\left(h_{t-i} - h\right)'$ and $h = \frac{1}{T} \sum_{t=1}^{T} h_t$.

Small Sample Properties. Since the model is non-linear and most of the test statistics rely on asymptotic results, we check their small sample properties. To do this, we select a parameter configuration θ_0 and simulate the data-generating process $\{\tilde{X}_{j1}(\theta_0), ..., \tilde{X}_{jT}(\theta_0)\}_{j=1}^{N}$. The assumed parameter configuration corresponds to the sample estimates of the model. Then, we estimate $\hat{\theta}$ using the moment conditions $Eh_t(\tilde{X}_t, \theta) = 0$ computed for each simulation by minimizing the quadratic criterion (14). We obtain $\{\hat{\theta}_1,\, ...,\hat{\theta}_N\}$. Under the null hypothesis that the estimators are unbiased in small samples, $lim_{N \to \infty} \frac{1}{N} \sum_{j=1}^{N} \hat{\theta}_j \to \theta_0$, for finite T. Then, we compute the empirical rejection ratio of the overidentifying no-arbitrage restrictions. We find that the empirical rejection frequency is 6.1%, as opposed to the theoretical value of 5%. The test slightly over-rejects the model in small samples: the asymptotic tests tend to be biased on the conservative side when used in small samples.

VI. Empirical Results

This section summarizes the results based on the estimation of a three factor model.

1. *Joint Test of the Overidentifying Restrictions.* Table 2 reports estimates of the parameters and their corresponding standard errors.[24] It is easy to verify that $2k_y > \sigma_y^2$, $\theta_y > \lambda > 1$, and $k_y > 0$ so that Y_t satisfies the conditions $[C1]$ and $[C2]$ for Y_t to be stationary and square-integrable. The equivalent conditions for the liquidity factors ℓ_{1t} and ℓ_{2t} are also satisfied. Moreover, the two conditions for the existence of a monetary equilibrium require $-k_y - k_{\ell_i} - (\sigma_{cy}\sigma_{\ell_i}\rho + \sigma_{\ell_i y}\sigma_{\ell_i}) < 0$ for $i = 1, 2$. These conditions are satisfied and, at the estimated parameter values, are equal to -0.35 and -0.22 respectively. Finally, a sufficient condition for $Cov(dH, dC) > 0$ is that $\sigma_y \geq \frac{v\sigma_c^2 - v\gamma c^* + \lambda\psi_c v^2}{\sigma_c^2 - \gamma c^* - v\psi_c}$ where $\psi_c = \sigma_{yc}\sigma_c - \sigma_{y\ell_i}\sigma_c\rho_{\ell_i,c}$ and $v = \frac{\sigma_y\psi_c + \sigma_c^2 - \gamma c^*}{2\lambda\psi_c}$ with $c^* = \max_i \sigma_{\ell_i}\sigma_c\rho_{c,\ell_i}$. The condition is satisfied.

To assess the model's overall goodness of fit we first run a joint GMM test on the model overidentifying restrictions. The test is based on all moment conditions. Under the null hypothesis that the model is correctly specified, the maximum $J_T(X_t, \hat{\theta})$ statistics is asymptotically chi-squared distributed with 17 degree of freedom:[25]

$$J_T(\hat{Y}, \hat{g}, \hat{\theta}) = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h\left(X_t, \hat{\theta}\right)\right]' W_T^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h\left(X_t; \hat{\theta}\right)\right] \sim \chi^2_{(17)}$$

The model implied p-value, based on all overidentifying moment conditions, is 36%. The model is not rejected.

Campbell and Cochrane (1999) and Menzly, Santos, and Veronesi (2004) choose a parameter configuration to match a set of unconditional moments of real equity returns. Some of these parameter values are different from our estimates. For instance, MSV choose $k_y = 0.16$ and $\sigma_y^2 = 0.62$, which are both higher than the values that we estimate.[26] On the one hand, since they calibrate a number of moments that is smaller than the number of their free parameters, the difference in parameter values is not surprising. Different parameter configurations imply the same moments in their exercise. Moreover, since the stationary distribution p of a diffusion process satisfies $\frac{d}{dx}\left[p(x)\sigma^2(x)\right] = 2\mu(x)p(x)$, the drift and volatility parameters μ and σ are identified only up to a scale factor. Thus, MSV could scale the choice of k and σ_y^2 by a factor ten without affecting the unconditional moments of the observable processes.[27] Clearly, however, even if the two parameter configurations produce similar *unconditional* moments, they generally imply different *conditional* moments and since bond prices are conditional moments of the stochastic discount factor, this difference plays a crucial role in our study. For this reason, in our empirical analysis and tests we will pay special attention to the conditional moment properties of interest rates. Moreover, their parameter choice violates condition

[24] The asymptotic covariance matrix of the parameters is based on the outer-product of the Jacobian of the log-likelihood function.

[25] See Hansen (1982).

[26] We estimate $k_y = 0.022$ and the model-implied σ_y^2 is equal to 0.07.

[27] To see this, note that the statistical distribution of Y_t is an Inverse-Gamma with parameters $-2(\frac{k}{\sigma^2} - 1)$ and $2\frac{k\theta}{\sigma^2}$.

[$C2$]. Thus, in the last section of our paper we investigate the equity risk premium that is implied by a model with habit persistence that produces realistic interest rates implications.

2. *Nominal Yield Curve.* The average fitting errors of the yield curve range between 27 and 60 basis points, see Table 4, Panel A. Since in the estimation the same three factors are required to fit the consumption process, the inflation rate, the money supply growth rate and the second moments of the yields, we also report the results when the model is estimated exclusively using yield curve moment conditions. The overall mean absolute error of the yield curve drops to 21 basis points, see Table 4, Panel B.

We find that the yield curve is higher during low surplus consumption (high habit stock) periods, i.e. recessions. In these states, investors' marginal utility is high implying a higher consumption demand. Figure 3 illustrates the relationship between the yield curve and the surplus consumption ratio for different levels of the money growth rate. This relationship is monotone both at high and low monetary growth levels. The closer the consumption to the habit, the higher the level of the yield curve. A one standard deviation change in the surplus consumption ratio induces a 60 basis points change in the level of the yield curve. This is about the same average yield difference between a three and eight year bond. Moreover, we find that the yield curve is steeper during periods of high monetary growth. For instance, when the money growth rate is one standard deviation above its long-run mean, the slope of the yield curve is 370 basis points. The average slope is 40 basis points. This can be due to the following. First, a high money growth rate anticipates high future inflation rates, which translates to higher current long-term yields. Second, since real monetary holdings are mean reverting, high current values, which are expected to revert to its long-term mean, reduce the surplus consumption ratio and therefore increase current future long-term yields.

3. *Real Interest Rates and Consumption.* We compare the behavior of the model-implied and empirical real yields using index-linked bonds. We use the approach suggested by Green and Odegaard (1997) and Buraschi and Jiltsov (2005) to control for the tax implications of the inflation-adjustment of the principal amount. We focus this part of the analysis on the sample period after January 29, 1997, which is the date of the first Tips issue. The model-implied mean and standard deviation of the 3 month real yield are equal to 2.13% and 1.13% respectively. The corresponding empirical values are equal to 2.88% and 0.47%. The model can reproduce the first moment of the 3 month real interest rate, but it clearly underestimates its volatility. The p-value of a GMM test for the joint null hypothesis that the first two moments are correctly specified is 5%. However, the null hypothesis that the second moment of the 3 month real yield is correctly specified is rejected.

The model fares better with long-term real yields. The model-implied mean and standard deviation of the 10 year real yield are equal to 2.51% and 0.44% respectively. The corresponding empirical values are equal to 2.84% and 0.34%. The p-value of a GMM test on the joint null hypothesis that the first two moments are correctly specified is 12%.

The model-implied mean and standard deviation of the consumption growth rate are equal to

2.4% and 1.7% respectively. The corresponding empirical values are equal to 1.9% and 1.9% (Table 1). The model can fit bond prices without an excessively volatile consumption process. We run a joint test for the null hypothesis that the model-implied first two moments correspond to the empirical values and find that the null hypothesis is not rejected with a p-value of 0.11.

4. *Inflation.* Figure 2 plots the model-implied and realized inflation rate. The model-implied mean and standard deviation of the inflation rate are equal to 4.03% and 2.33% respectively. The corresponding empirical values are equal to 4.70% and 3.23% (Table 1). The p-value of a GMM test for the null hypothesis that the model-implied first two moments are equal to their empirical counterparts is 0.25. The model-implied inflation forecasting errors are 29, 57, and 115 basis points at 3, 6 and 12 month horizons respectively (Table 6). For the sake of comparison, we compute the forecasting errors implied by an exogenous ARMA(1,1) specification and find that the corresponding values are 28, 53, and 110 basis points.

We run an orthogonality test on the model prediction errors to test the inflation process goodness of fit. Let $E_t\left(\pi_{t+1}\right| I_t)$ be the model-implied expected inflation rate. If the model is correctly specified, the prediction errors should be orthogonal to any function of x_t, measurable with respect to I_t. If the model is not correctly specified, some function of the explanatory variable $\phi\left(x_t\right)$ would improve the model forecasts, i.e. $E_t\left(\pi_{t+1}\right| I_t) + \theta'\phi\left(x_t\right)$. Consider the inflation forecast error $u_{t+1} = \pi_{t+1} - E_t\left(\pi_{t+1}\right| I_t) - \theta'\phi\left(x_t\right)$, and the null hypothesis $H_0 : \theta = 0$. Define $h\left(x_t, \theta\right)$ as a function of the prediction errors

$$h\left(x_t, \theta\right) = \begin{bmatrix} u_{t+1}\left(\theta\right) \\ u_{t+1}\left(\theta\right) \otimes \left[\xi\left(x_t\right)\right] \end{bmatrix}$$

Under the null hypothesis that the model is correctly specified $\theta_0 = 0$ and $Eh(x_t, \theta_0) = 0$. Using a standard GMM approach we observe that, under the null hypothesis, the following d_T-statistics is distributed as a χ^2

$$d_T = T \cdot \left[h_T(x_t, \theta_0)' W_T^{-1} h_T\left(x_t, \theta_0\right) - h_T\left(x_t, \theta\right)' W_T^{-1} h_T\left(x_t, \theta\right)\right]$$

where θ_0 is the parameter value restricted to be equal to zero and θ is the unrestricted parameter value, with $h_T\left(x_t, \theta\right) = \sum_{t=1}^{T} h_t\left(x_t, \theta\right)$.

The results are reported in Table 5. We find that the p-value of the orthogonality tests are 0.066, 0.156 and 0.374 at 3, 6 month and 1 year horizons respectively. Thus, we do not reject the null hypothesis that the inflation prediction errors are orthogonal to $\phi(x_t)$.

5. *The Term Structure of Second Moments.* We now turn to explore the extent to which the model can reproduce the time variation of the conditional second moments of the term structure of interest rates once we fix the parameter at their estimated values.[28] Let ϕ_t^n be the *Model-Implied Conditional Second Moment,* which is obtained by solving in closed form $E_t\left[\Delta y_{t+\Delta t}^n\right]^2$ using the

[28] Dai and Singleton (2001), Backus, Telmer and Wu (1999) document the trade-off of traditional models in explaining the first and second moments of interest rates.

model's structural restrictions. We run a regression the regression[29]

$$\left(\Delta y_{t+\Delta t}^{n}\right)^{2}=\alpha+\beta\times\phi_{t}^{n}+\varepsilon_{t+\Delta t}$$

We test the null hypothesis that $H_0 : \alpha = 0$ and $H_0 : \beta = 1$. The results are summarized in Table 9, Panel B. We fail to reject the null hypothesis that $H_0 : \alpha = 0$ for maturities above 3 years. Moreover, we do not the null hypothesis that $H_0 : \beta = 1$ for any maturities. The R^2 of the regressions range from 11% for the 10-year yield to maturity to 26% for the 3-month yield to maturity. These results can be compared with Duarte (2000). He runs a similar test both on a three factor CIR model and on his own model with a flexible specification for the price of risk. Based on the 1983-1998 sample period,[30] he rejects the null hypothesis that $H_0 : \beta = 1$ and reports a R^2 that ranges between 7% and 15% for the CIR model. The persistence of the habit stock as a state variable clearly helps explain the yield change volatility at the short end of the term structure as the R^2 are larger than those reported in Duarte (2000). We find it, however, less helpful in the long end of the term structure.

We also compute an asymptotic GMM test of the model's ability to reproduce the conditional second moments of bond yields. For each maturity n, we construct the test statistic $d_T = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right]' W_T^{-1} \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right]$ with $h_{t+\Delta t} = \left(\Delta y_{t+\Delta t}^{n}\right)^2 - \phi_t^n$. This statistic is distributed as a Chi-square with one degree of freedom. The results are reported in Table 9, Panel A. The *p-value* of the test ranges from 37% for the 3-month yield to 18% for the 10-year yield to maturity. A joint test based on all maturities, gives a p-value equal to 21%. We conclude that the model can produce sufficient time-variation in the conditional second moments.

VII. Nonlinear Interest Rates

Our model implies a nonlinear short-term interest rate process. To investigate the nonlinearity of interest rates, Conley, Hansen, Luttmer, and Scheinkman (1997) study the pull function $\wp(r)$, which is defined as the conditional probability that the process r_t reaches the value $r + \varepsilon$ before $r - \varepsilon$, if initialized at $r_0 = r$. Formally, let T_ε be the local hitting time $T_\varepsilon = \inf(t \geq 0; r_t = r + \varepsilon)$, then $\wp(r)$ is defined as $\wp(r) = \Pr\{T_{r+\varepsilon} < T_{r-\varepsilon} | r_0 = r\}$. In practice, $\wp(r)$ is computed as

$$\wp(r)=\frac{\mathcal{S}(r)-\mathcal{S}(r-\varepsilon)}{\mathcal{S}(r+\varepsilon)-\mathcal{S}(r-\varepsilon)}$$

where $\mathcal{S}(y) = \int^y s(x)dx$ with $s(x)$ being the scale function of the interest rate process, i.e. $s(x) \equiv \exp\left[-\int^x \frac{2\mu_r(v)}{\sigma_r^2(v)}dv\right]$. Solving the previous equation, $\wp(r) = \frac{1}{2} + \frac{\mu_r(r)}{2\sigma_r^2(r)}\varepsilon + \mathrm{o}(\varepsilon)$, where $\mu_r(r)$ and $\sigma_r(r)$ are the drift and local volatility of the interest rate process. Thus, $\wp(r)$ has a simple and intuitive interpretation: it is a conditional measure of mean reversion (scaled by twice the local variance). Conley, Hansen, Luttmer, and Scheinkman (1997) estimate $\wp(r)$ using the properties of subordinated

[29] The spirit of this regression analysis is similar to the one outlined in Duarte (2000). Our regression, however, applies to the volatility of the *changes* in bond yields, as opposed to that of the *level* of bond yields

[30] This time span does not include the period of high interest rate volatility, which is more difficult to predict.

diffusions for alternative interest rate specifications and suggest a test statistic to compare the model-implied $\wp(r)$ with its sample counterpart. They find statistical evidence of non-linearity in $\mu_r(r)$. In what follows, we use their methodology to investigate the difference between the model-implied pull function, computed at the estimated structural parameter values, and its empirical counterpart, estimated using semi-parametric methods.

Figure 1

To estimate the model-free $\wp(r)$, we assume a flexible polynomial specification of the local volatility $\sigma(r) = \sum_{i=0}^{\gamma} \sigma_i r^i$ and follow Conley, Hansen, Luttmer, and Scheinkman (1997) to estimate the drift $\mu(r)$. The results are shown in Figure 1. The pull function is positive for $r_t \leq 5.4\%$ and it goes above 10 for interest rate values below 4%. The model-implied $\wp(r)$ has a similar behavior and it remains inside the confidence bounds for most part of the support. We run a test of the null hypothesis that the pull function implied by the habit model is equal to the empirical one and find that the p-value is equal to 0.43. We do not reject the null hypothesis that the model is correctly specified. Second, we impose the restriction that the interest rate drift $\mu_r(r)$ is linear and recompute the pull function. The p-value is now equal to 0.02 and we find the pull function outside the confidence bounds for large sections of the support. We conclude that the type of non-linearity in interest rates implied by the model is consistent with the data in our sample period.

VIII. The Lead-lag Relation between Interest Rates, Consumption and Money

The lead-lag correlations between real interest rates, consumption and output are important properties in the real business cycle literature. Fiorito and Kollintzas (1994) and Chari, Christiano, and Eichenbaum (1995) report that real and nominal interest rates are positively (negatively) correlated with past (future) detrended output. Wachter (2005) focuses on a real interest rate model and finds that future real interest rates are predicted by the past average consumption. King and Watson (1996) and Boldrin, Christiano, and Fisher (2001) view the lead-lag relationship between consumption, output and interest rates as an important challenge to standard models with time-separable preferences, such as Cox, Ingersoll, and Ross (1985). They argue that the original model should be extended to include more general preferences and/or monetary policy shocks. The empirical evidence reported in these studies is consistent with the predictions of our model. To see this, observe that equation (11) implies that $\partial r_t/\partial Y_t > 0$. The relationship between Y_t and the nominal interest rate R_t given by equation (8) is more complicated since it involves the indirect effect of Y_t on the inflation rate. Nonetheless, it is possible to show that $\partial R_t/\partial Y_t > 0$. This implies that an increase in lagged consumption increases the habit stock, thus reducing S_t and increasing Y_t, with the resulting effect of increasing interest rates. Since money reduces transaction costs, lagged positive innovations in the money supply have a similar effect.

We investigate these relationships by regressing the short term real and nominal interest rates onto the Y_t process. The exact solution for Y_t is given in Proposition 1 as $Y(t) = \lambda + (Y_0 - \lambda)e^{\phi t}$

$\exp\{-\sigma_y(W_t - W_0)\} + k(\theta - \lambda)\int_0^t e^{\phi(t-s)} \exp\{-\sigma_y(W_t - W_s)\}ds$ where $\phi = \exp(-(k_y + \frac{1}{2}\sigma_y^2))$. We construct an empirical time-series for Y_t as a function of lagged observed innovations in consumption and monetary holdings by discretizing the previous solution. We obtain $\tilde{Y}_t$. Then, we study the following regressions:

$$r_{t+1} = \alpha_1 - \beta_1 \tilde{Y}_t(\phi, \gamma) + \varepsilon_{t+1}$$
$$R_{t+1} = \alpha_2 - \beta_2 \tilde{Y}_t(\phi, \gamma) + \eta_{t+1}$$

An empirical prediction of the model is that $H_0 : \beta_1 < 0$ and $H_0 : \beta_2 < 0$. We present the results both for a full grid of parameter configurations (ϕ, γ), to check their robustness against a range of parameter values, and for the parameter values obtained by estimating the structural model.

Real Interest Rate. The regression results are given in Table 3, Panel A. At the estimated parameter values of the structural economy, $k_y = 0.02$ and $\sigma_y = 0.027$ hence $\phi = 0.96$ and $\gamma = 0.51$. With this parameter configuration, we obtain $\beta_1 = -0.17$ and $R^2 = 15\%$. The slope coefficient is negative and significantly different from zero as predicted by the model. The result is robust to any parameter configuration (ϕ, γ) with $\phi > 0.90$. Moreover, we find that the result, both in terms of R^2 and statistical significance of the slope coefficient, is not very sensitive to γ.

Nominal Interest Rate. The regression results for the nominal interest rates are given in Table 3, Panel B. At the estimated parameter values of the structural economy, the slope coefficient is -0.32 with a t-statistic equal to -7.5. The R^2 is 34%. Can we use a real term structure model, similar to Cox, Ingersoll, and Ross (1985) and Wachter (2005), to explain the dynamics of the nominal yield curve? The simple answer is "no". Nominal interest rates are strongly influenced by monetary fundamentals. To see this, notice that when $\gamma = 0$ (see first column in Table 3, Panel A and B), the $\tilde{Y}_t$ process has very limited explanatory power of the interest rate process. The R^2 hardly reaches 1%. For this parameter value, money does not provide any transaction service and is not held in equilibrium. The model economy is real. When the real monetary aggregate is included, however, for parameter values above $\gamma = 0.40$ the R^2 exceeds 30%. This evidence is very useful since it suggests that a real term structure model cannot, on its own, explain the dynamics of the nominal yield curve. Moreover, the persistent effect generated by the habit stock help explain the relationship between interest rates and lagged consumption.

IX. The Expectations Hypothesis

The expectations hypothesis of interest rates, hereafter EH, is one of the most debated and studied financial relationships. If the EH were correct, at least in a statistical sense, one could use implied forward rates to obtain a simple unbiased proxy for the expected future spot rate. Most of the EH empirical evidence, however, indicates its rejection. This empirical evidence suggests the existence of a time-varying risk premium.[31] The extent and importance of the deviations are such that the

[31] Bekaert, Hodrick, and Marchall (2001) suggest a different explanation and investigate whether the violation of the EH in U.S. data may be the result of a peso problem, in which a high-interest-rate regime occurred less frequently in

size of the bias with respect to the EH predictions has been used in the empirical literature as a separate moment condition to build model specification tests. Such a metric is directly related to the properties of the conditional second moments of interest rates.[32]

We explore the extent to which monetary and habit factors can explain the time variation of the forward risk premium. Let the forward interest rate at time t for an instantaneous forward contract beginning at time $T = t + \tau$ be $f(t, T)$. The instantaneous forward rate is $-\frac{\partial}{\partial T} ln\, N(t, \tau)$. Taking the derivative of the log-price of the bond we obtain

$$f(t,\tau) = -\frac{\partial}{\partial \tau} lnN(t,\tau) = \rho - \sum_{i=1}^{2} \frac{\frac{\partial}{\partial \tau}\Lambda_{Yi}(\tau)\, Y_t + \frac{\partial}{\partial \tau}\Lambda_{\ell_i}(\tau)\, \ell_{it} + \frac{\partial}{\partial \tau}\Lambda_{\ell_i Y}(\tau)\, \ell_{it} Y_t + \frac{\partial}{\partial \tau}\Lambda_{0i}(\tau)}{(\Lambda_{Yi}(\tau)\, Y_t + \Lambda_{\ell_i}(\tau)\, \ell_{it} + \Lambda_{\ell_i Y}(\tau)\, \ell_{it} Y_t + \Lambda_{0i}(\tau))}$$

The nominal rate is given by

$$R_t = \frac{Y_t(\ell_{1t} + \ell_{2t})}{\left[\sum_{i=1}^{2} (\Gamma_{Yi} Y_t + \Gamma_{\ell_i} \ell_{it} + \Gamma_{\ell_i Y} \ell_{it} Y_t + \Gamma_{0i})\right]}$$

The EH postulates that the difference between the forward rate $f(t, \tau)$ and the expected future spot interest rate $E_t(R_{t+\tau})$ is constant:

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha$$

However, in our model the forward risk premium depends on the levels of the nominal risk factors ℓ_{it} and the consumption surplus S_t. Thus, the model may provide a possible explanation for the EH violation. The nonlinear dependence of the nominal rate on the model factors does not allow analytical solutions for $E_t(R_{t+\tau})$. However, we can compute $E_t(R_{t+\tau})$ numerically using standard methods. To assess the forward premium's time variation and dependence on the monetary and habit factor, we regress the forward premium onto the monetary factors ℓ_{it} and the inverse consumption surplus ratio Y_t.

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha + \beta_{1i}\ell_{it} + \beta_2 Y_t + \varepsilon_t$$

We then assess (a) whether the forward premium is constant by testing $H_0 : \beta_{1i} = \beta_2 = 0$, and (b) the relative contribution of the nominal and habit factors to the forward risk premium time variation. The results are given in Table 7. To gain insight into the reasons for the strong rejection of the EH, we decompose the total time-variation of the forward premium into two components: the nominal and habit factors. We find that at a three month horizon, 87% of the volatility of the forward premium is due to the nominal factors and 13% is due to the habit factor. At a one year horizon, the importance of the habit factor increases to 62%. Especially at short horizons, the rejection of the EH is mainly due to the time variation in the risk premium of nominal shocks.

the U.S. sample than was rationally anticipated.

[32] The Campbell and Shiller (1991) tests of the EH focus on the slope coefficient properties of a regression of future yield changes onto the current slope of the term structure. Since such a slope coefficient is a ratio between a conditional covariance and a conditional variance, the ability of a model to reproduce the empirical violations of the EH are a function of the ability of the model to reproduce the empirical conditional second moments of interest rates.

Campbell and Shiller regressions. Campbell and Shiller (1991) regress the change in the constant time-of-maturity yield onto the current slope of the yield curve. To determine the time variation of the forward risk premium, we explore the following question: "*If we generate term structure data using our structural model and run Campbell-Shiller (1991) type regressions, do we find the same pattern in the slope coefficients?*". Let y_t^τ be the yield at time t on a Treasury bond with maturity $t+\tau$. Given a sampling frequency equal to m units of time, consider the following regression

$$y_{t+m}^{n-m} - y_t^n = \alpha + \beta \underbrace{\left[\left(\frac{m}{n-m} \right) (y_t^n - y_t^m) \right]}_{S_t^{n,m}} + \varepsilon_t$$

The Expectations Hypothesis suggests that $\beta = 1$. Campbell and Shiller (1991) test this hypothesis and find not only that the slope coefficient is different from one, but is also negative. Thus, an increase in the slope of the term structure is followed by a decrease in long term yields. At a 7 year maturity horizon, the empirical slope coefficient is about -3. These results have been proven to be robust. In fact, a large sample of empirical literature now considers the slope coefficients of such regressions as moment conditions in themselves, from which to build tests of model specification.

We generate data using the model and run a Campbell-Shiller type regression for each simulated run. Then, we test whether the model-implied moments are significantly different from the empirical ones. Since the slope coefficient is the ratio of the covariance between the left and right hand side variable over the variance of the right hand side variable, we can design a GMM test using the slope coefficient as a moment condition. Let $\beta(\hat{\Theta})$ be the model-implied slope coefficient of the Campbell-Shiller regressions, given the set of estimated structural parameters $\hat{\Theta}$. Let $\hat{\beta}$ be the empirical slope coefficient obtained by re-running the Campbell-Shiller regression on the data-set. We simulate the model 1000 times and test whether the simulated moments are close to those obtained from the data.

Table 8 summarizes the results. Both the absolute levels of the slope coefficient and their patterns as a function of maturity closely mirror the Campbell and Shiller results. The slope coefficient at a one year horizon implied by the structural model is -0.02 compared to a value of -0.58 obtained empirically. As the horizon increases, the slope coefficients decrease as in Campbell and Shiller. At a 7 year horizon, the implied slope regression coefficient is -1.86 while the empirical Campbell-Shiller value is -2.13. We run Chi-square tests of the null hypothesis that $H0 : \beta(\hat{\Theta}) = 1$ and that the two sets of coefficients are equal, i.e. $H0 : \beta(\hat{\Theta}) = \hat{\beta}_{cs}$. We find that the implied values of the Campbell-Shiller regression coefficients strongly reject the expectations hypothesis at any confidence level. Additionally, the implied slope coefficients $\beta(\hat{\Theta})$ are not significantly different from those obtained by Campbell and Shiller, with p-values ranging between 0.07 and 0.44. The p-value of a joint test for all maturities is 0.14.

Why does the model succeed? Most traditional affine reduced-form models of the term structure assume that the market risk premium is proportional to the volatility of the latent factors. Duffee (2002), Duarte (2000), Dai and Singleton (2000), Backus, Telmer, and Wu (1999) show that this assumption is an important limiting feature. Our model differs in that the equilibrium price of risk is not directly proportional to the local volatility of the pricing factors. Moreover, it can change sign, allowing for more flexibility in the expected returns dynamics. The properties of the risk premium are such that the volatility of the returns can be high without necessarily implying a high expected bond return.

X. The Inflation Risk Premium

An additional important implication of the model regards the yield spread between nominal and index-linked bonds. Several articles in the empirical literature find that this spread exceeds the expected inflation rate, which indicates the existence of a large inflation risk premium. Using U.K. data, Risa (2001) estimates that the average inflation risk premium has been above 100 basis points; Tristani and Vestin (2005) find that in the Euro-area it has fluctuated over time between 20 and 100 basis points. Ang and Bekaert (2005) obtain similar values using U.S. Treasury data.[33]

This empirical evidence cannot be easily reconciled with standard asset pricing models. Lucas (2000), for instance, calibrates a cash-in-advance model and finds that in low inflation regimes the welfare costs of inflation are extremely small. Buraschi and Jiltsov (2005) show that in order for a structural model to support realistic inflation risk premia one needs to assume fiscal distortions and a large effective marginal tax rate.

In our model, the violation of the Fisher relationship between nominal and real interest rates is an equilibrium feature of the model and the inflation risk premium is positive even in the absence of fiscal distortions. The violation is generated by the fact that inflation reduces the real value of the monetary holdings that are used to finance the optimal consumption plan. With respect to traditional models with time-separable preferences, however, habit formation increases the volatility of the stochastic discount factor and its covariance with inflation. Thus, habit formation can potentially generate inflation risk premia that are consistent with those observed empirically even abstracting from fiscal distortions. To explore this issue, in what follows we fix the estimated values of the structural parameters and compute the model-implied inflation risk premia.

The inflation risk premium is defined as $COV_t\left[e^{-\rho\tau}\frac{u'(X_{t+\tau}-H_{t+\tau})}{u'_c(X_t-H_t)}; \frac{p^*_t}{p^*_{t+\tau}}\right]$, which is also equal to the difference between the value of a nominal bond $N(t,t)$ and the value of a real bond adjusted by the expected change in the general price level $B(t,\tau)\times E_t\left[\frac{P_t}{P_{t+\tau}}\right]$. When we use the estimated values of the structural parameters, we find that the average inflation rate risk premium is 44 basis points at a 8 year horizon. It ranges over time between 20 and 90 basis points. The average term structure of this premium, calculated over the entire sample, is presented in Panel D of Figure 4. The term structure is upward sloping. In the short run, inflation is influenced by the short-term history of monetary policy. In the longer run, a greater inflation uncertainty and longer bond duration translates into a higher premium on nominal bonds. Moreover, duration amplifies the price impact

[33] An additional reason for the growing interest of the literature in the estimation of the inflation risk premium is that expected inflation rates are often computed by subtracting the inflation risk premium from the yield spread between nominal and indexed-linked bonds.

of inflation on long term bonds so that long nominal bonds require higher risk premia.

Panel B of Figure 4 illustrates the dynamics of the inflation risk premia. During periods of high nominal interest rates and inflation, such as during the 1982 recession, the inflation risk premium increased. Panel C of Figure 4 shows the three dimensional evolution of the inflation risk premium in both time and maturity domains and it suggests that a structural model with habit persistence can support large and time-varying inflation risk premia that are consistent with those found by the empirical literature.[34]

XI. The Equity Premium

What is the trade-off between explaining interest rate properties and the equity risk premium? Traditional models with time separable preferences find it hard to match both the observed equity premium and bond premia. For sensible levels of risk aversion, observed equity excess returns are too high. Moreover, bond excess returns are too small with respect to their volatility.

Stambaugh and Kandel (1991) show that breaking the link between the atemporal risk aversion and the intertemporal rate of substitution allows the model to produce equity risk premium closer to the empirical evidence without worsening the risk free rate puzzle. In what follows, we investigate the magnitude of the trade-off between matching the properties of the term structure of interest rates and matching the level of equity risk premium. Bekaert, Engstrom, and Grenadier (2004) estimate moment conditions implied by an affine model with preference shocks to investigate the trade-off between matching equity and bond risk premia. They find that stochastic risk aversion helps to explain some important features in the data.

Let $W_i(t)$ be a contingent claim (equity) on the dividend $D_i(t)$ and let the dividend be a fraction w_i of the total consumable output, $D_i = w_i C$. Menzly, Santos, and Veronesi (2004) investigate the cross section of equity returns assuming that the dividend consumption ratio w_i is an industry specific stochastic process. We focus on the total wealth portfolio and assume w_i to be a constant. In equilibrium $\sum_i D_i = C$. Using the functional form of the stochastic discount factor, we have

$$\begin{aligned} W(t) &= \frac{1}{u_c'(t)} E_t \left[\int_t^\infty e^{-\rho(u-t)} u_c'(u) D(u) du \right] \\ &= w S_t C_t E_t \left[\int_t^\infty e^{-\rho(u-t)} \frac{1}{S_u} du \right] \end{aligned}$$

Under the assumption that the integral converges, which requires $\rho > 0$, we can apply Fubini's Theorem to invert the order of integration:

$$W(t) = w S_t C_t \left[\int_t^\infty e^{-\rho(u-t)} E_t \left[Y_u \right] du \right]$$

[34] It would be of great interest to extend the dataset to include also U.S. index-linked bonds. Unfortunately, the first issue of these securities occured in 1997, so that the lack of available data limits, thus far, the reliability of a joint estimation of the model.

Since the drift of the process is linear, $E_t[Y_u] = \theta_y + (Y_t - \theta_y)\, e^{-k_y(u-t)}$. Hence

$$\begin{aligned} W(t) &= wC_t \left[\int_t^{\infty} e^{-\rho(u-t)} S_t \left[\theta_y + (Y_t - \theta_y)\, e^{-k_y(u-t)} \right] du \right] = \\ &= wS_tC_t \left[\frac{\theta_y}{\rho} - \frac{\theta_y}{\rho + k_y} \right] + \frac{1}{\rho + k_y} C_t \\ &= wC_t \frac{1}{\rho + k_y} \left(1 + \frac{k_y \theta_y}{\rho} S_t \right) \end{aligned}$$

The risk premium on the total wealth portfolio is thus given by $dR_t^w = \frac{dW(t)+D(t)dt}{W(t)} - r(t)dt$. Let us define $A \equiv \left[\frac{\theta_y}{\rho} - \frac{\theta_y}{\rho + k_y} \right]$ and $B \equiv \frac{1}{\rho + k_y}$ so that $W_i(t) = w_i A S_t C_t + BC_t$. Using Ito's rule it can be shown that

$$dR_t^w = \mu^w dt + \sigma^w dW_t$$

$$\mu^w = \mu_c + \frac{BS_t}{A + BS_t} \left[k_y(1 - \theta_y S_t) + (1 - \lambda S_t)^2 \sigma_y^2 - (1 - \lambda S_t)\sigma_c \left[\sigma_{cy} - \sigma_{l_i y} \rho_{l_i c} \right] \right] - r_t \tag{15}$$

$$\sigma^w dW_t = \sigma_c dW_t^c - \left(\frac{BS_t}{A + BS_t} \right) (1 - \lambda S_t) \sigma_y' dW_t^y \tag{16}$$

The expected equity risk premium is the sum of two terms. The first term is directly proportional to the consumption growth rate. The second term is the product of two terms. The first term increases with the surplus ratio. The second term is non monotone in S_t. For large values of k_y and λ and small values of σ_c, the equity risk premium is decreasing in the surplus ratio S_t. More precisely, for $S_t < \frac{1}{2\lambda^2 \sigma_y^2} \left(k_y \theta_y + 2\lambda \sigma_y^2 \text{ - } \lambda \sigma_c \left(\sigma_{cy} - \rho_{cl_i} \sigma_{yl_i} \right) \right)$, the equity risk premium is decreasing in the surplus ratio S_t. The intuition is simple: the larger the surplus ratio is, the lower the curvature of the utility function is, the lower the indirect risk aversion is and therefore the lower the expected equity premium is. At the estimated values of the structural parameters, which are not based on equity return properties, the implied equity risk premium is 4% and the volatility of the equity risk premium is 5%. This compares with empirical values of 7% and 16% respectively. Since the results suggest the existence of a trade-off between fitting term structure moments and the average equity premium, we re-estimate the model imposing (15) and (16) as additional overidentifying restrictions. In the new estimation, the model is also required to reproduce the unconditional equity risk premium and its volatility. With these additional restrictions, the median yield absolute fitting errors increase to $14bp$, $16bp$, $16bp$, and $39bp$ for bonds with maturity $3m$, $1y$, $5y$, and $10y$ respectively (compared with Table 4, Panel B). The additional equity risk premium restriction increases the average yield curve fitting errors by about $2bp$. We also investigate the effect of these restrictions on the conditional second moment properties of the yield curve by recomputing the Campbell-Shiller linear projection coefficients. When we impose the restrictions on μ^w and σ^w in the estimation, the difference between model-implied and empirical linear projection coefficients increases:

	Joint Test	1	2	3	5	7	10
Empirical β		-0.579	-0.955	-1.238	-1.723	-2.135	-2.621
Model $\beta(\widehat{\theta})$		-0.020	-0.339	-0.652	-1.274	-1.865	-2.492
Restricted Model $\beta(\widehat{\theta})$		-0.015	-0.026	-0.553	-1.097	-1.650	-2.221
p-val for $H_0^{un} : \beta(\widehat{\theta}) = \beta_{CS}$		0.069	0.088	0.129	0.237	0.351	0.440
p-val for $H_0^{re} : \beta(\widehat{\theta}) = \beta_{CS}$		0.069	0.084	0.101	0.180	0.250	0.358
Joint Test $H_0^{re} : \beta(\widehat{\theta}) = \beta_{CS}$	0.11						

The joint test on all maturities produces a p-value equal to 11%. The restricted model is still not rejected. The absolute values of the projection coefficients, however, are lower. The restricted model finds it more difficult to reproduce the linear projection coefficients for horizons lower than two years. The intuition is simple. In order to match the equity risk premium, the restricted model must imply a higher indirect risk aversion. This increases interest rate volatility and reduces the absolute level of the linear projection coefficients. In turn, this reduces the short term performance of the model.

The implied relative risk aversion (RRA) can be obtained from the value function V as $RRA = -\frac{WV_{WW}(W)}{V_W(W)} = -\frac{\partial \ln V_W}{\partial \ln W}$. Since the envelope condition for the maximization problem implies that $V_W = u_c$, we obtain $RRA = -\frac{\partial \ln u_c}{\partial \ln C_0}\frac{\partial \ln C_0}{\partial \ln W_0} = \eta_0 \frac{\partial \ln C_0}{\partial \ln W_0}$ where η is the curvature of the utility function: $\eta_t = -\frac{C_t u_{cc}(C_t, X_t)}{u_c(C_t, X_t)} = -\frac{\partial \ln u_c(C_t, X_t)}{\partial \ln C_t}$. Using the first order conditions, it is possible to show that

$$RRA_t = \frac{\rho}{\rho + k_y}\left(Y_t + \frac{k_y}{\rho}\theta_y\right) = \frac{\rho}{\rho + k_y}\left(\frac{1}{S_t} + \frac{k_y}{\rho}\theta_y\right)$$

In the *unrestricted* model, the unconditional average of Y_t is θ_y. If we substitute $Y_t = \theta_y$, the average value of the RRA coefficient is 11. Most of the dynamics of indirect RRA range between 10 and 20 with the exception of the $1980 - 1982$ period, characterized by the short-term monetary experiments, in which the model-implied RRA coefficient reaches 28.

The implied average risk aversion for the *restricted* model is 28. The average curvature needed to reproduce, at the same time, also the unconditional equity risk premium is higher. Thus, even if the unrestricted model can match interest rates properties without assuming a large curvature of the utility function, the restricted model requires a larger average risk aversion. The difference between the restricted and unrestricted indirect RRA is statistically significant. This result highlights both the advantages and limitations of this particular class of habit models. Reverse-engineering the preference structure to obtain a habit process that is consistent with asset pricing moment conditions grants sufficient flexibility to the model to match, at the same time, conditional first and second moments of bond yields and the non-linear dynamics of the spot interest rate. The model, however, shows its fragility when pushed one step further: explaining, at the same time, conditional moments of bond yields and the equity risk premium.

XII. Conclusion

In the last ten years, models with time non-separable preferences have been the focus of several studies. Important examples of these models explore preferences with habit formation. Little is known, however, about their term structure of interest rates implications. This paper is the first to investigate a monetary model with (external) habit formation and derive closed-form solutions for the dynamics of both nominal and real yield curves.

The distinctive features of the model with respect to traditional specifications are that the price of risk is not a constant multiple of the interest rates volatility and that it is state dependent. In bad (good) states of the world, the implied curvature of the indirect utility function is higher (lower). This is useful to help explain the changes in the observed term premium over the business cycle. Second, interest rates are correlated with both current and lagged monetary and consumption innovations. Third, the drift of the short-term interest rate is non-linear.

Moreover, the paper documents new empirical evidence of the extent to which habit persistence can help explain term structure of interest rates dynamics. We find the following.

First, tests of the overidentifying yield curve pricing restrictions do not reject a model with habit formation. We find that, at the estimated parameter values of the structural model, the model can simultaneously reproduce both the persistence of the conditional second moments of changes in bond yields and their conditional first moments. We run an asymptotic GMM test based on the second moments of yield changes and find that the null hypothesis that the model is correctly specified is not rejected at any horizon between 3 months and 10 years.

Second, habit formation helps reproduce both the sign and magnitude of the interest rate deviations from the expectations hypothesis described by Campbell and Shiller (1991). The model-implied linear projection coefficients are negative and increasing in absolute value with the regression horizon.

Third, we investigate whether, at the estimated parameter values, habit persistence helps explain the lead-lag correlation between interest rates and money highlighted in the macroeconomics literature (King and Watson, 1996). We find that a predictive regression of future nominal interest rates on the model-implied nominal habit stock produces a R^2 in excess of 30%.

Fourth, since the model can account for deviations from the Fisher hypothesis, we investigate the spread between nominal and real interest rates and estimate the inflation risk premium. We find that the inflation risk premium accounts for about one fourth of the nominal versus real interest rate spread. This premium is upward sloping and time varying. The average inflation risk premium is 44 basis points at an eight year horizon and it ranges between 20 and 90 basis points. We find that this time variation plays a key role in explaining the rejection of the expectations hypothesis.

We also document the limitations of the model and show the extent of the trade-off between explaining the equity risk premium and the characteristics of interest rates dynamics. We find that the equity risk premium implied by the unrestricted model is 4%. This is lower than the empirical value of 7%. When we restrict the model to also match, at the same time, the equity risk premium, most of the p-values of tests based exclusively on interest rate data decline substantially. The model is still not statistically rejected. Its performance, however, in reproducing short-term Campbell-Shiller linear projection coefficients is weaker.

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Appendix. Proofs

Proposition 1.
A. Strong form solution of the diffusion processes.
Consider $dY_t = k(\theta - Y_t)dt - \sigma_y(Y_t - \lambda)dW_t$. Let $Z_t = Y_t - \lambda$, so that $dZ_t = k(\theta_z - Z_t)dt - \sigma_y Z_t dW_t$ and $\theta_z = \theta - \lambda$. Let us make a change of variable using the function $\omega(t) = \exp\{\left[k + \frac{1}{2}\sigma_y^2\right] t + \sigma_y(W(t) - W(0))\}$. Using Ito's rule and noticing that $d\omega_t = \omega_t(k + \sigma_y^2)dt + \omega_t\sigma_y dW_t$ we obtain:

$$\begin{aligned} d(\omega_t Z_t) &= \omega_t dZ_t + Z_t d\omega_t + \langle d\omega_t dZ_t \rangle \\ &= k\theta_z\omega_t dt \end{aligned}$$

which can be directly integrated to yield

$$Z_t = \frac{1}{\omega_t}\left[Z_0\omega_0 + k\theta_z \int_0^t \omega_s ds\right]$$

Transforming back to $Y(t)$:

$$\begin{aligned} Y(t) &= \lambda + \frac{1}{\omega_t}\left[(Y_0 - \lambda) + k(\theta - \lambda)\int_0^t \omega_s ds\right] \\ \omega_s &= \exp\{(k + \frac{1}{2}\sigma_y^2)s + \sigma_y(W_s - W_0)\} \end{aligned}$$

B. Existence of a stationary density.
Sufficient conditions for the existence of a stationary density are that $\sigma^2(y) > 0$ in the interior of the support (λ, ∞) of the process and that both boundaries are entrance boundaries, i.e. $\int_\lambda^\infty m(y)dy < \infty$ and $\int_\lambda^x s(y)dy = \int_x^\infty s(y)dy = \infty$, $\forall x \in (\lambda, \infty)$ [Theorem 5.7 and 5.13 of Karatzas and Shreve (1991), p.335; see for applications Conley, Hansen, Luttmer, and Scheinkman (1997) and Ait-Sahalia (1996)] where $s(y)$ and $m(y)$ are the scale function and speed density of the process

$$\begin{aligned} s(y) &= \exp\left[-\int^y \frac{2\mu(v)}{\sigma^2(v)}dv\right], \qquad L < y < U \\ m(y) &= \frac{1}{\sigma^2(y)s(y)} \end{aligned}$$

Substituting the drift and volatility of the dY_t process, $s(y) = \exp\left[-\int^y \frac{2k(\theta - v)}{\sigma_y^2(v-\lambda)^2}dv\right]$. Integrating by parts we have:

$$s(y) = (y - \lambda)^{\frac{2k}{\sigma_y^2}} \exp\left[\frac{2k(\theta - y)}{\sigma_y^2(y - \lambda)}\right]$$

For the lower boundary as $y \to \lambda$, if $\frac{2k\theta}{\sigma_y^2} > 0$ the scale function $s(y)$ is dominated by $\exp\left[\frac{2k(\theta-y)}{\sigma_y^2(y-\lambda)}\right]$ and $\int_0^x (y - \lambda)^{\frac{2k}{\sigma_y^2}} \exp\left[\frac{2k(\theta-y)}{\sigma_y^2(y-\lambda)}\right] = \infty$. For the upper boundary, as $y \to \infty$, if $\frac{2k\theta}{\sigma_y^2} > 0$, $s(y)$ is dominated by $(y - \lambda)^{\frac{2k}{\sigma_y^2}}$, whose integral is $\frac{1}{\frac{2k}{\sigma_y^2}+1}(y - \lambda)^{\frac{2k}{\sigma_y^2}+1}$. Thus, if $2k + \sigma_y^2 > 0$, we have $\int_x^\infty (y - \lambda)^{\frac{2k}{\sigma^2}} \exp\left[\frac{2k(\theta-y)}{\sigma_y^2(y-\lambda)}\right] = \infty$. The last condition requires that $\int_0^\infty m(v)dv < \infty$

$$\begin{aligned} \int_0^\infty m(v)dv &= \int_0^\infty \frac{1}{\sigma_y^2(v-\lambda)^2}(v - \lambda)^{-\frac{2k}{\sigma^2}} \exp\left[-\frac{2k(\theta - v)}{\sigma_y^2(v - \lambda)}\right] dv = \\ &= \frac{1}{\sigma_y^2}\int_0^\infty \exp\left[-\frac{2k(\theta - v)}{\sigma_y^2(v - \lambda)}\right](v - \lambda)^{\frac{-2k-2\sigma_y^2}{\sigma_y^2}} dv \end{aligned}$$

For the upper boundary, as $v \to \infty$, the exponential converges to $\exp\frac{2k}{\sigma_y^2}$; the integral of $v^{\frac{-2k-2\sigma_y^2}{\sigma_y^2}}$ is $-\frac{\sigma_y^2}{2k+\sigma_y^2}v^{\frac{-2k-\sigma_y^2}{\sigma_y^2}}$ which converges to a finite value if $-2k - \sigma_y^2 < 0$ or $2k + \sigma_y^2 > 0$. For the lower boundary, as $v \to \lambda$, if $\frac{2k\theta}{\sigma_y^2} > 0$ the exponential goes to zero and it dominates the behavior of the integrand.

C. Functional form of the stationary density.
Consider first the case $\lambda = 0$. If it exists, the stationary density $p(x)$ of the previous diffusion process is equal to the normalized speed function, i.e. $p(x) = \mathcal{N}\cdot m(x)$, where $\mathcal{N}$ is the normalization constant. Given the previous solution for $m(x)$, the stationary distribution must take form $p(x) = \mathcal{N}\cdot x^a \exp\left(\frac{b}{x}\right)$. This is an Inverted-Gamma density with $a = -2(1 + \frac{k}{\sigma^2})$ and $b = -2\frac{k\theta}{\sigma^2}$.

The solution can be easily verified by checking that $p(x)$ solves the Fokker-Planck equation $\frac{d}{dy}\left[\mu(y)p(y)\right] - \frac{1}{2}\frac{d^2}{dy^2}\left[\sigma^2(y)p(y)\right] = 0$, which can be rewritten (see Wong (1964)) as $\left[p(x)\sigma^2(x)\right]' = 2\mu(x)p(x)$. Substituting the guess $p(x) = \mathcal{N}\cdot x^a \exp\left(\frac{b}{x}\right)$, we obtain:

$$\left(\sigma^2 a + 2\sigma^2\right)\left[x^{a+1}\exp(\frac{b}{x})\right] - b\sigma^2\left[x^a\exp(\frac{b}{x})\right] = 2k\theta\left[x^a\exp(\frac{b}{x})\right] - 2k\left[x^{a+1}\exp(\frac{b}{x})\right]$$

Matching the coefficients, we have $\sigma^2(2+a) = -2k$ and $-b\sigma^2 = 2k\theta$. This implies that $a = -2(1+\frac{k}{\sigma^2})$ and $b = -2\frac{k\theta}{\sigma^2}$. Notice that the Inverse Gamma distribution is well defined if and only if $b < 0$ and $a < -1$. These are satisfied if Condition 1 and 2, respectively, are satisfied: $k\theta > 0$ and $2k + \sigma^2 > 0$.

For $\lambda > 0$, the diffusion can be easily obtained from the previous one using a change of variables $x = y - \lambda$, the result is $p(y) = \mathcal{N}\cdot(y-\lambda)^a \exp\left(\frac{b}{y-\lambda}\right)$. This is an Inverted-Gamma density with $a = -2(1+\frac{k}{\sigma^2})$ and $b = -2\frac{k(\theta-\lambda)}{\sigma^2}$.

D. Second Moments. Let us consider the canonical representation of Y_T^2. Taking the expected value and differentiating with respect to T we obtain:

$$\frac{dE_0\left(Y_T^2\right)}{dT} = \nu_0(T) + \nu_1 E_0\left(Y_T^2\right)$$

with $\nu_1 = (\sigma^2 - 2k)$ and $\nu_0(T) = 2\left(k\theta - \lambda\sigma^2\right)E_t\left(Y_T\right) + \lambda^2\sigma^2$. The previous differential equation admits the following solution:

$$E_0 Y^2(T) = e^{(\sigma^2-2k)T}Y_0^2 + \int_0^T \nu_0(s)e^{(\sigma^2-2k)(T-s)}ds$$

If Condition 2 is satisfied, the result folllows.

Lemma 1. (Conditional Moments of Product)
Consider a linear system of two mean-reverting Ito processes ξ_{1t} and ξ_{2t}

$$\begin{aligned} d\xi_{1t} &= k_{\xi_1}\left(\theta_{\xi_1} - \xi_{1t}\right)dt + \left(\xi_{1t} - \lambda_{\xi_1}\right)\left[\upsilon dW_t^2 + \sigma_{\xi_1}dW_t^1\right] \\ d\xi_{2t} &= k_{\xi_2}\left(\theta_{\xi_2} - \xi_{2t}\right)dt + \left(\xi_{2t} - \lambda_{\xi_2}\right)\sigma_{\xi_2}dW_t^2, \qquad E(dW_t^1\cdot dW_t^2) = \rho dt \end{aligned}$$

We will prove that the conditional expectation of their product $q_t = \xi_{1t}\xi_{2t}$ is equal to

$$E_t\left[q_{t+\tau}\right] = A_3(\tau;\Theta_{\xi_1},\Theta_{\xi_2})q_t + A_1(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{1t} + A_2(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{2t} + A_0(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) \tag{17}$$

The diffusion of $q_t = \xi_{1t}\xi_{2t}$ is

$$\begin{aligned} dq_t &= \xi_{2t}\left[k_{\xi_1}\left(\theta_{\xi_1} - \xi_{1t}\right)dt + \left(\xi_{1t} - \lambda_{\xi_1}\right)\left[\upsilon dW_t^2 + \sigma_{\xi_1}dW_t^1\right]\right] + \xi_{1t}\left(k_{\xi_2}\left(\theta_{\xi_2} - \xi_{2t}\right)dt + \sigma_{\xi_2}\left(\xi_{2t} - \lambda_{\xi_2}\right)dW_t^2\right) \\ &\quad + \rho\sigma_{\xi_2}\left(\xi_{2t} - \lambda_{\xi_2}\right)\sigma_{\xi_1}\left(\xi_{1t} - \lambda_{\xi_1}\right)dt + \left(\xi_{2t} - \lambda_{\xi_2}\right)\sigma_{\xi_2}\left(\xi_{1t} - \lambda_{\xi_1}\right)\upsilon dt \end{aligned}$$

The stochastic process q_t follows

$$dq_t = \left[a_3 + h_{\xi_2}\xi_{2t} + h_{\xi_1}\xi_{1t} + h_q q_t\right]dt + \Sigma_q dW$$

with

$$\left\{\begin{array}{ll} h_{\xi_2} = k_{\xi_1}\theta_{\xi_1} - \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_1} - \upsilon\sigma_{\xi_2}\lambda_{\xi_1} & a_1 = k_{\xi_2}\theta_{\xi_2} \\ h_{\xi_1} = k_{\xi_2}\theta_{\xi_2} - \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_2} - \upsilon\sigma_{\xi_2}\lambda_{\xi_1} & a_2 = k_{\xi_1}\theta_{\xi_1} \\ h_q = -k_{\xi_1} - k_{\xi_2} + \rho\sigma_{\xi_2}\sigma_{\xi_1} + \upsilon\sigma_{\xi_2} & a_3 = \rho\sigma_{\xi_2}\sigma_{\xi_1}\lambda_{\xi_2}\lambda_{\xi_2} + \upsilon\sigma_{\xi_1}\lambda_{\xi_1}\lambda_{\xi_2} \end{array}\right. \tag{18}$$

Consider the following three dimensional process $f_t = \left[\xi_{2t}, \xi_{1t}, q_t\right]$. Then we can describe the dynamics of the process as

$$df_t = \left(A_0 + A_1 f_t\right)dt + \Sigma_f dW$$

where

$$A_0 = \left[\begin{array}{c} a_1 \\ a_2 \\ a_3 \end{array}\right], \qquad A_1 = \left[\begin{array}{ccc} -k_{\xi_2} & 0 & 0 \\ 0 & -k_{\xi_1} & 0 \\ h_{\xi_2} & h_{\xi_1} & h_q \end{array}\right]$$

The system is linear, and the expected value $E_t\left[f_{t+\tau}\right]$ can be calculated as

$$E_t\left[f_{t+\tau}\right] = \Psi\left(t+\tau\right)f_t + \int_t^{t+\tau}\Psi\left((t+\tau) - s\right)A_0 ds$$

with $\Psi(\tau) = U\exp\left(\Lambda\cdot\tau\right)U^{-1}$ where Λ is a diagonal matrix of eigenvalues of A_1 and U is the matrix of associated eigenvectors. We can find that

$$\Lambda = \left(\begin{array}{ccc} h_q & & \\ & -k_{\xi_1} & \\ & & -k_{\xi_2} \end{array}\right), \qquad U = \left(\begin{array}{ccc} 0 & 0 & -\frac{h_q + k_{\xi_2}}{h_{\xi_2}} \\ 0 & -\frac{h_q + k_{\xi_1}}{h_{\xi_1}} & 0 \\ 1 & 1 & 1 \end{array}\right)$$

Simple matrix multiplication gives us

$$\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1} = \begin{pmatrix} e^{-k_{\xi_2}\tau} & 0 & 0 \\ 0 & e^{-k_{\xi_1}\tau} & 0 \\ \frac{\left(e^{h_q\tau}-e^{-k_{\xi_2}\tau}\right)h_{\xi_2}}{h_q+k_{\xi_2}} & \frac{\left(e^{h_q\tau}-e^{-k_{\xi_1}\tau}\right)h_{\xi_1}}{h_q+k_{\xi_1}} & e^{h_q\tau} \end{pmatrix}$$

Let us define vector $e_3 = (0,0,1)$. Then the expected value of q_t is

$$E_t[q_{t+\tau}] = E_t[e_3 f_{t+\tau}] = e_3\Psi(\tau) f_t + \int_t^{t+\tau} e_3\Psi(t+\tau-s) A_0 ds$$

After some algebra we obtain

$$\begin{aligned} E_t[q_{t+\tau}] &= q_t e^{h_q\tau} + \xi_{1t}\frac{\left(e^{h_q\tau}-e^{-k_{\xi_1}\tau}\right)h_{\xi_1}}{h_q+k_{\xi_1}} + \xi_{2t}\frac{\left(e^{h_q\tau}-e^{-k_{\xi_2}\tau}\right)h_{\xi_2}}{h_q+k_{\xi_2}} \\ &+ a_3\frac{1-e^{h_q\tau}}{h_q} + a_2\frac{h_{\xi_1}}{h_q+k_{\xi_1}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_1}\tau}}{k_{\xi_1}}\right] \\ &+ a_1\frac{h_{\xi_2}}{h_q+k_{\xi_2}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_2}\tau}}{k_{\xi_2}}\right] \end{aligned} \tag{19}$$

Substituting back the values for $a_1, a_2, a_3, h_{\xi_2}, h_{\xi_1}$ and h_q using set of equations (18) we obtain the solution in terms of the original parameters). That is

$$E_t[q_{t+\tau}] = A_3(\tau;\Theta_{\xi_1},\Theta_{\xi_2})q_t + A_1(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{1t} + A_2(\tau;\Theta_{\xi_1},\Theta_{\xi_2})\xi_{2t} + A_0(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) \tag{20}$$

where

$$\begin{cases} A_3(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = e^{h_q\tau}, \quad A_1(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = \frac{\left(e^{h_q\tau}-e^{-k_{\xi_1}\tau}\right)h_{\xi_1}}{h_q+k_{\xi_1}}, \quad A_2(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = \frac{\left(e^{h_q\tau}-e^{-k_{\xi_2}\tau}\right)h_{\xi_2}}{h_q+k_{\xi_2}} \\ A_0(\tau;\Theta_{\xi_1},\Theta_{\xi_2}) = a_3\frac{1-e^{h_q\tau}}{h_q} + a_2\frac{h_{\xi_1}}{h_q+k_{\xi_1}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_1}\tau}}{k_{\xi_1}}\right] + a_1\frac{h_{\xi_2}}{h_q+k_{\xi_2}}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_{\xi_2}\tau}}{k_{\xi_2}}\right] \end{cases} \tag{21}$$

with Θ_{ξ_i} being the structural parameters of the diffusion processes $\Theta_{\xi_i} \equiv [k_{\xi_i}, \theta_{\xi_i}, \sigma_{\xi_i}, \lambda]$.
■

Proposition 2. (General Price Level)
The general equilibrium price level is obtained from the equilibrium rate of substitution between the money stock m_t and consumption.[35] We have

$$\frac{1}{P_t} = E_t\left[\int_t^\infty e^{-\rho(s-t)}\frac{u_m(C_s, m_s, H_s)}{u_c(C_t, m_t, H_t)}\frac{1}{P_s}ds\right]$$

In the case of the log-utility function $u(c,m,H) = \log(Cm^\gamma - H)$, we obtain

$$\begin{aligned} \frac{1}{P_t} &= \gamma\frac{C_t}{Y_t}\int_t^\infty e^{-\rho(s-t)}E_t\left[Y_s\frac{1}{M_s}\right]ds \\ &= \gamma\frac{C_t}{Y_t}\frac{1}{L_t M_t}\int_t^\infty e^{-(\rho+\mu_M)(s-t)}E_t[Y_s L_s]\,ds \end{aligned} \tag{22}$$

To solve for the price level we need to solve for the expectation under the integral. For simplicity, let us first consider the univariate case. Let $q_t = Y_t L_t$ and $f_t = [L_t, Y_t, q_t]$. Using Ito's rule, it is easy to show that

$$df(t) = [A_0 + A_1 f(t)]\,dt + \Sigma dW_t$$

with

$$A_0 = \begin{bmatrix} a_1 \\ a_2 \\ a_3 \end{bmatrix}, \qquad A_1 = \begin{bmatrix} -k_\ell & 0 & 0 \\ 0 & -k_y & 0 \\ h_\ell & h_Y & h_q \end{bmatrix}$$

$$\begin{aligned} a_1 &= k_\ell\theta_\ell, \quad a_2 = k_y\theta_y, \quad a_3 = 0 \\ h_Y &= k_\ell\theta_\ell \\ h_\ell &= k_y\theta_y + (\lambda\sigma_{yc}\sigma_\ell\rho + \lambda\sigma_{y\ell}\sigma_\ell) \\ h_q &= -k_y - k_\ell - (\sigma_{yc}\sigma_\ell\rho + \sigma_{y\ell}\sigma_\ell) \end{aligned}$$

[35] See Bakshi and Chen (1996) for an analytical derivation of this continuous time first order conditions obtained as the continuous time limit of a discrete time economy.

Let $\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1}$, where Λ is the diagonal matrix of eigenvalues of A_1 and U the associated eigenvectors matrix. From Lemma 2, we have

$$E_t[f_{t+\tau}] = \Psi(t+\tau) f_t + \int_t^{t+\tau} \Psi((t+\tau)-s) A_0 ds,$$

Notice that

$$\Lambda = \begin{pmatrix} h_q & & \\ & -k_Y & \\ & & -k_\ell \end{pmatrix}, \qquad U = \begin{pmatrix} 0 & 0 & -\frac{h_q+k_\ell}{h_2} \\ 0 & -\frac{h_q+k_y}{h_1} & 0 \\ 1 & 1 & 1 \end{pmatrix}$$

with $\Psi(\tau) = U \exp(\Lambda \cdot \tau) U^{-1} = \begin{pmatrix} e^{-k_\ell \tau} & 0 & 0 \\ 0 & e^{-k_y \tau} & 0 \\ \frac{(e^{h_q\tau}-e^{-k_\ell\tau})h_\ell}{h_q+k_\ell} & \frac{(e^{h_q\tau}-e^{-k_y\tau})h_Y}{h_q+k_y} & e^{h_q\tau} \end{pmatrix}$, therefore using the result (20) we have

$$E_t[Y_s L_s] = \sum_{i=1}^{2} (A_q(\tau) q_t + A_Y(\tau) Y_t + A_\ell(\tau) \ell_{it} + A_0(\tau))$$

where

$$\begin{cases} A_q(\tau) = e^{h_q\tau}, \quad A_Y(\tau) = \frac{(e^{h_q\tau}-e^{-k_y\tau})h_Y}{h_q+k_y}, \quad A_\ell(\tau) = \frac{(e^{h_q\tau}-e^{-k_\ell\tau})h_\ell}{h_q+k_\ell} \\ A_0(\tau) = a_3 \frac{1-e^{h_q\tau}}{h_q} + a_2 \frac{h_Y}{h_q+k_y}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_y\tau}}{k_y}\right] + a_1 \frac{h_\ell}{h_q+k_\ell}\left[\frac{1-e^{h_q\tau}}{-h_q} - \frac{1-e^{-k_\ell\tau}}{k_\ell}\right] \end{cases} \tag{23}$$

Hence, the inverse price level is

$$\frac{1}{P_t} = \gamma \frac{C_t}{Y_t} \frac{1}{L_t M_t} \int_0^\infty e^{-(\rho+\mu_M)\tau} \left[\sum_{i=1}^{2} (A_q(\tau) q_t + A_Y(\tau) Y_t + A_\ell(\tau) \ell_{it} + A_0(\tau))\right] d\tau$$

In order for the integral to converge, the parameter $A_i(\tau)$ need to be bounded. In addition to Conditions $[C1]$ and $[C2]$, this requires additional constraints on the size of the covariance terms between the liquidity shocks and the dY process:

$$h_q = -k_y - k_{\ell_i} - (\sigma_{yc}\sigma_{\ell_i}\rho_{\ell_i c} + \sigma_{y\ell_i}\sigma_{\ell_i}) < 0$$

Note that all expression for A's are of the form $e^{\zeta s}$ or $\frac{1-e^{\zeta t}}{\zeta}$. Thus, for convenience let us calculate the following integrals for a generic value ζ. We will later substitute their values as a function of the structural parameters.

$$\int_0^\infty e^{-(\rho+\mu_M)\tau} e^{\zeta\tau} d\tau = \int_0^\infty e^{-(\rho+\mu_M-\zeta)\tau} d\tau = \frac{1}{\rho+\mu_M-\zeta}$$

$$\int_0^\infty e^{-(\rho+\mu_M)\tau} \left[\frac{1-e^{\zeta\tau}}{\zeta}\right] d\tau = \frac{1}{\zeta}\left[\frac{1}{\rho+\mu_M} - \frac{1}{\rho+\mu_M-\zeta}\right] = -\frac{1}{(\rho+\mu_M)(\rho+\mu_M-\zeta)}$$

Using the result, let us consider the first term inside the integral

$$\Gamma_q \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_q(\tau) d\tau = \int_0^\infty e^{-(\rho+\mu_M)\tau} e^{h_q\tau} d\tau = \frac{1}{\rho+\mu_M-h_q}$$

Similarly,

$$\Gamma_\ell \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_\ell(\tau) d\tau = \frac{h_\ell}{h_q+k_\ell}\left[\frac{1}{\rho+\mu_M-h_q} - \frac{1}{\rho+\mu_M+k_\ell}\right]$$

$$\Gamma_Y \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_Y(\tau) d\tau = \frac{h_Y}{h_q+k_y}\left[\frac{1}{\rho+\mu_M-h_q} - \frac{1}{\rho+\mu_M+k_y}\right]$$

$$\Gamma_0 \equiv \int_0^\infty e^{-(\rho+\mu_M)\tau} A_0(\tau) d\tau = a_3\left[-\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)}\right] + a_2 \frac{h_Y}{h_q+k_y}\left[\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)} - \frac{1}{(\rho+\mu_M)(\rho+\mu_M+k_y)}\right]$$
$$+ a_1 \frac{h_\ell}{h_q+k_\ell}\left[\frac{1}{(\rho+\mu_M)(\rho+\mu_M-h_q)} - \frac{1}{(\rho+\mu_M)(\rho+\mu_M+k_\ell)}\right]$$

Summarizing

$$\frac{1}{P_t} = \gamma \frac{C_t}{Y_t} \frac{1}{L_t M_t} \left[\Gamma_q(\tau) q_t + \Gamma_Y(\tau) Y_t + \Gamma_\ell(\tau) \ell_t + \Gamma_0(\tau) \right]$$

In the multivariate case, $L_t = \sum_i \ell_{it}$, the result is easily generalizable to:

$$\frac{1}{P_t} = \gamma \frac{C_t}{Y_t} \frac{1}{L_t M_t} \left[\sum_{i=1}^{2} \left(\Gamma_{q_i}(\tau) q_{it} + \Gamma_{Y_i}(\tau) Y_t + \Gamma_{\ell_i}(\tau) \ell_{it} + \Gamma_{0i}(\tau) \right) \right]$$

where the additional index i refers to the parameters of the process ℓ_{it}.

The integral of future values of the inverse consumption surplus ratio $E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} Y_s ds$ and monetary factors $E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} \ell_{is} ds$ can be obtained using similar methods. Under the assumption that the two integrals converge, which require $\rho + \mu_M > 0$, we apply Fubini's theorem to invert the order of integration. Moreover, the linearity of the drift of dY_t implies $E_t(Y_s) = \theta_y + (Y_t - \theta_y) e^{-k_y(s-t)}$ so that

$$\begin{aligned} E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} Y_s ds &= \frac{\theta_y}{\rho + \mu_M} + \frac{(Y_t - \theta_y)}{\rho + \mu_M + k_y} \\ E_t \int_t^\infty e^{-(\rho+\mu_M)(s-t)} \ell_{is} ds &= \frac{\theta_\ell^i}{\rho + \mu_M} + \frac{\left(\ell_{is} - \theta_\ell^i\right)}{\rho + \mu_M + k_\ell^i} \end{aligned}$$

Proposition 3. (High Order Conditional moments)
Consider $d\ell(t) = k(\theta - \ell)dt + \sigma(\ell - \ell_0) dW_t$, From Ito's rule

$$d\left[\ell(t)^n\right] = n\ell(t)^{n-1} d\ell(t) + \frac{n(n-1)}{2} \ell(t)^{n-2} \left[\sigma_\ell(\ell_t - \ell_0)\right]^2 dt$$

Thus

$$\begin{aligned} \frac{d}{dt} E_0 \ell(t)^n &= E_0 \ell(t)^n \left[-nk + \frac{n(n-1)}{2} \sigma_y^2 \right] \\ &+ E_0 \ell(t)^{n-1} \left[nk\theta - \lambda n(n-1) \sigma_y^2 \right] \\ &+ E_0 \ell(t)^{n-2} \left[\frac{n(n-1)}{2} \ell_0^2 \sigma_y^2 \right] \end{aligned}$$

Let $V_0(t) \equiv E_0 \ell(t)^n$, then integrating between 0 and T

$$\begin{aligned} V_0(T) - V_0(0) &= \int_0^T \Psi_0(s) ds + \int_0^T \Psi_1 V_t(s) ds \\ \Psi_0(s) &= E_0 \ell(s)^{n-1} \left[nk\theta - \lambda n(n-1) \sigma_y^2 \right] + E_0 \ell(s)^{n-2} \left[\frac{n(n-1)}{2} \ell_0^2 \sigma_y^2 \right] \\ \Psi_1 &= \left[-nk + \frac{n(n-1)}{2} \sigma_y^2 \right] \end{aligned}$$

differentiating with respect to T

$$V_0'(T) = \Psi_0(T) + \Psi_1 V(T)$$

which is known to have solution $E_0 \ell(T)^n = e^{\Psi_1 T} \ell(0)^n + \int_0^T \Psi_0(s) e^{\Psi_1 (T-s)} ds$. Notice the dependence of $\Psi_0(T)$ on the conditional moments $E_0 \ell(T)^{n-1}$ and $E_0 \ell(T)^{n-2}$. The first conditional moment satisfies the following ODEs

$$\frac{dE_t\left(\ell_T\right)}{dT} = k\theta - kE_t\left(\ell_T\right)$$

which has solutions $E_t\left(\ell_T\right) = \theta + \left(\ell_t - \theta\right) e^{-k(T-t)}$. All the other moments can be computed recursively.

The conditional variances of Y_T, ℓ_T and central cross moment $E_t\left[\left(Y_T - E_t\left(Y_T\right)\right)\left(\ell_T - E_t\left(\ell_T\right)\right)\right]$ can be constructed using the same approach.

A. Conditional Moments of Nominal Yields

Nominal yields are $y(t,\tau) = -\frac{\ln N(t,\tau)}{\tau}$, a non-linear function $\phi(Z_t, \Theta)$ of the state vector $Z_t = [Y, (t); \ell(t)]$. Thus, since we have closed-form solutions for all the moments of $z(t)$, we can compute the conditional first moments of $y(t,\tau)$ by Taylor expansion

$$y(T,\tau) = y(t,\tau) + \phi'(y(t,\tau))[y(T,\tau) - y(t,\tau)] + \sum_{n=2} \frac{1}{n!} \frac{\partial^n}{\partial y^n} \phi(y(t,\tau))[y(T,\tau) - y(t,\tau)]^n$$

$$E_t y(T,\tau) = y(t,\tau) + \phi'(y(t,\tau))[E_t y(T,\tau) - y(t,\tau)] + \sum_{n=2} \frac{1}{n!} \frac{\partial^n}{\partial y^n} \phi(y(t,\tau)) E_t [y(T,\tau) - y(t,\tau)]^n$$

The second moments are obtained using the delta method, since the conditional variance-covariance matrix $Var(Z_{t+\Delta t}|Z_t)$ is known:

$$Var(y^\tau_{t+\Delta t}|y^\tau_t) = \left[\frac{\partial y(Z_{t+\Delta t}, \Theta, \tau)}{\partial Z_{t+\Delta t}}\right]' Var(Z_{t+\Delta t}|Z_t) \left[\frac{\partial y(Z_{t+\Delta t}, \Theta, \tau)}{\partial Z_{t+\Delta t}}\right] \tag{24}$$

Let $\Psi_t = \sum_{i=1}^2 \left(\frac{\Gamma_{1i}}{S_t} + \Gamma_{2i}\ell_{it} + \frac{\Gamma_{3i}\ell_{it}}{S_t} + \Gamma_{0i}\right)$, and $\Phi_t = \sum_{i=1}^2 \left(\frac{\Lambda_{1i}(\tau)}{S_t} + \Lambda_{2i}(\tau)\ell_{it} + \frac{\Lambda_{3i}(\tau)\ell_{it}}{S_t} + \Lambda_{0i}(\tau)\right)$,

$$N(t,\tau) = e^{-\rho\tau} \frac{\sum_{i=1}^2 \left(\frac{\Lambda_{1i}(\tau)}{S_t} + \Lambda_{2i}(\tau)\ell_{it} + \frac{\Lambda_{3i}(\tau)\ell_{it}}{S_t} + \Lambda_{0i}(\tau)\right)}{\sum_{i=1}^2 \left(\frac{\Gamma_{1i}}{S_t} + \Gamma_{2i}\ell_{it} + \frac{\Gamma_{3i}\ell_{it}}{S_t} + \Gamma_{0i}\right)}$$

thus

$$\begin{aligned} \tfrac{\partial N}{\partial Y} &= e^{-\rho\tau} \left\{ \tfrac{\sum_{i=1}^2 (\Lambda_{1i}(\tau) + \Lambda_{3i}(\tau)\ell_{it})}{\Psi_t} + (-1)\left(\tfrac{\sum_{i=1}^2 \Phi_t}{\Psi_t^2}\right)\left(\textstyle\sum_{i=1}^2 (\Gamma_{1i} + \Gamma_{3i}\ell_{it})\right)\right\} \\ \tfrac{\partial N}{\partial \ell_{it}} &= e^{-\rho\tau} \left\{ \tfrac{\sum_{i=1}^2 (\Lambda_{2i}(\tau) + \Lambda_{3i}(\tau)Y_t)}{\Psi_t} + (-1)\left(\tfrac{\sum_{i=1}^2 \Phi_t}{{\Psi_t}^2}\right)\left(\textstyle\sum_{i=1}^2 (\Gamma_{2i} + \Gamma_{3i}Y_t)\right)\right\} \end{aligned} \tag{25}$$

By Ito's rule:

$$dy = \frac{\partial y}{\partial t} + \frac{\partial y}{\partial Y} dY_t + \frac{1}{2}\frac{\partial^2 y}{\partial Y^2}\langle dY\rangle^2 + \sum_{i=1}^n \left[\frac{\partial y}{\partial \ell_{it}}(d\ell_{it}) + \frac{1}{2}\frac{\partial^2 y}{\partial \ell_{it}}\langle d\ell_{it}\rangle^2 + \frac{\partial^2 y}{\partial Y_t \partial \ell_{it}}(d\ell_{it})(dY_t)\right]$$

Therefore, the dynamics of bond yields is given by

$$\begin{aligned} dy(t,\tau) &= \mu_{yt} dt + \left(\frac{\partial y}{\partial Y}\sigma_{cy}(Y_t - \lambda)\right) dW^c_t + \left(\frac{\partial y}{\partial \ell_i}\sigma^i_\ell \ell_{it} + \frac{\partial y}{\partial Y}\sigma_{iY}(Y_t - \lambda)\right) dW^{\ell_i}_t \\ &= \mu_{yt} dt + \sigma_{yc}(Y_t, \ell_{it}) dW^c_t + \sigma_{yi}(Y_t, \ell_{it}) dW^{\ell_i}_t \end{aligned}$$

Table i: Summary Statistics

This table presents summary statistics of the dataset used in the estimation. It is based on observations between January 1960 and December 2000. *Inflation* is the observed inflation rate calculated as the 12 months percentage change in the CPI index. *Money growth* is the observed 12 months percentage change in the M2 money stock. *Consumption Growth* is the real per-capita consumption growth. The other values are the yields to maturity of nominal bonds at different maturities. The real yield data is from 1997 to 2000. The real yield implied by the model is based on the model fitted to the data from 1960 to 2000. The p-values of the test of the whether mean and volatilty of the time series are statistically different from their empirical counterparts are given in the last columns.

	Mean Model	Mean Empirical	Volatility Model	Volatility Empirical	p-value
Nominal Yields					
3 month	6.24%	6.28%	2.11%	1.79%	0.44
2–year	6.92%	6.99%	1.31%	1.38%	0.55
10-year	7.52%	7.55%	0.89%	0.95%	0.73
Real Yields					
3 months	2.13%	2.88%	1.13%	0.47%	0.05
2-year	2.32%	2.87%	0.97%	0.34%	0.11
10-year	2.51%	2.84%	0.44%	0.34%	0.12
Inflation	4.03%	4.70%	2.33%	3.23%	0.25
Money Growth	5.76%	6.09%	2.54%	3.52%	0.18
Consumption	2.40%	1.90%	1.70%	1.94%	0.11

Table ii: Parameter Estimates

This table presents the estimates of the structural parameters. The estimation is based on the asset pricing restrictions for eight nominal bonds with maturities ranging from 3 month to 10 years, as well as the processes for the money supply M2, the inflation and the habit. The estimated model has three factors. The inverse consumption surplus factor follows

$$dY_t = k_Y\left(\theta_Y - Y_t\right)dt - \left(Y_t - \lambda\right)\left[\sigma_{cy}dW_t^c + \sigma_{ly}\,dW_t^l\right]$$

We assume two liquidity shocks ℓ_{it} affecting the money supply following $d\ell_{it} = k_{l_i}\left(\theta_{l_i} - \ell_{it}\right)dt + \sigma_{\ell_i}\ell_{it}dW_t^{\ell_i}, \quad i = 1, 2$.The Brownian motions W_t^l and W_t^c are assumed to be correlated. In parenthesis we report the p-values of the Likelihood Ratio test.

ρ	k_Y	θ_Y	σ_{cy}	λ
0.0150	0.0218	11.0012	0.1580	10.4007
(0.01)	(0.02)	(0.01)	(0.01)	(0.03)
σ_{l_1y}	σ_{l_2y}	ρ_{l_1c}	ρ_{l_2c}	σ_c
0.0631	0.0511	0.2380	0.3185	0.0157
(0.01)	(0.01)	(0.01)	(0.02)	(0.02)
k_{l_1}	θ_{l_1}	σ_{l_1}	k_{l_2}	θ_{l_2}
0.3662	0.8218	0.0109	0.2401	3.1978
(0.02)	(0.01)	(0.01)	(0.01)	(0.01)
σ_{l_2}	μ_c	μ_m	γ	
0.0106	0.0135	0.0432	0.5133	
(0.01)	(0.03)	(0.01)	(0.02)	

Test of Overidentifying Restrictions

$J_T = 18.38, \quad pval = 0.36, \quad df = 17$

Table iii: Lead-lag Relation between Interest Rates and Habit

We run the following regressions

$$Panel\ A: \quad r_{t+1} = \alpha_1 + \beta_1 \tilde{Y}_t(\phi, \gamma, n) + \varepsilon_{t+1}$$

$$Panel\ B: \quad R_{t+1} = a_1 + b_1 \tilde{Y}_t(\phi, \gamma, n) + \varepsilon_{t+1}$$

where r_{t+1} and R_{t+1} are the realized real and nominal interest rates. $\tilde{Y}_t$ is the discretized time-series process of Y_t obtained from the solution in Proposition 1. $\phi = \exp[-(k + \frac{1}{2}\sigma_y^2)]$

Panel A: Predictability **Real** Interest Rate by Money-Adjusted Habit

$R^2(\phi, \gamma)$

$\phi \setminus \gamma$	0.000	0.050	0.100	0.400	0.600	0.800	1.000
0.200	0.001	0.003	0.005	0.018	0.026	0.032	0.037
0.500	0.001	0.003	0.005	0.018	0.024	0.029	0.032
0.700	0.000	0.000	0.001	0.007	0.011	0.013	0.015
0.900	0.019	0.019	0.019	0.015	0.013	0.012	0.011
0.950	0.087	0.098	0.105	0.103	0.095	0.089	0.084
0.980	0.194	0.219	0.233	0.223	0.206	0.193	0.183

$\beta(\phi, \gamma)$ with T-stat in parenthesis

$\phi \setminus \gamma$	0.000	0.050	0.100	0.400	0.600	0.800	1.000
0.200	0.794	1.162	1.462	2.246	2.261	2.149	1.999
	(0.353)	(0.537)	(0.706)	(1.422)	(1.702)	(1.891)	(2.024)
0.500	0.257	0.355	0.434	0.611	0.598	0.557	0.511
	(0.396)	(0.573)	(0.733)	(1.388)	(1.631)	(1.790)	(1.900)
0.700	0.026	0.071	0.106	0.193	0.194	0.182	0.168
	(0.079)	(0.226)	(0.358)	(0.891)	(1.084)	(1.210)	(1.296)
0.900	-0.217	-0.208	-0.196	-0.118	-0.087	-0.067	-0.053
	(-1.437)	(-1.452)	(-1.449)	(-1.303)	(-1.204)	(-1.129)	(-1.072)
0.950	-0.357	-0.355	-0.341	-0.209	-0.153	-0.119	-0.096
	(-3.211)	(-3.425)	(-3.554)	(-3.528)	(-3.373)	(-3.246)	(-3.149)
0.980	-0.389	-0.382	-0.359	-0.202	-0.146	-0.113	-0.091
	(-5.096)	(-5.507)	(-5.731)	(-5.568)	(-5.288)	(-5.078)	(-4.925)

Panel B: Predictability **Nominal** Interest Rate by Money-Adjusted Habit

$R^2(\phi, \gamma)$

$\phi \setminus \gamma$	0.000	0.050	0.100	0.400	0.600	0.800	1.000
0.200	0.059	0.078	0.098	0.201	0.245	0.275	0.296
0.500	0.048	0.069	0.091	0.203	0.252	0.285	0.308
0.700	0.030	0.050	0.072	0.193	0.246	0.283	0.309
0.900	0.006	0.024	0.052	0.229	0.303	0.350	0.381
0.950	0.004	0.029	0.068	0.294	0.373	0.419	0.448
0.980	0.006	0.039	0.085	0.313	0.382	0.421	0.445

$b(\phi, \gamma)$ with T-stat in parenthesis

$\phi \setminus \gamma$	0.000	0.050	0.100	0.400	0.600	0.800	1.000
0.200	-5.495	-6.114	-6.576	-7.201	-6.723	-6.113	-5.525
	(-2.595)	(-3.031)	(-3.434)	(-5.206)	(-5.921)	(-6.406)	(-6.744)
0.500	-1.434	-1.645	-1.802	-2.017	-1.877	-1.700	-1.532
	(-2.336)	(-2.828)	(-3.280)	(-5.242)	(-6.025)	(-6.558)	(-6.931)
0.700	-0.574	-0.706	-0.806	-0.964	-0.900	-0.815	-0.733
	(-1.834)	(-2.390)	(-2.898)	(-5.080)	(-5.943)	(-6.531)	(-6.945)
0.900	-0.114	-0.228	-0.314	-0.442	-0.402	-0.354	-0.311
	(-0.775)	(-1.646)	(-2.434)	(-5.662)	(-6.845)	(-7.618)	(-8.148)
0.950	-0.072	-0.187	-0.267	-0.342	-0.295	-0.251	-0.216
	(-0.642)	(-1.786)	(-2.805)	(-6.713)	(-8.017)	(-8.829)	(-9.369)
0.980	-0.067	-0.156	-0.211	-0.232	-0.194	-0.162	-0.138
	(-0.817)	(-2.082)	(-3.174)	(-7.008)	(-8.164)	(-8.858)	(-9.311)

Table iv: Goodness of Fit by maturity

This table presents fitting errors for the model measured in basis points. The fitting errors are defined as the difference between the model generated nominal spot rate and the observed nominal rate during the sample period. The maturity of the bonds range between 3 months and 10 years. Panel A reports the estimation results when all moment conditions are used. These moments include asset pricing restrictions as well as restrictions from the money, inflation and habit process. Panel B reports the fitting errors of an estimation based excusively on the yield curve restrictions of the model.

Panel A: Term Structure plus Macro Variable Fit

	Min error	Max Error	Mean Absolute Error	Median Absolute Error
3m	-411.7	254.9	61.6	33.3
6m	-473.0	173.1	78.9	58.8
1y	-377.4	246.3	54.4	30.7
2y	-278.9	300.8	47.8	27.1
3y	-219.7	314.8	46.6	31.2
5y	-182.4	321.2	41.6	26.8
7y	-155.2	314.0	45.6	32.1
10y	-185.5	275.5	71.2	62.5

Panel B: Term Structure Fit Only

	Min error	Max Error	Mean Absolute Error	Median Absolute Error
3m	-36.4	42.7	12.8	11.7
6m	-80.6	-1.7	37.6	38.0
1y	-72.4	94.4	19.7	14.6
2y	-83.1	130.0	32.8	28.2
3y	-68.4	117.3	31.4	26.1
5y	-61.7	70.5	17.6	13.1
7y	-47.0	23.1	10.1	8.7
10y	-54.1	-26.8	38.4	38.3

Table v: Orthogonality Tests

The table shows the results of the orthogonality tests of the prediction errors of growth rates in consumption, price index and monetary holdings. We test the null hypothesis $H_0 : \theta = 0$ in the GMM framework with the following moment restrictions

$$\begin{bmatrix} u_{t+12}(\theta) \\ u_{t+12}(\theta) \otimes [\xi(x_t)] \end{bmatrix}$$

In the case of the consumption equation, let $c_{t+1} = \ln\left(\frac{C_{t+1}}{C_t}\right)$, the prediction errors are defined as

$$u_{t+1} = c_{t+1} - E_t\left[c_{t+1}|I_t\right] - \theta' \phi(x_t)$$

We test the null hypothesis using the following statistics d_T

$$d_T = T \cdot \left[h_T(x_t, \theta(H_0))' W_T^{-1} h_T(x_t, \theta(H_0)) - h_T(x_t, \theta^*)' W_T^{-1} h_T(x_t, \theta^*)\right]$$

which is χ^2 distributed under the null hypothesis. We consider the following set of lagged explanatory variables: $\phi(x_t) = const$, c_{t-1}, c_{t-1}^2. We report the value of the GMM d_T statistics with their corresponding Chi-square p-values in parenthesis.
Panel B and C present the results of the same orthogonality tests for the inflation and money growth. We report the value of test statistic d_T for three time horizons (3 months, 6 months and 1 year).

Panel A: Orthogonality test of consumption

	3 months	6 months	1 year
d_T	3.319	3.173	4.922
	(0.345)	(0.366)	(0.178)

Panel B: Orthogonality test of inflation

	3 months	6 months	1 year
d_T	7.180	5.220	3.115
	(0.066)	(0.156)	(0.374)

Panel C: Orthogonality test of money

	3 months	6 months	1 year
d_T	39.257	11.538	6.098
	(0.000)	(0.009)	(0.107)

Table vi: Forecasting Errors

The table presents the forecast errors of the model for the growth rates in consumption, inflation rate and money growth. The model-implied forecast errors are compared to an ARMA(1,1) specification. The errors are calculated as the mean absolute deviation of the n-periods forecast. The unit of measure is basis points. The sample period is 1960-2000.

Panel A: Consumption Growth

	3 months	6 months	1 year
Model	38 bp	72 bp	119 bp
ARMA(1,1)	34 bp	60 bp	109 bp

Panel B: Inflation Rate

	3 months	6 months	1 year
Model	29 bp	57 bp	115 bp
ARMA(1,1)	28 bp	53 bp	110 bp

Panel C: Money growth

	3 months	6 months	1 year
Model	45 bp	90 bp	178 bp
ARMA(1,1)	41 bp	87 bp	170 bp

Table vii: Test of Expectation Hypothesis

The table presents the results of test for the unbiased expectation hypothesis. We linearize the forward premium and investigate the following specification:

$$f(t,\tau) - E_t(R_{t+\tau}) = \alpha + \beta_{1i} l_{ti} + \beta_2 S_t + \varepsilon_t$$

Columns 1 and 2 present the results of the test of the unbiased expectation hypothesis i.e. whether forward premium is constant at different horizons (different values of τ) from 3 months to 5 years. Columns 3 and 4 quantify the relative contribution of monetary and habit factors to the total variable of the forward premium.

	$J - stat$	$p - value$	Relative Factor Contribution	
			Monetary	Habit
3 months	6.14	0.00	87%	13%
6 months	8.06	0.00	73%	27%
1 year	9.60	0.00	62%	38%
2 years	16.57	0.00	64%	36%
3 years	33.29	0.00	61%	39%
5 years	65.51	0.00	57%	43%
Joint Test (All Maturities)		p-$Value = 0.00$		

Table viii: Campbell and Shiller Regressions

This table reports the Campbell and Shiller regressions. The main regression equation is

$$R_{t+m}^{n-m} - R_t^n = \alpha + \beta\left(\frac{m}{n-m}\right)(R_t^n - R_t^m) + \varepsilon_t$$

where R_t^n is the yield of a bonds with maturity n at time t. The expectation hypothesis implies that the coefficient β is equal to 1. The value of m is taken to be one month. The first row shows the results of Cambell and Shiller regressions on a sample ranging between 1960 and 2000. The second row shows the values of the same β coefficient *implied* by the structural model at the estimated values of the structural parameters. Standard errors are given in parenthesis.

	1y	2y	3y	5y	7y	10y
Empirical β	-0.579	-0.955	-1.238	-1.723	-2.135	-2.621
Model β	-0.020	-0.339	-0.652	-1.274	-1.865	-2.492
Standard Error	0.378	0.456	0.519	0.628	0.706	0.860
p-val for $H0 : \beta(\theta) = \beta_{CS}$	0.069	0.088	0.129	0.237	0.351	0.440
p-val for $H0 : \beta(\theta) = 1$	0.003	0.002	0.001	0.000	0.000	0.000

Joint Test (All Maturities) $H0 : \beta(\theta) = \beta_{CS}$ *p-Value* $= 0.14$

Joint Test (All Maturities) $H0 : \beta(\theta) = 1$ *p-Value* $= 0.002$

Table ix: Conditional Volatility

Panel A: An Asymptotic GMM Test

We test the correct specification of the model implied conditional volatility. Given the closed-form model solution for the second non-central moment of yield changes, denoted as $\mathcal{M}^V(Y_t, g_{it}, \theta)$, we construct a GMM test based on the following moment conditions

$$h_{t+\Delta t} = (\Delta y_t^\tau)^2 - \mathcal{M}^V(Y_t, g_{it}, \theta)$$

From which we construct the following Chi-square statistics:

$$d_T = \left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right]' W_T^{-1}\left[\frac{1}{\sqrt{T}}\sum_{t=1}^{T} h_{t+\Delta t}\right]$$

	3m	6m	1y	2y	3y	5y	7y	10y
d_T	0.808	1.249	1.601	1.749	1.921	2.078	2.005	1.764
$p-value$	0.369	0.264	0.206	0.186	0.166	0.149	0.157	0.184

Joint Test (All Maturities) *p-Value* $= 0.21$

Panel B: Predictive Power

This table shows how well the model-implied conditional volatility of yield changes predicts the future realized volatility. We solve for the second moment implied by the model and run the following regression

$$(\Delta y_{t+\Delta t}^n - E_t[\Delta y_{t+\Delta t}^n])^2 = \alpha + \beta \times \Phi_t + \varepsilon_{t+\Delta t}$$

where Φ_t is the *Model-Implied Conditional Second Moment of* Δy. We test the null hypothesis that $H_0 : \alpha = 0$ and $H_0 : \beta = 1$. The p-value for $H_0 : \alpha = 0$ are given in parenthesis under the respective values for α. The p-values for $H_0 : \beta = 1$ are given in the last row before the R^2.

	3m	6m	1y	2y	3y	5y	7y	10y
α	-0.003	-0.003	-0.002	-0.001	-0.001	-0.000	-0.000	0.000
β	1.202	1.142	1.046	1.004	0.971	0.896	0.828	0.771
R^2	0.259	0.237	0.204	0.189	0.182	0.158	0.133	0.110
$P-value, H0 : \beta = 1$	0.209	0.382	0.780	0.980	0.858	0.526	0.303	0.186
$P-value, H0 : \alpha = 0$	0.002	0.003	0.011	0.035	0.052	0.215	0.594	0.695

Joint Test (All Maturities) $H0 : \beta = 1$ *p-Value* $= 0.42$
Joint Test (All Maturities) $H0 : \alpha = 0$ *p-Value* $= 0.07$

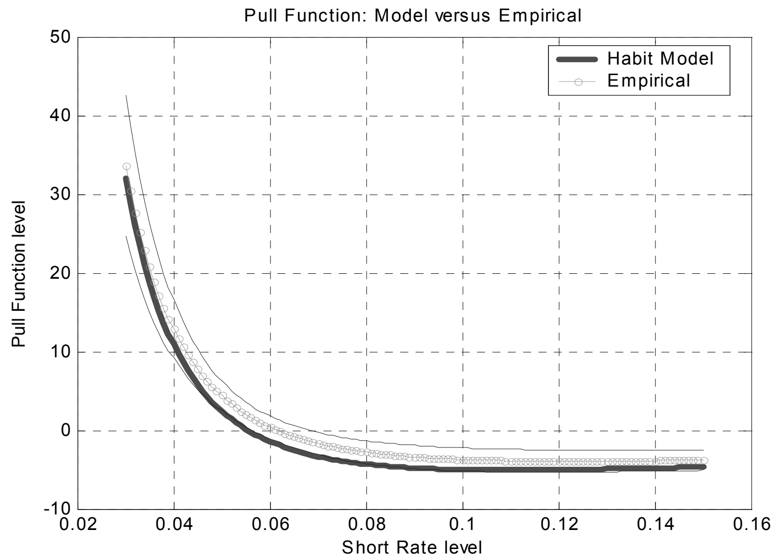


FIGURE 1: **Pull Function**

The plots shows the empirical pull function based on the flexible semiparametric approach by Conley et. al. (1997), under the a flexible parametrization of the local volatility $\sigma(r_t) = \sum_{i=0}^{\gamma} \sigma_i r_t^i$, and the habit model-implied pull function. The two thin solid lines are the 95% confidence bounds around the empirical functions.

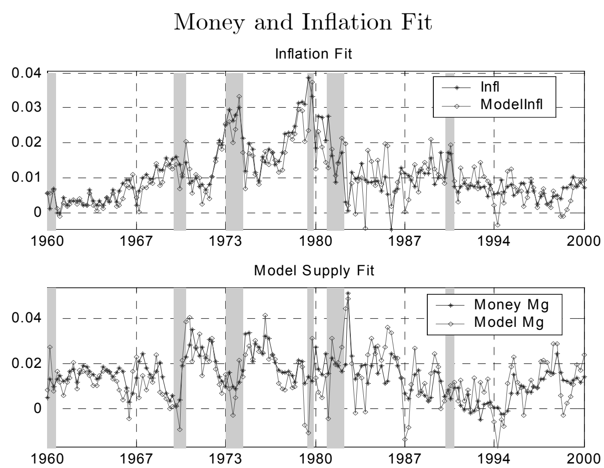
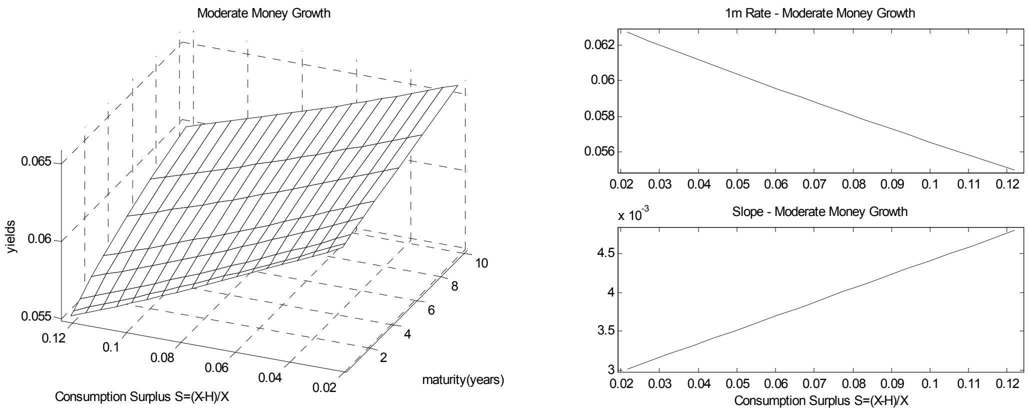


FIGURE 2: **Quality of Fit for Macro Variables**

This figure plots the model-implied time series of inflation and money supply together with their empirical counterparts. Gray boxes on the graph show the periods of US recessions compiled and reported by NBER.

Panel A: Historical Average Money Growth



Panel B: High Money Growth

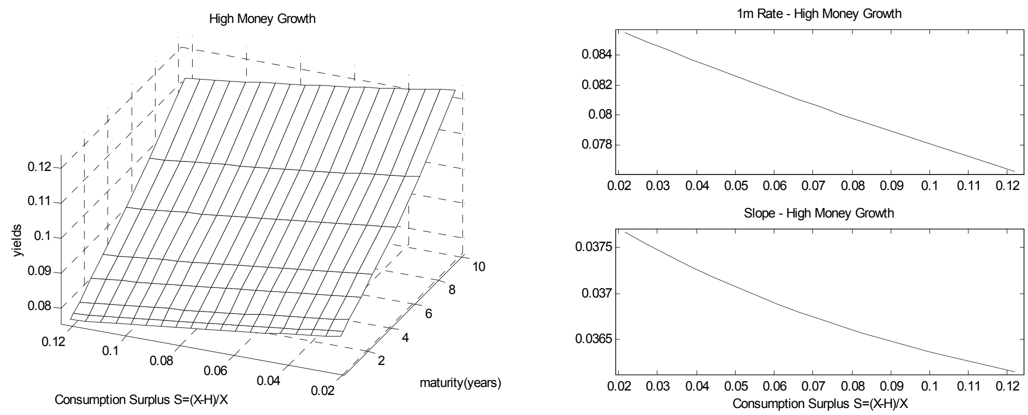


FIGURE 3: **Sensitivity of Bond Yields to Habit Level**

This figure illustrates the bond yields sensitivity to the inverse of the surplus ratio, i.e. Y_t. We consider two regimes, moderate money growth (historical mean value, Panel A) and high money growth (2 standard deviations higher than the historical mean, Panel B).

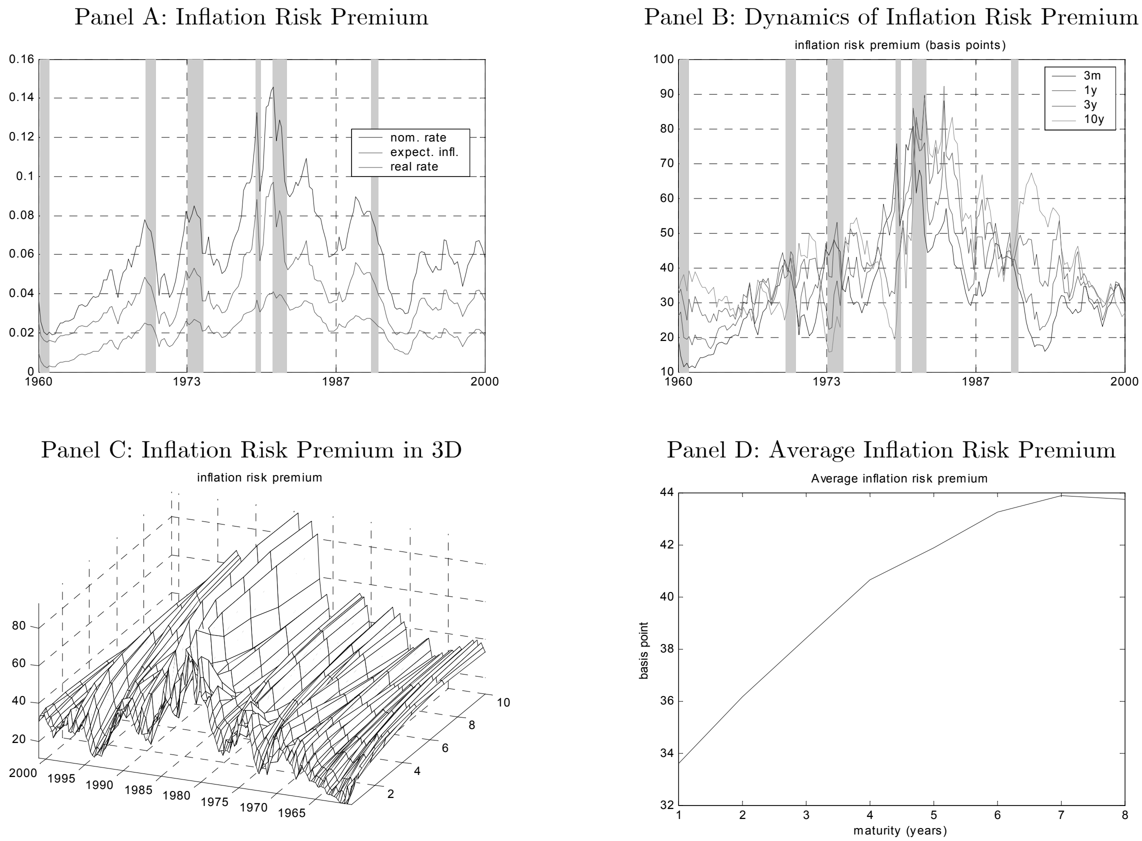


FIGURE 4: **Inflation Risk Premium**

Panel A plots the inflation risk premium over time with respect to nominal and realized real interest rates. Panel B shows the time variation of the inflation risk premium for different time horizons. U.S. recession periods are marked as gray boxes. Panel C shows the three dimensional evolution of the term structure of inflation risk premia. Panel D shows the average term structure of inflation risk premia over the entire sample.