A Simple Model of Dynamic Public Goods Contribution*

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Abstract

This paper addresses the question of individual provision of public goods in a repeated setting. We consider an environment, in which a group of infinitely lived individuals faces a new problem to solve in each period. A particular group member might be able to solve a problem of one period but not of the other. This framework is applicable to numerous economic phenomena such as open-source software developing, online review writing, problem solving in teams, among others. These phenomena share a public good nature, i.e., all group members equally benefit from the solved problem while a cost is carried by a member who provides the solution. We characterize a Markov equilibrium with cooperation, in which group members able to solve the problem choose to do so. We find that increasing the group size has non-monotonic effect on individual incentives to cooperate. When the group gets very large, the benefit from having large scale is offset by members' free riding.

1 Introduction

A number of activities which can be performed by a single individual have a public good nature. Information revealing, knowledge building, problem solving, idea generating in

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organizations and teams are some of those. Indeed, consider a group of individuals facing a particular problem or requiring a specific piece of information. Some members of the group might be able to solve the problem or might have the required information. If just one of those reveals the solution or the necessary piece of information then the entire group enjoys the outcome. That individual therefore provides a public good for his peers. Obviously, a potential contributor has incentives to withhold the solution or information and not to carry a cost of revealing it.

Note moreover that there is usually a dynamic component in such environments. Within organizations, work teams face new problems or request new information every period. And different skills, expertise and ability are required to solve any new problem. So a particular team member might be able to solve a problem, or might have a necessary piece of information, in one period but not in the other. So he might want to costly reveal the solution when he has one in the hope that someone else reveals theirs when he doesn't have one. However, given that in each period there are several individuals able to solve the problem, he might want to save a cost of revealing the solution when he has one in the hope that someone else reveals anyway in this period, and then he is safe in the eyes of a next period "contributors". It follows therefore that a free-riding problem arises here. Then the important questions are the following. Can cooperation be sustained when team members are sufficiently patient? Do individuals have more or less incentives to free ride when the group size expands?

While a number of studies have formally analyzed free-riding problems in dynamic settings, we are not aware of any paper, which studies individually provided public goods in dynamic framework. The present paper contributes to this line. In particular, we build a repeated setting, in which every period a team incurs a problem. A particular member of the team might have required ability and expertise to solve a current period problem, and this is his private information. In other words, every individual knows if he has necessary skills to resolve the problem but doesn't know who else might have this ability too. If at least one member with this ability reveals the solution all team members equally benefit from the outcome. Providing the solution implies a certain cost for a provider.

We restrict the members' strategies to be Markovian with one period recall. In other words, the members condition their actions on the previous period outcome, i.e., either the problem is solved or not. Our goal is to characterize the equilibria with cooperation, in which once a member has the solution he shares it with his peers. We consider the following trigger strategy: an individual cooperates if the problem in the previous period is solved but punishes his peers (i.e., doesn't reveal the solution when he has one) for several period if the problem is not solved in the previous period. We determine the conditions under which there is an equilibrium with cooperation with every member revealing the solution once he gets one.

We turn then to a further analysis of an equilibrium with cooperation for parameter values where such equilibrium exists. Our objective is to analyze individual incentives for cooperation as the group size grows. Intuitively, there are two forces at work in this case. On the one hand, the bigger the team the more members are able to solve the problem in each period and, therefore, the more chances for the problem to be solved. It follows that the continuation payoff is larger and the members are more likely to cooperate, which creates a "large-scale" effect. On the other hand, the bigger the team the more members are able to solve the problem in each period and, therefore, the more chances to be undetected once deviating (since someone else is likely to reveal the solution). It implies that the members are more likely to deviate due to this "free-riding" effect. We find that the former effect dominates in smaller teams while the latter in larger teams. So the members of small teams have more incentives to cooperate as the team size grows. However, members of big teams have more incentives to shirk as the team expands. The latter finding is in line with Olson's (1965) argument that smaller groups may be more cohesive than larger groups. Our contribution here is to emphasize that in the case of individually provided public goods this argument holds for groups of medium and large size, in which "free-riding" effect offsets "large-scale" effect. In very small groups, new members are potential contributors to individually provided public good in the coming periods and, therefore, considerably increase the continuation payoff and incentives to cooperate.

The results of this paper are related to the literature on dynamic public good games (Fershtman and Nitzan 1991, Battaglini and Harstad 2012, Battaglini et al. 2012a, 2012b, Harstad 2012, among some others). TO MENTION MORE PAPERS This literature analyzes the Markov equilibria of dynamic free-riding games for different collective decision settings. The contribution of this paper is to study the problem of individually provided public goods, which has not been addressed in the literature to the best of our knowledge.

The rest of the paper is organized as follows. Section 2 outlines a model. Section 3 conducts the analysis of a stage game. Section 4 studies the repeated setting. Section 5 discusses several extensions of the baseline model. Finally, Section 6 concludes the paper.

2 The Model

Time is discrete and continues forever. Consider a group of N risk-neutral individuals endowed with discount factor $\delta \in (0, 1)$. In each period $t = 0, 1, 2, ...$, there is a chance for individuals to make contribution to public goods. If the contribution is made, each contributor receives private benefit a and each member of the rest of the group receives private benefit 1. Making contribution requires cost $c > a$ and so a contributor's private benefit is not large enough to compensate his cost. There are two elements determining whether contribution will take place. First, in each period, each group member receives a personal shock, which is i.i.d. across members and across periods. With probability $q \in (0, 1)$, a member is able to make the contribution, and with probability $1 - q$, a member is unable to make the contribution. We call those members who are able to contribute informed members and those who are unable to contribute uninformed members. Second, whether the public goods are eventually contributed also depends on the informed members' willingness to contribute. For simplicity, we assume that the contribution is anonymous and the group members only observe whether contribution takes place and potentially how many people make contribution in period t $=, 0, 1, 2, ...$, but not the the identity of a contributor. Furthermore, we rule out the possibility that group members use contracting or side payments to coordinate contributions.[1]

The timing of events within one period is as follows. First, informed members are randomly drawn by nature and each group member knows whether they themselves can contribute or not but not other members' ability to contribute. Second, informed members simultaneously and independently decide whether to contribute to the public goods. Finally, all members observe the outcome and enjoy their private benefits if contribution has taken place.

Many public goods problems can fit into our framework. We provide the following examples to help illustrating the potential applications.

Example 1 (Online Reviews) *In each period, there is a new restaurant available for visiting. Ex-ante, the quality of being good is* p *and being bad is* $1-p$*. If the quality is good, each consumer gains benefit* 1 *from visiting. If the quality is bad, each consumer who visits incurs a loss* -1*. Some consumers by accident(with probability* q*) sample the restaurant and thus become informed about the quality. Thus,* $a=0$*. They can choose whether to post a review about the restaurant quality by incurring cost* c*.*

Example 2 (Open Source Software) *There is a preexisting software application, the code of which is open, can potentially be developed. In each period, each programmer comes up with a idea of how to improve the application, e.g., fixing bugs, improving efficiency, etc, with probability* q*. For those programmers who have the idea, it costs* $c>a=1$ *to make the improvement and share with other members. If the improvement has been made, each member enjoys benefit* 1*.*

[1] Green(1987) studied a general equilibrium model where a continuum of traders have uncertainty in their individual endowment. He characterized the optimal incentive-compatible allocation and show that this allocation can be supported by bond exchange between the trades and an intermediary.

Example 3 (Team of Specialists) *There is a team consisting N members. Each member has privately-known speciality or knowledge in solving certain kinds of tasks. In each period, a random task comes in. With probability q, a member is bale to solve the problem at cost $c > a = 1$. If the problem is successfully solved, each team member gets benefit 1.*

Several features in our set-up are worth emphasizing. First, each member's contributions are *strict* substitutes within a period in the sense that if a member has already contributed, there is no additional public value another member can add by contributing. This is true if the public goods contribution is in the form of fine information provision or problem solving. If the information has already been provided, providing the same information again does not benefit anyone. If a problem has already been solved, there is no point to work on it anymore. There are other public goods problems where individual contributions are weak substitutes or even complements, but these are not the focus in the current paper. Second, informed members do not gain any benefit as a result of social effects such as social image, moral satisfaction or joy of giving. These effects can easily be included into our framework. However, our main object is to show cooperation among group members can emerge through a novel yet fully rational channel. Finally, we assume that the cost of contributing to the public goods exceeds a contributor's private benefit. The opposite case has been well analyzed in the literature by using static models. The focus of those papers is typically on the free-riding problem and the equilibria involve players using mixed strategy. In our dynamic version, we show that cooperation can emerge even when individual incurs current loss by contributing and free-riding problem is still present.

We turn now to the analysis of a stage game.

3 Efficiency and Equilibrium in Stage Game

The first-best outcome in a single period will be that one informed member makes the contribution as long as $N_t \geq 1$ and

$$N - 1 \geq c - a. \tag{1}$$

It is obviously inefficient if more than one informed member makes contribution as the public goods are strict substitutes. However, the first best in which just one informed member contributes is not achievable due to the lack of coordination among group members. in this section, we therefore only consider the second best in which every group member commits to contributing once he is able to do so. We will consider the possibility of using *public randomization device* to improve efficiency in the later discussion.

Denote by N_t the realized number of informed members in period t. Assume that all informed members contribute to the public goods. Then the total surplus for uninformed members is $N - N_t$. The *ex-ante* total surplus is

$$\mathrm{W} = N\left[1 - (1-q)^N - q(1-a+c)\right]. \tag{2}$$

(See Appendix E for details.) Given the lack of coordination among informed members, we use the following efficiency criterion. Can having all informed members commit to contribute generate a positive surplus? This criterion is met if and only if

$$\frac{1-(1-q)^N}{q} \geq 1 - a + c. \tag{3}$$

Rearrange (3) to

$$(1-q)\left[1-(1-q)^{N-1}\right] \geq q(c-a). \tag{4}$$

The L.H.S. of (4) is a member's expected benefit of being an uninformed member, while the

R.H.S. of (4) is the expected net cost of being an informed member. The efficiency criterion requires the former to be greater than the later. In what follows, we assume that condition (3) is satisfied, and focus on the group members' incentives to cooperate so that the efficiency criterion can be fulfilled by the equilibrium play.

Now consider a one-shot game, which replicates one-period events of the benchmark model. It is straightforward to show that in a one-shot game, no informed members chooses to contribute to the public goods since there the cost of contributing exceeds the private benefit, $c > a$, and he does not gain any future payoff. There is then a unique equilibrium in which all informed members (if any) choose not to contribute. The stage total surplus is zero.

4 Dynamic Public Goods Contribution

We turn now to a repeated version of the stage game. In a repeated setting, the group members can condition their actions on the history of the game. Every member knows whether or not contribution took place in the previous period. However, this is an imperfect signal about his peers' previous actions. Indeed, if no public goods was contributed then it might be due either to the informed members' decision to shirk or to the lack of informed members. In general, the group members can condition their actions on the public history, private history and his private information in the current period.

The prescribed equilibrium (pure) strategy is similar to the one in Green and Porter(1983) in which the play is divided into cooperating phase and punishing phase. Each informed member makes contribution in the cooperating phase and does not make contribution in the punishing phase. The punishment will last for T periods and the members switch back to cooperation afterwards. We use the state variable w_t to denote the state in period t. $w_t = 0$ if period t belongs to punishing phase while $w_t = 1$ if t belongs to cooperating phase. At the end of each period $t = 0, 1, 2......$, all members observe whether the public goods have been contributed in period t. This observation serves as a public signal. We thus define the following binary signal space: $\mathbf{Y} = \{\underline{y}, \overline{y}\}$, and $y_t = \overline{y}$ if $w_{t-1} = 1$ and the contribution is made in period $t-1$; $y_t = \underline{y}$ either if $w_{t-1} = 1$ but the contribution is not made in period $t-1$, or if $w_{t-j-1} = 0$ for all $j = 1, 2, ...T$ and $w_{t-T-2} = 1$.[2] Then, the public history is $Y_t = (y_0, y_1, ...y_{t-1})$ and the set of all public histories is $\mathcal{H} = \bigcup_{t=0}^{\infty} Y_t$. The strategies we consider throughout this article, as standard in the literature of repeated games with imperfect public monitoring, are restricted to those public strategies. That is, a member's strategy only depends on the public history, but not his private history.[3] The transition between states can be summarized as follows:

- $w_t = 1$ when $t = 0$.
- If $w_{t-1} = 1$ and $y_t = \overline{y}$, $w_t = 1$. If $w_{t-1} = 1$ and $y_t = \underline{y}$, $w_t = 0$.
- If $w_{t-1} = 0$ and $y_t = \overline{y}$, $w_t = 1$. If $w_{t-1} = 0$ and $y_t = \underline{y}$, $w_t = 0$.

The following automaton represents the proposed equilibrium strategy.

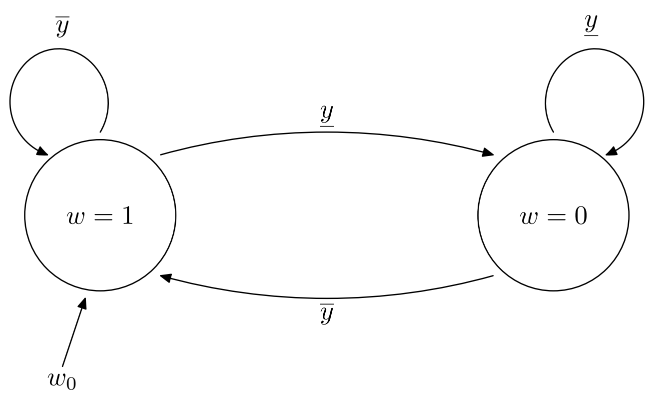

Figure 1: The automaton representing the strategy profile.

Consider an informed member in period t. We check first his incentive to deviate in the punishment phase, i.e., when $w_t = 0$. If he contributes to the public goods in the

[2]If the economy is very large, the number of contributors may correctly reflect the number of informed members. In section 5.2, we discuss this situation by fixing the number of informed members.

[3]It is well known that if all players other than i is playing a public strategy, then player i has a public strategy as a best reply.

punishment phase then the subsequent play will not be affected but he will incur a loss $a - c$ by contributing. This deviation is therefore unprofitable. It follows that an informed member has no incentive to deviate in the punishment phase.

We turn next to an informed member's incentive to deviate in the cooperation phase, i.e., when $w_t = 1$. If he chooses to contribute then his expected payoff is

$$a - c + \delta V\left(w_{t+1}\right) = a - c + \delta V\left(1\right), \tag{5}$$

where $V\left(\cdot\right)$ is the members' value function defined at the beginning of each phase, before members learn whether they are able to contribute or not. Consider now the case in which the informed member deviates and chooses not to contribute. If there are other informed members in the group, those will cooperate and the public goods will be contributed anyway, i.e., $w_{t+1} = 1$. If there are no other informed members then the public provision will be absent, i.e., $w_{t+1} = 0$. Denote by α the probability that the informed member under consideration is a single informed member in the group. This probability is equal to

$$\alpha = \left(1 - q\right)^{N-1}.$$

Then his expected payoff from deviating is given by

$$\delta\left[\left(1 - \alpha\right) V\left(1\right) + \alpha V\left(0\right)\right]. \tag{6}$$

So the informed member has no incentive to deviate in the cooperation phase if his expected payoff from contributing (5) is higher than that from shirking (6) which yields

$$\delta \alpha \left(V\left(1\right) - V\left(0\right)\right) \geq c - a. \tag{7}$$

Consider now the member's value function $V\left(\cdot\right)$. In period t, the member gets informed with probability q and stays uninformed with probability $1-q$. If $w_t = 1$ (i.e., contribution was made in period $t-1$) and the member gets informed then following the aforementioned strategy he contributes in period t, and therefore $w_{t+1} = 1$. If $w_t = 1$ but this member stays uninformed in period t then contribution might still be made in period t by other informed members in case there are any (i.e., with probability $1-\alpha$). Then the member under consideration will receive private benefit of 1, and the next period state will be $w_{t+1} = 1$. However, with probability α no other group member gets informed, and so the public goods are not provided in the current period and $w_{t+1} = 0$. It follows that the value function in state $w_t = 1$ is recursively defined by

$$V(1) = q(a - c + \delta V(1)) + (1-q)\left[(1-\alpha)(1 + \delta V(1)) + \alpha\delta V(0)\right]. \tag{8}$$

Suppose now that the public goods were not supplied in period $t-1$, and the current state is therefore $w_t = 0$. Then the game enters the punishment phase, and no contribution will be made for T periods. The group members will not contribute even when they are informed. After T-period punishment, the game returns to the cooperation phase. Thus, the value function in state $w_t = 0$ is recursively defined as

$$V(0) = \sum_{\tau=0}^{T-1} \delta^{\tau} \cdot 0 + \delta^T V(1) = \delta^T V(1). \tag{9}$$

Substituting (9) into (8) yields the value function in state $w_t = 1$:

$$V(1) = \frac{(1-q)(1-\alpha) - q(c-a)}{1 - \delta q - \delta(1-q)(1 - \alpha + \delta^T \alpha)}. \tag{10}$$

Therefore, the non-deviation condition of the cooperation phase (20) simplifies to

$$\delta\left(1-\delta^T\right) \underbrace{\alpha}_{\substack{\text{free-riding}\\ \text{effect}}} \underbrace{V(1)}_{\substack{\text{large-scale}\\ \text{effect}}} \geq c - a, \tag{11}$$

with $V(1)$ given in (10). Note that the probability that the active player under consideration is a single informed member in the group, $\alpha = (1-q)^{N-1}$, decreases with group size N. For fixed $V(1)$, the larger group size N the harder the non-deviation condition (11) is to hold. Indeed, for large groups, the chance of having multiple informed members is high. Then, an unilateral deviation may well pass undetected and therefore unpunished as there are other informed members cooperating. This is a typical "*free-riding effect*" in the team problem. However, $V(1)$ is not constant in group size N. From (10), it is easy to check that $V(1)$ is decreasing in α and thus increasing in N. The continuation payoff $V(1)$ measures the value of being cooperative. As the group size grows, this continuation payoff tends to increase. Intuitively, in larger groups, the probability that contribution takes place in future periods is higher. This implies a higher expected payoff in periods where a group member is unable to contribute and relies on his peers to contribute. Thus, the larger the group size the more incentives the members have to cooperate. This effect refereed to as a "*large-scale effect*" arises in our framework because in each period the ability to provide a public good shifts from some group members to others. If all group members could provide a public good in all periods, this "*large-scale effect*" would not exist.

Substituting (10) into (11) and simplifying yields

$$\alpha \left[\frac{(1-q)(1-\alpha)}{c-a} - 1 \right] \geq \left(\frac{1}{\delta} - 1 \right) \frac{1}{1-\delta^T}.$$

Note that the left-hand side of this non-deviation condition is constant in discount factor δ while the right-hand side is decreasing from positive infinity to $1/T$ over the range of δ. So we get the following result. (Proofs of this and other propositions are given in the Appendix.)

Proposition 1 *A symmetric Public Perfect Equilibrium (PPE) in which informed members commit to contribute can be sustained if discount factor δ exceeds a critical value $\overline{\delta}$. The* *threshold $\overline{\delta}$ exists if and only if the following condition holds:*

$$\alpha\left[\frac{(1-q)(1-\alpha)}{c-a}-1\right]\geq\frac{1}{T}. \tag{12}$$

If the left-hand side of (12) is positive then we can always find a large enough T such that condition (12) is satisfied. Therefore, the necessary and sufficient condition for the existence of such T (and so the existence of an equilibrium with cooperation) is

$$(1-q)(1-\alpha)\geq c-a. \tag{13}$$

We compare now this condition with the efficiency condition, $(1-q)(1-\alpha) \geq q(c-a)$, derived in Section 3. Note that $(1-q)(1-\alpha)/q \geq (1-q)(1-\alpha)$ for all $q \in (0,1)$ and therefore for some parameter values (in particular, for $(1-q)(1-\alpha) < c-a < (1-q)(1-\alpha)/q$) the efficiency condition holds while the existence condition does not hold. Thus, the efficiency may not be always achieved in our framework. The trigger strategy may not provide enough incentive for pioneers to choose to cooperate even if they are patient and the punishment phase is arbitrarily long. Indeed, in the equilibrium, a member's periodic expected payoff $(1-q)$ is discounted by the probability of having multiple informed members in the group, $1-\alpha$.

We are interested in the group-size effect, that is how the change of N affects the difficulty of sustaining cooperation. Previous literature, both theoretical and empirical, found ambiguous effect caused by large group size. The negative results were believed to be caused by the classical free-riding problem, while the positive results were often attributed to some behavioral elements such as social image or joy of giving which increase with the number of recipients. As a complementary contribution, our fully rational framework provides a unified explanation for both positive and negative group-size effect.

Proposition 2 *Group-size effect is positive in small groups but negative in large groups in the following sense:*

- *When N is too small or too large, a cooperating equilibrium may not exist.*
- *When N is too small or too large, the cooperation-required discount factor is larger.*

Obviously, when group gets bigger, each informed member has larger incentive to choose to a free rider, as their deviation is harder to be detected in such a large group. Also, the incentives for cooperation are provided to informed members through intertemporal rewarding of cooperative behavior. This channel can work well only when the possibility of existing informed members in the future period is high, which is increasing with the group size. Thus, both large-scale effect and free-riding effect are amplified in large groups. While large-scale effect is good for cooperation, free-riding effect destroys members' incentive to write reviews. Proposition (2) shows that the large-scale effect dominates in the small group while free-riding effect dominates in the large group.

5 Extensions

5.1 Public Randomization Device

In our basic model, the individual contributions are *strict* substitutes. Every informed member's contribution is identical and there is no additional social benefit that an informed member can add, if at least one other informed member has already contributed. Obviously, this gives rise to the inefficiency problem when all members try to cooperate but cannot coordinate. In this section, we consider the use of a public randomization device with which the inefficiency can be alleviated.

Definition 1 *Suppose that M informed members commit to contribute in period t. A Public Randomization Device(PRD) is a lottery under which, with probability $1/M$, only one of the M informed members will be randomly selected to make actual contribution.*

Under PRD, at most one informed member makes actual contribution. For example, in the online-review problem, if a consumer wants to contribute a review but when he log in the review site, he finds another review with the same information has already been posted. Then, there is no need for this consumer to write another review. However, who will be the very first to post a review is randomly determined.

Suppose a PRD is used. Now, whenever there is at least one informed member in period t and all of them follow the cooperating strategy, the periodic total surplus is $(N-1)+(a-c)$. The state variable is simply $w_t \in \{0,1\}$ as defined earlier. When every informed member commits to contribute in period t, an individual informed member contributes with probability 1 if he is the only informed member, contributes with probability 1/2 if there are two informed members in total, contributes with probability 1/3 if there are three informed members in total, and so on so forth. Anticipating all other informed members commit to contribute, an individual informed member's expected payoff of making contribution is

$$-\sum_{N_t=1}^{N}\left[\binom{N-1}{N_t-1} q^{N_t-1}(1-q)^{N-N_t+1}\left(\frac{c-a}{N_t}\right)\right]+\delta V(1) = -\underbrace{(c-a)\left[\frac{1-(1-q)\alpha}{Nq}\right]}_{\text{current loss}} + \underbrace{\delta V(1)}_{\text{continuation payoff}}. \tag{14}$$

Lemma 1 *The current loss is strictly decreasing in N.*

If an individual informed member chooses not to contribute, the expected payoff is

$$\delta[(1-\alpha)V(1)+\alpha V(0)]. \tag{15}$$

The value functions can be recursively defined as follows

$$V(1) = q\left[-(c-a)\left(\frac{1-(1-q)\alpha}{Nq}\right)+\delta V(1)\right]+(1-q)\Big[(1-\alpha)(1+\delta V(1))+\alpha\delta V(0)\Big] \tag{16}$$

$$V(0) = \delta^T V(1). \tag{17}$$

Substitute (17) into (16), we can solve for $V(1)$.

$$V(1) = \frac{(1-q)(1-\alpha) - [(c-a)(1-(1-q)\alpha)/N]}{1-\delta q - \delta(1-q)\left(1-\alpha+\alpha\delta^T\right)}. \tag{18}$$

Note that α decrease in N. In the expression of (18), the first term in the numerator strictly increases in N. Also, Lemma 1 indicates the second term in the numerator strictly decrease in N. Finally, provided that $\delta^T < 1$, the denominator strictly decreases in N. Thus, $V(1)$ strictly increases in N.

To sustain cooperation, any agent should not have incentive to deviate to shirk whenever he becomes informed. This requires

$$-(c-a)\left[\frac{1-(1-q)\alpha}{Nq}\right] + \delta V(1) \geq \delta[(1-\alpha)V(1) + \alpha V(0)], \tag{19}$$

or equivalently,

$$(c-a)\underbrace{\left[\frac{1-(1-q)^N}{Nq}\right]}_{\substack{\text{current large-scale}\\ \text{effect}}} \leq \delta(1-\delta^T)\underbrace{(1-q)^{N-1}}_{\substack{\text{free-riding}\\ \text{effect}}} \quad \underbrace{V(1)}_{\substack{\text{continuation large-scale}\\ \text{effect}}} \quad . \tag{20}$$

We now provide some comparative statics regarding these three effects.

	$q\uparrow$	$N\uparrow$	$q\to 0$	$q\to 1$	$N\to 1$	$N\to\infty$
current large-scale effect	$\downarrow$	$\downarrow$	1	$\frac{1}{N}$	c	0
continuation large-scale effect	?	$\uparrow$	0	$\frac{c}{N(1-\delta)}$	$\frac{-cq}{1-\delta q-\delta^{T+1}(1-q)}$	$\frac{(1-q)(1-s)p}{1-\delta}$
free-riding effect	$\downarrow$	$\downarrow$	1	0	1	0

Table 1: Comparative Statics.

It is immediate from the above table that the cooperation cannot be sustained in equilibrium if (i)q is too small or too large or (ii)N is too small. The the incentive is provided through intertemporal channel. An informed member can get benefit from cooperating in the current period only when there is some chance that, in some future period, he is uninformed but there exist other informed members. He is not likely to be uninformed when q is too large and he is not likely to obtain help from other members if q is too small. From Table 1, we can see that a too big N causes huge free-riding problem, but might be offset by the current large-scales effect. However, when N is too small, the continuation large-scale effect is too small to provide enough incentive to cooperate.

The cooperating equilibrium emerges if the expected payoff in (14) is higher than the one in (15), which is equivalent to

$$\left(\frac{1}{\delta}-1\right)\left(\frac{1}{1-\delta^T}\right) \leq \alpha\left[\frac{Nq(1-q)(1-\alpha)}{(c-a)(1-(1-q)\alpha)}-1\right]. \tag{21}$$

This condition holds only when players are patient enough.

Proposition 3 *Suppose the following condition holds.*

$$1/T \leq \alpha\left[\frac{Nq(1-q)(1-\alpha)}{(c-a)(1-(1-q)\alpha)}-1\right].$$

Then, there always exists a critical discount factor $\overline{\delta}$ so that cooperating equilibrium can be sustained for any $\delta \geq \overline{\delta}$.

The proof of Proposition 3 is very similar to the proof of Proposition 1, so we skip it. Since T can be arbitrarily large, to have (3) satisfied and so $\overline{\delta}$ exists, it is sufficient and necessary to have

$$\frac{Nq(1-q)(1-\alpha)}{1-(1-q)\alpha} \geq c-a. \tag{22}$$

We can compare this condition with the condition under first-best benchmark, second-best benchmark and the equilibrium without PRD.

Proposition 4 *We obtain the following properties regarding efficiency when PRD is present:*

- *PRD facilitates cooperation, compared to the no-coordination case.*
- *The cooperative equilibrium under PRD achieves better efficiency than the second-best benchmark when q is relatively large.*
- *The first-best benchmark is not always achieved with PRD.*

Finally, we show that the non-monotonic group-size effect can still hold when PRD is used. This result basically requires the R.H.S of (21) is non-monotonic. We plot the R.H.S of (21) when $c - a = 0.1$ and $q = 0.2$ to help illustrate.

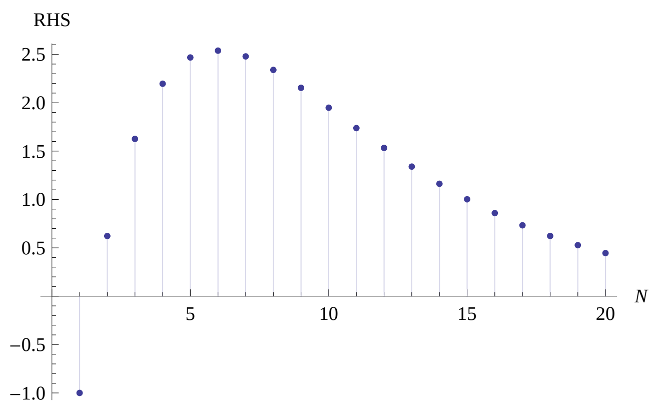

Figure 2: Group-size effect.

5.2 Fixed Number of informed members

In some situations, a more reasonable assumption regarding the number of informed members is that it is proportional to the total number of group members and constant over time. But the identity of informed members is still i.i.d. across periods. This assumption excludes the possibility of having no informed members in the group. We still denote the group size as N. The number of informed members is now assumed to be $\tilde{N} = N/k$ where k denotes the ration of the number informed members to the number of all players.

Let us start by considering the case where $\tilde{N} = 1$. The prescribed strategy is now the grim trigger strategy under which whenever deviation happens, all players return to the static Nash play for ever. The informed member's payoff from cooperating is

$$a - c + \delta V(1), \tag{23}$$

while the payoff from shirking is

$$\delta V(0). \tag{24}$$

Obviously, free-riding effect is absent in this case as there is one and only one informed member and his deviation will be surely detected. In the cooperating phase, with probability k, a player under consideration becomes informed in period $t+1$ and needs to make contribution, and with probability $1-k$ another player becomes informed and the player under consideration will obtain benefit of 1. The value function at state $w_t = 1$ can be explicitly written as

$$V(1) = k(a - c + \delta V(1)) + (1-k)(1 + \delta V(1)). \tag{25}$$

$V(1)$ can be easily solved as

$$V(1) = \frac{1 - k + k(a-c)}{1-\delta}. \tag{26}$$

The value function at state $w_t = 0$ is simply

$$V(0) = 0. \tag{27}$$

Thus, cooperation can be sustained as an equilibrium outcome if and only if

$$a - c + \delta V(1) \geq \delta v(0). \quad \Leftrightarrow \quad \frac{\delta(1-k)}{1 - \delta + k\delta} \geq c - a. \tag{28}$$

The L.H.S. of (28) is increasing in δ. The more patient the players, the more likely the cooperation can be sustained. Notice that L.H.S. of (28) is a decreasing function of k while k decreases in N. Thus, when $\tilde{N} = 1$, the group-size effect is always positive.

Now let us consider the case where $\tilde{N} \geq 2$. If the signal space is still restricted to be binary, it is straightforward to show that no cooperation can be sustained in a pure-strategy equilibrium. Intuitively, since an informed members knows he is not the only informed member in the group, his deviation will not be detected, given the other informed members are cooperating. Then, his best response is to shirk. However, if the state space can be extended to $\{1, 2, ..., \tilde{N}\}$ and a punishment will be triggered whenever $y_t \neq \tilde{N}$, the cooperating equilibrium will be restored. In the appendix, we also consider the symmetric mixed strategy equilibrium in the $\tilde{N} \geq 2$ case.

5.3 Heterogeneous Players

6 Conclusion

Appendix

A Proof of Proposition 1

The non-deviation condition is

$$c \leq \frac{\delta(1-\delta^T)[(1-q)(1-\alpha)-q(c-a)]}{1-\delta q-\delta(1-q)(1-\alpha)-\delta^{T+1}(1-q)\alpha}.$$

The denominator can be simplified to $1-\delta+\alpha(1-q)(\delta-\delta^{T+1})$ and thus positive. Multiply both sides of the above condition by the denominator and we get

$$F(\delta;T) \equiv \left(\frac{1}{\delta}-1\right)\left(\frac{1}{1-\delta^T}\right) \leq \alpha\left[\frac{(1-q)(1-\alpha)}{c-a}-1\right]. \tag{29}$$

It is easy to verify that $\lim_{\delta\to 0} F(\delta;T) = +\infty$, $\lim_{\delta\to 1} F(\delta;T) = \frac{1}{T}$ and

$$\frac{\partial F(\delta;T)}{\partial\delta} = \frac{-(1-\delta^T)+T\delta^T(1-\delta)}{\delta^2(1-\delta^T)^2}. \tag{30}$$

The denominator of the fraction in (30) is strictly positive. The sign of the fraction is then solely determined by the sign of the numerator. Note that the numerator is continuous in δ , $\lim_{\delta\to 0} -(1-\delta^T)+T\delta^T(1-\delta) = -1$ and $\lim_{\delta\to 1} -(1-\delta^T)+T\delta^T(1-\delta) = 0$. If the numerator is a strictly increasing function in δ, its value will be strictly negative for all $\delta \in (0,1)$. This is confirmed by

$$\frac{\partial}{\partial\delta}[-(1-\delta^T)+T\delta^T(1-\delta)] = \left(T+T^2\right)\left(\delta^{T-1}-\delta^T\right) > 0.$$

Then, $F(\delta;T)$ strictly decreases from infinity to $1/T$ when δ increases from 0 to 1. Thus, when $\alpha\left[(1-q)(1-\alpha)/(c-a)-1\right] \geq 1/T$, we can always find a unique intersection between $F(\delta;T)$ and $\alpha\left[(1-q)(1-\alpha)/(c-a)-1\right]$ at $\overline{\delta} \in (0,1)$, which is the critical discount factor. *Q.E.D.*

B Proof of Proposition 2

First, a cooperating equilibrium exists if

$$\alpha\left[\frac{(1-q)(1-\alpha)}{c-a}-1\right]\geq F(\overline{\delta};T) \tag{31}$$

When q is fixed, the L.H.S. of (31) increases in α when $\alpha<[1-(c-a)/(1-q)]/2$ and decreases in α when $\alpha>[1-(c-a)/(1-q)]/2$. Since α is a decreasing function of N, we know that the L.H.S. of (31) decreases in N when $\alpha<[1-(c-a)/(1-q)]/2$ and increases in N when $\alpha>[1-(c-a)/(1-q)]/2$. This means the value of the L.H.S. first increases in N and then decreases. The harshest possible punishment that the equilibrium strategy can impose is $T=\infty$ which equals the R.H.S. of (31) to $(1/\delta)-1$. When $N\to 2$, the L.H.S.$\to$ $(q-c+a)(1-q)^2/(c-a)$ which could be either positive or negative negative. When $N\to\infty$, the L.H.S.$\to 0$ which is lower than $1/\delta-1$. Therefore, when N is relatively large or small, a cooperative equilibrium may not exist.

Second, suppose that the cooperative equilibrium exists and so the threshold discount factor $\overline{\delta}$ exists. $\overline{\delta}$ is determined by

$$\alpha\left[\frac{(1-q)(1-\alpha)}{c-a}-1\right]\geq F(\overline{\delta};T) \tag{32}$$

Differentiate both side of (32) w.r.t N, we have

$$F_\delta\frac{d\overline{\delta}}{dN}=\alpha\ln(1-q)\left[\frac{(1-2\alpha)\,(1-q)}{c-a}-1\right]. \tag{33}$$

Since both F_δ and $\alpha\ln(1-q)$ are negative, $d\overline{\delta}/dN{\geq}0$ when $(1-2\alpha)\,(1-q)/(c-a){\geq}1$ and $d\overline{\delta}/dN{<}0$ when $(1-2\alpha)\,(1-q)/(c-a){<}1$. Also notice that $(1-2\alpha)\,(1-q)$ is an increasing function of N. Thus, $d\overline{\delta}/dN$ is negative only when N is relatively small and is positive when N is relatively large. *Q.E.D.*

C Proof of Lemma 1

To prove the current loss is strictly decreasing in N, it suffices to show that $d[(1-(1-q)\alpha)/N]/dN < 0$. The derivative is

$$\frac{d[(1-(1-q)\alpha)/N]}{dN} = \frac{(1-q)^N[1-N\ln(1-q)]}{N^2-1}.$$

To have it negative, we need to show that $(1-q)^N[1-N\ln(1-q)]-1<0$. Observe that $(1-q)^N[1-N\ln(1-q)]$ achieves its maximum when $N=1$(N should take positive integers) because its derivative $-N(1-q)^N(\ln(1-q))^2<0$. Substitute $N=1$ into $(1-q)^N[1-N\ln(1-q)]$, the expression becomes $(1-q)(1-\ln(1-q))$ whose value is strictly between 0 and 1 provided that $q\in(0,1)$. Thus, $(1-q)^N[1-N\ln(1-q)]-1<0$ and the current loss strictly decreases in N. *Q.E.D.*

D Proof of Proposition 4

First, we compare (22) with (13). Note that $Nq/[1-(1-q)\alpha]$ is an increasing function of q and $\lim_{q\to 0} Nq/[1-(1-q)\alpha]=1$. Thus the L.H.S. of (22) is always larger than the L.H.S. of (13), which means whenever condition in (13) is satisfied, the condition in (22) is automatically satisfied.

Second, we compare (22) with the second-best benchmark, (4). Move the q on the R.H.S. of (4) to the L.H.S., so we get

$$\frac{(1-q)(1-\alpha)}{q} \geq c-a. \tag{34}$$

Note that $1/q$ is strictly decreasing from infinity at $q=0$ to 1 at $q=1$, while $Nq/[1-(1-q)\alpha]$ is strictly increasing from 1 at $q=0$ to infinity at $q=1$. Thus, the L.H.S. of (22) and the L.H.S. of (34) have a unique intersection at some $\overline{q}\in(0,1)$. When $q>\overline{q}$, the L.H.S. of (22) is bigger, (34) implies (22). In fact, since there is at most one informed member is contributing in the cooperating equilibrium with PRD, the efficiency level it achieves is higher than the one in the second-best benchmark.

Finally, we compare (22) with the first-best benchmark, (1). The L.H.S. of (22) can be modified to

$$\frac{Nq(1-q)(1-\alpha)+1-(1-q)\alpha}{1-(1-q)\alpha}-1. \tag{35}$$

Given the fact that

$$\underbrace{N-1+Nq\alpha(1-q)+\alpha(1-q)}_{>0}+\underbrace{Nq(1-q)-N\alpha(1-q)}_{>0}>0 \tag{36}$$

we have the L.H.S. of (22) is strictly smaller than the L.H.S. of (1). Then, with some parameters, the first-best outcome cannot be achieved even by using PRD. *Q.E.D.*

E Total Surplus in Stage Game

When every group member commits to contributing whenever he is able to, the expected total surplus in a single period is given by

$$\mathrm{W}=\sum_{N_t=1}^{N}\left[\begin{pmatrix}N\\N_t\end{pmatrix}q^{N_t}(1-q)^{N-N_t}\left((N-N_t)+N_t(a-c)\right)\right],$$

or equivalently

$$\begin{aligned}\mathrm{W}&=\sum_{N_t=0}^{N}\left[\begin{pmatrix}N\\N_t\end{pmatrix}q^{N_t}(1-q)^{N-N_t}\left((N-N_t)+N_t(a-c)\right)\right]-N(1-q)^N\\&=N\left[1-(1-q)^N\right]-(1-a+c)\sum_{N_t=0}^{N}\left[\begin{pmatrix}N\\N_t\end{pmatrix}q^{N_t}(1-q)^{N-N_t}N_t\right].\end{aligned}$$

Note that $\begin{pmatrix} N \\ N_t \end{pmatrix} N_t$ equals $\begin{pmatrix} N-1 \\ N_t-1 \end{pmatrix} N$ for $N_t \geq 1$. Then the total surplus is

$$\begin{aligned} \mathrm{W} &= N\left[1-(1-q)^N\right] - Nq(1-a+c)\left(\sum_{N_t=1}^{N} \begin{pmatrix} N-1 \\ N_t-1 \end{pmatrix} q^{N_t-1}(1-q)^{N-N_t}\right) \\ &= N\left[1-(1-q)^N - q(1-a+c)\right]. \end{aligned}$$

F Symmetric Mixed-Strategy Equilibrium in the Case of $\tilde{N} \geq 2$

Denote $\sigma \in [0,1]$ the probability that an informed member chooses to contribute in the cooperating phase. Let $\overline{y}$ be a heuristic cut-off value of y so that any $y < \overline{y}$ triggers a punishment, and cooperation continues otherwise. In the cooperating phase, if an informed member contributes, his expected payoff is

$$a - c + \Pr(y_t \geq \overline{y} \mid +)\delta V(1) + (1 - \Pr(y_t \geq \overline{y} \mid +))\,\delta V(0), \tag{37}$$

where

$$\Pr(y_t \geq \overline{y} \mid +) = \sum_{j=\overline{y}-1}^{\tilde{N}-1}\left[\begin{pmatrix} \tilde{N}-1 \\ j \end{pmatrix} \sigma^j (1-\sigma)^{\tilde{N}-1-j}\right]$$

is the probability that $y_t \geq \overline{y}$ conditional on that the informed member under consideration contributes. If this informed member shirks, the expected payoff is

$$\Pr(y_t \geq \overline{y} \mid -)\delta V(1) + (1 - \Pr(y \geq \overline{y} \mid -))\,\delta V(0), \tag{38}$$

where

$$\Pr(y_t \geq \overline{y} \mid -) = \sum_{j=\overline{y}}^{\tilde{N}-1} \left[\begin{pmatrix} \tilde{N}-1 \\ j \end{pmatrix} \sigma^j (1-\sigma)^{\tilde{N}-1-j} \right]$$

is the probability that $w_t \geq \overline{w}$ conditional on that the informed member under consideration shirks. The value function $V(1)$ can be written as

$$\begin{aligned} V(1) =& k\Big[a - c + \Pr(y_t \geq \overline{y} \mid +)\delta V(1) + (1 - \Pr(y_t \geq \overline{y} \mid +))\delta V(0)\Big] \\ &+ (1-k)\Big[\Pr(y_t \geq \overline{y} \mid -)(1 + \delta V(1)) + \Pr(1 \leq y_t < \overline{y} \mid -)(1 + \delta V(0)) \\ &+ \Pr(y_t = 0 \mid -)\delta V(0)\Big]. \end{aligned} \tag{39}$$

The value function $V(0)$ can be written as

$$V(0) = \delta^T V(1). \tag{40}$$

For simplicity, let us consider the most severe punishment so that $T = \infty$ and $V(0) = 0$. We then can solve $V(1)$ as

$$V(1) = \frac{k(a-c) + (1-k)\left(1 - \Pr(y_t = 0 \mid -)\right)}{1 - k\delta\Pr(y_t \geq \overline{y} \mid +) - (1-k)\delta\Pr(y_t \geq \overline{y} \mid -)}. \tag{41}$$

Since we are looking for a symmetric mixed-strategy equilibrium, we must have

$$a - c + \Pr(y_t \geq \overline{y} \mid +)\delta V(1) = \Pr(y_t \geq \overline{y} \mid -)\delta V(1). \tag{42}$$

This gives

$$V(1) = (c-a) \Bigg/ \left(\delta \begin{pmatrix} \tilde{N}-1 \\ \overline{y} \end{pmatrix} \sigma^{\overline{y}} (1-\sigma)^{\tilde{N}-1-\overline{y}} \right) \tag{43}$$

The equilibrium mixed strategy is then solved by equating (41) and (43). Let us denote

$$P = \binom{\tilde{N}-1}{\overline{y}} \sigma^{\overline{y}}(1-\sigma)^{\tilde{N}-1-\overline{y}}.$$

Equating (41) and (43) yields

$$-k = \frac{(1-\delta \Pr(y_t \geq \overline{y} \mid -))}{P} - \frac{\delta(1-k)[1-(1-\sigma)^{\tilde{N}-1}]}{c-a} - \delta. \tag{44}$$

Holding $\tilde{N}$ fixed, let us examine how the change of k affects σ, and thus how the change of N affects σ. Note that $1-(1-\sigma)^{\tilde{N}-1}$ increases in σ. Thus, having both $\Pr(y_t \geq \overline{y} \mid -)$ and P increase in σ is a sufficient condition for $d\sigma/dk > 0$. This will be achieved when $\overline{y} > \tilde{N}/2$ if $\tilde{N}$ is even and $\overline{y} \geq \tilde{N}+1/2$ if $\tilde{N}$ is odd.

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