Does the absence of *competition in markets* foster *competition for the market?* A dynamic approach to aftermarkets

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In this paper, we investigate dynamic price competition when firms strategically interact in two distinct but interrelated markets: a primary market and an aftermarket, where indirect network effects arise. We set up a differential game of two-dimensional price competition and we conclude that the absence of price competition in the aftermarket (competition *in* the market) fosters dynamic price competition in the primary market (competition *for* the market). We also investigate the impact of network sizes on firms' prices in the primary market concluding that, in equilibrium, larger firms have incentives to compete more fiercely for new "uncolonized" consumers.

Key Words: dynamic competition, differential games, Linear Markov Perfect Equilibrium, aftermarkets, network effects

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1. INTRODUCTION

In recent years, well-known competition policy cases such as *Xerox* , *Kodak* or *Chrysler* have raised the problem of incentives to compete in industries where firms are involved in two distinct but interrelated markets: *(i)* primary markets (where firms sell an "*original equipment*" like a printer, an hardware, an helicopter, a car...*);* and *(ii)* aftermarkets (where firms provide complementary goods or services (CGS) such as ink cartridges, software programmes, helicopter spare parts, post-sale services like maintenance or reparation services...).

In these competition policy cases, it has been argued that competition in aftermarkets was practically absent because consumers were significantly *locked in* to equipment's manufacturers and, consequently, equipment manufacturers had the monopoly in the provision of CGS to their installed basis of consumers.

In this paper, we explicitly deal with the interrelations between primary markets and aftermarkets . Assuming that consumers are totally locked in to the equipment's producers (no direct competition in the aftermarket), we raise the following questions: *(i)* to what extent the absence of competition in the aftermarket stimulates competition for new and uncolonized consumers in the primary market? *(ii)* in the absence of competition in aftermarkets, what are the determinants of the intensity of competition in primary markets?

These questions are akin to the ones raised in the literature of standards competition, which (mostly informally) points out the existence of a trade-off between *competition in markets* and *competition for markets* (see Besen and Farrell (1994) or Geroski (2003)). *Competition for markets* is entailed by firms' competition for the rents associated with an uncolonized market (dynamic competition) and it is associated with competition between incompatible standards (at the end, only one standard prevails). Conversely, *competition in markets* corresponds to competition in a well-established market, where several firms have access to the technology behind the prevalent standard (static competition). Hence, there is a trade-off between *competition for markets* and *competition in markets*: intense competition in standards can only take place when firms expect quasi-monopolistic rents for prevalent standard (and these are only possible when competition in markets is weak).

In the case of aftermarkets, an akin distinction may be pointed out. In particular, one might think of competition for new uncolonized consumers in the primary market as a phenomenon of competition *for* the market. And, analogously, competition in the aftermarket, where firms compete in a well installed market could be conceived as a phenomenon of competition *in* the market. In this context, the questions we wish to address in this paper can be reformulated as follows: *(i)* does the absence of competition *in* the market fosters competition *for* the market? *(ii)* in the absence of competition *in* the market, what determines the intensity of competition *for* the market?

Considering the intrinsically dynamic nature of this problem, we deal with the previous questions in the context of a simple differential linear-quadratic game of two-dimensional price competition (in the primary market and the aftermarket) with *proprietary* but *partially compatible indirect* network effects (the utility of consumption in the aftermarket depends on the number of costumers who own the same equipment version and, to a smaller extent, on the number of costumers owning a distinct version of the original equipment). We consider that both consumers and firms are forward looking agents and we derive the Linear Markov Perfect Equilibrium (LMPE) of the linear-quadratic differential game (when it exists). Relying on linear strategies, we analytically compute the solution of the dynamic game and we identify *(i)* consumers' linear Markovian expectation rules; *(ii)* firms' linear Markovian price strategies (subscription fees) and *(iii)* firms' price strategies in the primary market.

Our formulation considers that, in the primary market, firms offer two different versions of a similar equipment, whose *expected life-time benefit per se* is considered to be exogenous and independent of the equipment version[1]. Conversely, in the aftermarket, we assume that firms offer differentiated CGS. In particular, we consider that CGS present different intrinsic characteristics (horizontal differentiation) and, they also exhibit differences in quality (vertical differentiation) entailed by *indirect* network effects. Indeed, indirect network effects constitute a quite frequent phenomenon in industries where firms simultaneously interact in primary markets and aftermarkets. In fact, very often, firms with larger market shares in the primary market benefit from stronger learning experience/ learning economies, which allow them to offer a higher quality service in the aftermarket. For example, in the case of reparation services, when consumers are totally locked in, a firm with a large basis of costumers, intervenes more often in the aftermarket and, consequently, it detects more rapidly the typical failures of its equipment and the better way to deal with these failures. Similarly, the greater the number of costumers of a certain printer, the more likely it will be to find a good post-sale assistance (the equipment manufacturer can offer a larger network of post-sale service providers and, in addition, their technicians become more experienced (on-the job-training). However, quite often (at least) part of these improvements entailed by indirect network effects can be appropriated by the rival firm (diffusion of

[1] This value might be either positive or null. When it is positive, we are implicitly assuming that the equipment has an intrinsic positive value, which means that consumers might use it even when they do not buy complementary goods/ services (e.g. a car, an helicopter have a positive value for consumers even if these don't buy temporary check up services).

When this value is negative, consumers only benefit from using the equipment when this is used together with CGS (e.g. a printer is not useful without cartridges or a computer is not useful without software).

unprotected knowledge). To account for this possibility we consider that network effects are proprietary but partially compatible.

Main findings: Our first main finding is that the absence of competition *in* the market actually fosters competition *for* the market. In particular, we have show that, when consumers are totally locked in to their equipment, the equilibrium prices in the primary markets are lower and, in certain circumstances, they can even be negative. A direct implication of this result is that the absence of competition *in* markets is not necessarily welfare detrimental: it leads firms to invest in price-cutting strategies in primary markets and, as a consequence, it fosters the rhythm of technological diffusion.

Our second main finding concerns the determinants of the intensity of price competition in the primary market (competition *for* the market). Our results reveal that competition in the primary market is more intense, *(i)* the greater the relative size of the aftermarket; *(ii)* the stronger the intensity of indirect network effects; *(iii)* the smaller the degree of compatibility between networks; and *(iv)* the more importance firms attach to future earnings (the lower the effective discount rates).

Finally, we also show that the incentives to compete *for* the market (i.e. set lower prices in the primary market) are positively related with the size of firms' basis of consumers: the larger the firms' basis of consumers, the greater are monopolist rents from the aftermarket and the more advantageous it becomes to invest in price-cutting strategies to capture new consumers in the primary market.

Related literature: Our model shares some features with existing models investigating dynamic price competition in the presence of network effects, namely Cabral (2007), Doganoglu (2003), Mitchell and Skrzypacz (2006) and Laussel et al (2004). Our framework shares many similarities with the one used by Cabral (2007), namely with respect to *(i)* the strategic role played by subscription prices (in our case, equipment prices) on firms' dynamic choices and *(ii)* the forward looking behaviour of economic agents. However, our model differs from Cabral (2007) with respect to the fact that we explicitly model firms' behaviour in the aftermarket (considering that consumption of CGS exhibits decreasing marginal utilities) while he assumes aftermarket outcomes to be exogenously determined. Additionally, the two papers present substantial methodological differences. While Cabral (2007) focus on Markov Perfect Equilibrium (without imposing linearity constraints), we focus on a narrower equilibrium concept (the LMPE). The simpler nature of the LMPE, allows us to derive market outcomes analytically, while Cabral (2007) has to rely on simulation techniques to characterize dynamic price competition.

Doganoglu (2003) sets up a discrete model of dynamic price competition in the presence of network effects and horizontal differentiation. However, differently from us, Doganoglu (2003) considers competition in spot prices (there is no aftermarket) and myopic, instead of forward-looking, agents.

Mitchell and Skrzypacz (2006) built a dynamic model of price competition where the value of the product depends on current and past market shares. Our model differs from Mitchell and Skrzypacz (2006) at the level of consumers' expectation rules and the nature of price competition (Mitchell and Skrzypacz (2006) do not consider an aftermarket, investigating price competition in terms of spot prices).

Our framework is also very close to the one pointed out in Laussel et al (2004). However, differently from us, they focus on negative network externalities (congestion effects) and, they do not consider firms' interaction in an aftermarket.

The paper is organized as follows. In section 2, we describe the main ingredients of the model. Section 3 points out consumers' linear Markov expectation rules and firms' linear Markov price strategies, which are used in section 4 to compute the LMPE of the game. In section 5, we provide a detailed analysis of our results, stressing the implications on the nature of price competition between firms. Finally, section 6 concludes and identifies some lines for future research.

2. THE MODEL

2.1. Assumptions and notation

We consider two firms (say firm 1 and 2) involved in two interrelated markets: a primary market (where each firm sells a different version of a certain original equipment) and an aftermarket (where firms provide exclusive/ incompatible CGS). At each point of time t, firm 1 and 2 interact in these two markets: in the primary market, they strategically post equipment prices and, in the aftermarket they choose the unit price of the CGS, exploiting their own basis of (locked in) costumers. The price of the original equipment sold by firm i at period t is denoted by $p_i(t)$; and the unit price of CGS offered by firm i is denoted by $v_i(t)$.

The timing of consumers' decisions is the following: at each point of time t, a flow of new consumers (given by μ) arrives to this industry and each new consumer decides which equipment he/she is willing to buy. Such decision is considered to be irreversible in the sense that switch of brands never takes place due to prohibitively high switching costs (like learning costs, habit formation,...). Thus, at each point of time t, older consumers (who already own an equipment) only participate in the aftermarket, where they choose the level of CGS they are willing to buy. Finally, we consider that all consumers face a constant probability of *"death"* (μ) and, accordingly, at each point of time t, some of the current equipment owners exit the industry at a rate equal to μ (the total size of the market is stationary).

Concerning products' characteristics, in the primary market, we consider that firms offer a technically similar equipment, in the sense that all consumers get the same *expected life-time utility* ($\vartheta \geq 0$) from the equipment *per se* (independently of the equipment version). When $\vartheta = 0$, consumers only benefit from the equipment when this is used together with CGS purchase in the aftermarket. We consider that the CGS offered by each firm are considered to be horizontally differentiated (CGS offered by each firm have different intrinsic characteristics) as well as vertically differentiated (due to existence of proprietary (though partially compatible) indirect network effects).

Concerning the differences in the intrinsic characteristics of the CGS, we focus on horizontal differentiation *a la* Hotelling. In particular, we consider that the spectrum of all possible variants of CGS is represented by the interval $[0, 1]$ and we assume that CGS offered by firm 1 are located at 0, while CGS offered by its rival are located at 1. In the side of consumers, there is a unit mass of consumers, uniformly distributed in the interval $[0, 1]$ according to their most preferred variant of CGS. A consumer located at $x \in]0, 1[$ bears a utility loss due to the discrepancy between the services rendered by his/her ideal firm and the services actually rendered by the existent firms (*"travel costs"*). We model such utility loss *a la* d' Aspremont *et al* (1979), considering quadratic travel costs: the consumer $x \in [0, 1]$ bears a travel cost of τx^2 when he/she purchases CGS from firm 1 and $\tau(1 - x)^2$ if he/she purchases CGS from firm 2 instead.

Concerning the differences in the quality of CGS entailed by indirect network effects, we focus on the case of proprietary but partially compatible network effects. Accordingly, we consider that the total number of equipment owners indirectly exerts a positive influence on the utility obtained from consumption of CGS in the aftermarket. However, the benefit exerted by those consumers who own the same equipment version is greater than the benefit exerted by those consumers who own a different version of the equipment.

We assume that all agents are forward-looking, i.e., at any time of the game (in particular, by the time of purchasing the original equipment), each consumer's information set includes the rational expectations formulated about interactions that will occur in subsequent stages of the game. Similarly, firms are able to perfectly anticipate other agents' future decisions.

In this context, we analyze agents behaviour and we characterize the LMPE of the linear quadratic game above mentioned, stressing its implications with respect to the equilibrium paths of prices in the primary market (equipment' prices) and analyzing to which extent the absence of competition in the market stimulates competition for the market.

Notice that, we focus on the LMPE for two reasons. Firstly, our differential game being linear quadratic, it is natural to search for the simplest possible equilibrium strategies, i.e. the linear ones. Secondly, even though it is known that, in general, there may exist non-linear strategies that solve linear-quadratic differential games, such sophisticated strategies are technically difficult (if not impossible) to compute analytically and, in the majority of cases, only computer simulation can provide some insights on their possible configurations.[2]

2.2. Consumers' behavior

2.2.1. Aftermarket

At each moment t, equipment owners (old consumers) have to decide on the level of CGS (k_i) they are willing to purchase in the aftermarket. Conditional on firms' market shares in the primary market, a consumer who owns the version i ($i = 1, 2$) of the original equipment obtains the following instantaneous net utility[3] from purchasing (at period t) a level of k_i CGS from firm i :

$$\begin{aligned} & U(x, k_i(t), D_i(t)) & (1) \\ = \; & \Upsilon + [D_i(t) + \phi(1 - D_i(t))] [\alpha k_i(t) - \frac{1}{2} k_i^2(t)] - v_i(t) k_i(t) - \tau(x - x_i)^2 & (2) \end{aligned}$$

where $\Upsilon > 0$ is a constant that measures the intrinsic *stand alone* value of CGS offered by firm i in the aftermarket; $D_i(t)$ denotes the instantaneous market share of firm i in the primary market; $\phi \in]0, 1[$ measures the *"degree of compatibility"* between rival networks, $[D_i(t) + \phi(1 - D_i(t))]$ measures the total (indirect) network effect, $\alpha > 0$ is a constant that captures the intensity of network effects (for a fixed size of the aftermarket); $v_i(t)$ corresponds to the unit price of CGS offered by firm i ($i = 1, 2$) at moment t and x_i denotes the location of firm i in the aftermarket: more precisely, $x_1 = 0$ and $x_2 = 1$.

If this consumer (who owns an equipment version i and is located at x) decided to purchase CGS from firm j (instead of firm i), he/ she obtains a net instantaneous utility of

$$U(x) = -v_j(t) k_j(t) - \tau(x - x_j)^2,$$

which is always negative because a consumer who owns an equipment version i cannot get any benefit from CGS offered by firm j as these are totally incompatible with the equipment version of this consumer (lock in phenomenon). For this reason, a consumer who owns an equipment version i will always prefer to buy the CGS offered by firm i (in detriment of firm j).

Notice also that the utility function in (1) exhibits three important properties. Firstly, it explicitly accounts for the indirect network effects

[2] In particular, Cabral (2007) uses simulation techniques to analyze price competition in the presence of network externalities.

[3] Notice that the utility function specified in (1) explicitly accounts for the indirect network effects arising in this industry and, accordingly, it corresponds to a reduced form of consumers utility.

created by the interrelations between primary markets and aftermarkets: consumers' utility from buying CGS from firm i depends on the size of the market of this firm in the primary market since manufacturers with larger market shares in the primary markets benefit from more intense learning and experience economies, being able to offer higher-quality CGS.

Secondly, from (1) follows that networks are (at least partially) compatible. In the context of our model, compatibility between networks means that manufacturers might benefit (at least partially) from the rival's learning and experience economies. The parameter $\phi \in \left]0,1\right[$ measures the extent to which firms might appropriate the rival's indirect network effect and, in our framework, it is considered to be exogenously determined. When $\phi \to 0$, owners of an equipment i do not benefit from the total number of owners of equipment j because manufacturers cannot appropriate the rivals' indirect network effects. In this case, indirect network effects take the form of typical variant-specific network effects[4]. In contrast, when $\phi \to 1$, owners of equipment i benefit from the total number of consumers in the industry (independently of the equipment version owned by these consumers). Consequently, in this case, indirect network effects take the form of industry-specific network effects[5].

Finally, from (1) follows that $U'(k_i) > 0$ but $U''(k_i) < 0$ and, as a result, for given indirect network effects (i.e. for given market shares in the primary market), consumption of CGS exhibits decreasing marginal returns.

Considering the utility function in (1), the problem faced by equipment owners (old consumers) at each point of time t consists in determine the level of CGS (k_i) they are willing to acquire from the firm to which they are locked in. The solution to this problem is given by the level of consumption of CGS (k_i) that maximizes the utility function in (1). Therefore, at each moment t, the individual demand of CGS by a owner of an equipment i is simply

$$k_i(v_i(t), D_i(t)) = \alpha - \frac{v_i(t)}{D_i(t) + \phi(1 - D_i(t))}, \tag{3}$$

where the unit price ($v_i(t)$) in equation (1) is posted by firm i at moment t. The individual demand in (3) implicitly assumes that, at each moment t, firms are not able to commit to the prices of future complementary goods or services. For some aftermarkets, this lack of commitment assumption seems to be quite reasonable: for example, in the case of printers we observe that printers manufactures do not usually commit themselves to the prices they will charge for the cartridges sold to consumers in the future.

[4] For example, in the case of helicopters, this situation would arise when equipment manufacturers cannot imitate rival's quality improvements entailed by learning economies associated with the exploitation of its own market of helicopters.

[5] For example, in the case of helicopters, this situation would arise in case of perfect diffusion of quality improvements resulting from learning economies entailed by indirect network effects.

In these circumstances, firms post $v_i(t)$ with the only purpose of maximizing instantaneous profits obtained in the aftermarket:

$$\pi_\iota^A(v_i(t); D_i(t)) = \left\{ \alpha v_i(t) - \frac{v_i^2(t)}{D_i(t) + \phi(1 - D_i(t))} \right\} D_i(t), \qquad (4)$$

where, without loss of generality[6], production costs are normalized to zero.

Therefore, firm i's optimal pricing policy in the aftermarket is given by

$$v_i^*(D_i(t)) = \frac{\alpha \left[D_i(t) + \phi(1 - D_i(t)) \right]}{2} \qquad (5)$$

and, at equilibrium, firm i's profits in the aftermarket (conditional on manufacturers' market shares in the primary market) are simply:

$$\pi_\iota^A(v_i^*(t); D_i(t)) = \frac{\alpha^2}{4} \left[(1 - \phi) D_i(t)^2 + \phi D_i(t) \right]. \qquad (6)$$

When the price of CGS is posted according to (5), a consumer that owns the variant i of the original equipment and is located at position x obtains the following utility from consumption of CGS offered by manufacturer i:

$$U(x, D_i(t)) = \Upsilon + \left[D_i(t) + \phi(1 - D_i(t)) \right] \frac{\alpha^2}{8} - \tau(x - x_i)^2. \qquad (7)$$

Notice that, the higher the number of consumers who own an equipment of version i, *(i)* the higher the price of CGS charged by firm i (equation (5)); *(ii)* the higher the profits obtained by this firm (equation (6)) and, finally, *(iii)* the higher[7] the net instantaneous utility obtained by each owner of an equipment version i (equation (7)). More precisely,

$$\frac{\partial v_i^*(D_i(t))}{\partial D_i(t)} = \frac{\alpha}{2}(1 - \phi) > 0 \qquad (8)$$

$$\frac{\partial \Pi_\iota^{A*}(D_i(t))}{\partial D_i(t)} = \frac{\alpha^2}{2}(1 - \phi) D_i(t) + \frac{\alpha^2}{4}\phi > 0 \qquad (9)$$

$$\frac{\partial U_i^*(x, D_i(t))}{\partial D_i(t)} = \frac{\alpha^2}{8}(1 - \phi) > 0 \qquad (10)$$

[6] The qualitative nature of our results would not change if one considers constant and symmetric marginal costs.

[7] Indeed, the higher quality of the CGS provided by firm i more than compensates the higher price charged by this firm as a result of an increasing in the number of consumers who own its equipment.

2.2.2. Primary market

In the primary market, firms sell different versions of the same original equipment to new consumers (those who don't own an original equipment). More precisely, at each moment t, a set of new (potential) consumers arrives to this industry at a rate of μ. At the moment of entry, new (potential) consumers have to choose which equipment (if any) they are willing to buy. Consumers' choices will be determined by *(i)* the *expected life-time utility* obtained from the equipment *per se,* which is exogenously given by $\vartheta \geq 0$ for both firms; *(ii)* the equipment price $p_i(t), i = 1, 2$, which is posted by firms at each period of time; and *(iii)* the *expected life-time utility* obtained from future consumption of CGS. The *expected life-time utility* obtained from future consumption of CGS corresponds to the stream of instantaneous utilities obtained from future consumption of CGS. More precisely, it will be determined by *(i)* the intrinsic characteristics of CGS offered by each firm (horizontal differentiation); *(ii)* the expected evolution of firms' market shares in the primary market (which determines the degree of indirect network effects that firm i and j will be able to reach in the aftermarket); and, finally *(iii)* the expected price to be paid for CGS eventually purchased in the future.

Under these circumstances, a consumer located at $x \in [0, 1]$ who considers to buy the equipment version produced by firm i, expects to get the following *life-time utility* from future consumption of CGS offered by firm i:

$$C_i(x, t) = \int_t^{\infty} [U(x, D_i(t))] e^{-(r+\mu)(v-t)} dv \tag{11}$$

where $U(x, D_i(t))$ is given by equation (7); $r > 0$ is the discount rate and μ is the probability that this consumer exits the industry. The sum $r + \mu$ is called the *effective discount rate*. Notice that:

$$C_i(x, t) \geq \frac{\Upsilon + \phi \frac{\alpha^2}{8} - \tau}{r + \mu},$$

since *(i)* D_i is bounded below by 0, and *(ii)* $(x - x_i)^2$ is bounded above by 1.

The *expected net life-time surplus* ($V_i(x, t)$) obtained by a new consumer x who purchases the version i of the original equipment is simply given by the sum of the *life-time utility* obtained from the equipment *per se* ($\vartheta \geq 0$) and the expected *life-time utility* from future consumption of CGS ($C_i(x, t)$) net of the price to be paid for this equipment ($p_i(t)$):

$$V_i(x, t) = \vartheta + C_i(x, t) - p_i(t). \tag{12}$$

We assume that

$$\frac{\Upsilon + \phi \frac{\alpha^2}{8}}{r + \mu} > \frac{2\tau}{r + \mu}, \tag{13}$$

which means that the *"travel cost"* is relatively small in comparison with the benefits obtained from consumption of CGS. This implies that the expected *life-time utility* from consumption of CGS is always positive and it can be shown that condition (13) is sufficient to ensure that all consumers expect to have non-negative *net life-time surplus* in a Nash equilibrium[8], guaranteeing that all the consumers arriving in this industry have incentives to buy a piece of the original equipment. This constitutes a fairly standard assumption[9] that rules out the rather trivial case of local monopolies where some of the new consumers would not buy any of the available versions of the equipment.

Consider a consumer who enters the market at moment t, buying the original equipment produced by firm i and let us define the *expected lifetime gross benefit* from future consumption of CGS as $\frac{\Upsilon}{r+\mu} + \frac{\alpha^2}{8}\Lambda_i(t)$, where

$$\Lambda_i(t) = \int_t^{\infty} \left[D_i(v) + \phi(1 - D_i(v))\right] e^{-(r+\mu)(v-t)} dv \geq 0, \qquad (14)$$

and $\frac{\Upsilon}{r+\mu} + \frac{\alpha^2}{8}\Lambda_i(t)$ represents the expected value of the discounted stream of future benefits obtained with the consumption of CGS gross of "travel costs" for a consumer who decided to buy equipment i at moment t.

Notice that $\Lambda_i(t)$ has two important properties. First,

$$\Lambda_i(t) \in \left[\frac{\phi}{r+\mu}, \frac{1}{r+\mu}\right] \qquad (15)$$

because $D_i(t) \in [0,1]$, and second,

$$\Lambda_1(t) + \Lambda_2(t) = \frac{1+\phi}{r+\mu} \qquad (16)$$

because $D_1(t) + D_2(t) = 1$.

Furthermore, considering (14), equation (12) can be more conveniently written as

$$V_i(x,t) = \vartheta + \frac{\Upsilon}{r+\mu} + \frac{\alpha^2}{8}\Lambda_i(t) - \frac{\tau(x-x_i)^2}{r+\mu} - p_i(t). \qquad (17)$$

In this context, new consumers arriving in the industry at period t will compare the expected *net life-time surplus* associated with each equipment version and, obviously, they will buy the equipment from the firm which provides them the higher expected *net life-time surplus.*

Since consumers are forward-looking agents who formulate rational expectations, they are able to perfectly forecast the future time path of firms'

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[8] Condition (13) establishes that $\frac{\Upsilon+\phi\frac{\alpha^2}{8}}{r+\mu} > \frac{2\tau}{r+\mu}$. It can be shown that $\frac{\tau}{\delta+\mu} \geq \widehat{p}_i$ (for further detail, see proposition 6 afterwards). Consequently, condition (13) assures that, at the Nash equilibrium, all consumers have incentives to buy an original equipment.

[9] See for instance d' Aspremont et al (1979).

market shares and, henceforth, they perfectly anticipate the *net life-time surplus* they can get from each of the alternative equipment versions.

As a consequence, at a given moment of time t, a new consumer located at x buys the original equipment offered by firm 1's if

$$\frac{-\tau x^2}{r+\mu} + \frac{\alpha^2}{8}\Lambda_1(t) - p_1(t) \geq \frac{-\tau(x-1)^2}{r+\mu} + \frac{\alpha^2}{8}\Lambda_2(t) - p_2(t). \tag{18}$$

Conversely, when the reverse inequality holds, this consumer prefers to buy the equipment version offered by firm 2.

The consumer located at $x = \tilde{x}$, such that condition (18) holds with equality, is indifferent between buying the equipment version offered by 1 or the one offered by firm 2. For such consumer, the differential in the equipment prices charged by the two firms:

$$p_1(t) - p_2(t),$$

is exactly compensated by the gap in the expected *life-time utilities* obtained from future consumption of CGS:

$$\frac{\alpha^2}{8}\left(\Lambda_1(t) - \Lambda_2(t)\right) - \left(\frac{\tau\tilde{x}^2}{r+\mu} - \frac{\tau(\tilde{x}-1)^2}{r+\mu}\right).$$

For given $p_1(t)$, $p_2(t)$, $\Lambda_1(t)$ and $\Lambda_2(t) = \frac{1+\phi}{r+\mu} - \Lambda_1(t)$[10] , there exists a unique number $\widetilde{x}(t) = \widetilde{x}(p_1(t), p_2(t), \Lambda_1(t))$ such that expression (18) holds with equality:

$$\widetilde{x}(t) = \left[\frac{1}{2} + \frac{(p_2(t) - p_1(t))(r+\mu) + \frac{\alpha^2}{4}\Lambda_1(t)(r+\mu) - \frac{\alpha^2}{8}(1+\phi)}{2\tau}\right] \tag{19}$$

When $\widetilde{x}(t) \in [0,1]$, consumers entering the market at t and located to the left of $\widetilde{x}(t)$ choose to buy equipment version 1. Conversely, consumers entering the market at t but located to the right of $\widetilde{x}(t)$ choose to buy version 2. When $\widetilde{x}(t) < 0$, the entire set of new consumers at time t would prefer version 2 to version 1. In contrast, when $\widetilde{x}(t) > 1$, the entire set of new consumers at time t would prefer version 1 to 2. In this paper, we focus on the competitive solutions, where both firms keep positive market shares. Thereby, we rule out the two last cases, assuming that

$$-\tau \leq (p_2(t) - p_1(t))(r+\mu) + \frac{\alpha^2}{4}\Lambda_1(t)(r+\mu) - \frac{\alpha^2}{8}(1+\phi) \leq \tau \tag{20}$$

Given that, at each moment t, new consumers come to the industry at a rate equal to μ, from equation (19) it follows that, provided that (20)

[10] By equation (16)

holds[11], the number of new consumers of equipment version 1 at moment t is equal to $d_1(t) = \mu\widetilde{x}(t)$ and the number of new consumers choosing to buy version 2 at moment t is given by $d_2(t) = \mu[1 - \widetilde{x}(t)]$. Note that $d_1(t) + d_2(t) = \mu$, given the full market coverage assumption associated with condition (13).

2.3. Firms' profits

The firm i's cash flow at time t is the sum of the revenues obtained in the aftermarket at time t (given by equation (6)) and the revenues obtained in the primary market (sales of new equipment). For simplicity reasons and without loss of generality[12] we consider null production costs. Thus, firm i's net cash flow is simply

$$\pi_i(t) = \frac{\alpha^2}{4}\left[(1-\phi)D_i(t)^2 + \phi D_i(t)\right] + p_i(t)d_i(t), \tag{21}$$

where $d_i(t)$ is the instantaneous demand of the equipment version offered by firm i[13].

Considering equation (19) and denoting $\frac{\alpha^2}{4}$ by a (and from now on, we shall interpret a as a measure of the *intensity of network effects* for a given size of the aftermarket), $d_i(t)$ can be written as follows

$$d_i(t) = \mu\left[\frac{1}{2} + \frac{(p_j(t) - p_i(t))(r+\mu) + a\Lambda_i(t)(r+\mu) - \frac{a}{2}(1+\phi)}{2\tau}\right], \tag{22}$$

where $i, j = 1, 2, \;\; i \neq j$.

From equation (22), we can determine the net rate of variation in the size of the "locked in market" of firm i (i.e., in the firm i's instantaneous market share, $D_i(t)$), which consists on the inflow of new consumers who bought firm's i version of the equipment ($d_i(t)$) net of the instantaneous exit flow ($\mu D_i(t)$) :

$$\dot{D}_i(t) = \frac{dD_i(t)}{dt} = d_i(t) - \mu D_i(t). \tag{23}$$

Given the instantaneous profit function (equation (21)), at each point of time, firms will strategically choose the equipment price, taking into

[11] If condition (20) does not hold, then we have to deal with an instantaneous market where all new consumers *strictly* prefer the same firm, say firm i.

[12] Considering symmetric positive constant marginal costs of production would not change the qualitative nature of our results.

[13] If we wish to be completely general, we could proceed as follows. Given the set $S = \{0, 1, \widetilde{x}\}$, define $mid\{0, 1, \widetilde{x}\}$ to be $\widetilde{x}$ if $\widetilde{x} \in [0, 1]$, to be 0 if $\widetilde{x} < 0$, and to be 1 if $\widetilde{x} > 1$. Then the flow demand facing firm 1 is $d_1(t) = \mu mid\{0, 1, \widetilde{x}(t)\}$ and the flow demand facing firm 2 is $d_2(t) = \mu - d_1(t)$.In what follows, we shall assume $\widetilde{x} \in [0, 1]$ and ignore this complication.

account that consumers are irreversibly locked in to the firm that produces the equipment they have chosen to buy by the moment they have entered the market.

More precisely, each firm i will choose the path of equipment prices, $p_i(t)$, that maximizes the present value of the expected future stream of cash flows

$$\Pi_i = \int_0^{\infty} \pi_i(t) e^{-rt} dt, \tag{24}$$

subject to the market shares' transition equation, given by (23).

In the following section, we give a step further and we analyze the consumers' linear Markov expectation rules and the firms' linear Markov subscription price strategies.

3. LINEAR MARKOV EXPECTATIONS AND STRATEGIES

Consumers' expectations and firms' price strategies are said to be Markovian when they only depend on the *state vector*, which conveys all the payoff relevant information. In our framework, the state vector at moment t is the vector of the size of firms' "locked-in" networks at time t, i.e. $(D_1(t); D_2(t))$. Consequently, both consumers' expectations and firms' strategies are exclusively based on the vector $(D_1(t), D_2(t))$ and, accordingly, they can be written as a function of these two variables. In particular, in the case of linear Markov perfection, they can be expressed as linear functions of the state vector. The next subsections provide a formal definition of consumers' Markov expectation rules and firms' Markov price strategies.

3.1. Linear Markov expectation rules

At time t, new consumers formulate expectations on the *life-time surplus* obtained from each of the alternative equipment versions. Such expectations are determined by consumers' expectations $\Lambda_1^e(t)$ and $\Lambda_2^e(t)$ of the actual values $\Lambda_1(t)$ and $\Lambda_2(t)$, which, according to equation (14), are determined by the trajectories of the firms' market shares ($D_1(t)$ and $D_2(t)$). In the presence of forward-looking agents, as it is the case here, consumers perfectly anticipate future market shares and

$$\Lambda_i^e(t) = \Lambda_i(t).$$

In this context, a Markov expectation rule is a *pair* of functions $(F_1(.), F_2(.))$ that generate the values $\Lambda_1^e(t)$ and $\Lambda_2^e(t)$ from what consumers currently observe, namely the market shares $D_1(t)$ and $D_2(t)$. This pair of functions maps any observed point $(D_1(t), D_2(t))$ on the unit simplex

$$\Delta = \{(D_1(t), D_2(t)) \mid 0 \leq D_i(t) \leq 1 \ , \ D_1(t) + D_2(t) = 1\}$$

to a point $(\Lambda_1^e(t), \Lambda_2^e(t))$, implying

$$\Lambda_i^e(t) = F_i(D_1(t), D_2(t))$$

In the special case where the expectation rule is linear, the following Lemma holds:

LEMMA 1. *The linear Markov expectation rules formulated by forward-looking consumers must be of the form*

$$\Lambda_i^e(t) = \delta_i + b_i D_i(t) \equiv f_i(D_i(t)) \text{ for } i = 1, 2 \tag{25}$$

where

$$\frac{\phi}{r+\mu} \leq \delta_i \leq \frac{1+\phi}{r+\mu} - b \tag{26}$$

$$b_1 = b_2 = b > 0 \tag{27}$$

$$b = \frac{1+\phi}{r+\mu} - (\delta_1 + \delta_2) \tag{28}$$

Proof. see the Appendix. ∎

Lemma 1 shows that, in a competitive framework, forward-looking consumers expect a positive correlation between the current number of consumers owning a certain equipment version and the future "locked in" market dominated by this firm (i.e., $\partial \Lambda_i^e(t)/\partial D_i(t) > 0$). Therefore, under rational expectations, equipment owners extrapolate from current market shares that, for given travel costs and subscription fees, *the larger is the current market share of a given firm i, the greater will be the expected life-time surplus,* $V_i(x, t)$, of buying the equipment provided by this firm.

3.2. Linear Markov price strategies

A firm i is said to adopt a Markov price strategy when the equipment price it charges is exclusively based on its knowledge of the size of its current "locked in" market ($D_i(t)$) and the size of the the rival's "locked in" market. In other words, under Markov pricing strategies, the state vector $(D_i(t); D_j(t))$ conveys all the payoff-relevant information and, consequently, it is the only determinant of the price of equipment charged by firms.

In our framework, a Markov pricing strategy is denoted by $P_i(.)$ and it is a function that maps any observed point $(D_1(t), D_2(t))$ on the unit simplex Δ to a point $p_i(t)$ on the real number line. Thus

$$P_i(D_1(t), D_2(t)) = p_i(t).$$

Again, since $D_1 + D_2 = 1$, we can write

$$P_j(D_i, D_j) = P_j(1 - D_j, D_j) \equiv \widetilde{p}_j(D_j) \equiv \widetilde{p}_j(1 - D_i).$$

In the special case of linear Markovian strategies, the equipment' prices set by firm 1 and 2 must be of the form

$$p_i(t) = \widetilde{p}_i(D_i(t)) = \eta_i + s_i D_i(t), i = 1, 2. \tag{29}$$

A linear Markovian price strategy, $\widetilde{p}_i^*(D_i)$, is said to be the *best reply* of firm i to both the rival firm's price strategy, $\widetilde{p}_j(1-D_i)$, and the consumers' expectation rules $(f_i(D_i), f_j(D_j))$, if for any initial market share in the equipment' market _$D_i(0)$_ firm i chooses a price strategy $\widetilde{p}_i^*(D_i)$, yielding a time path that maximizes the discounted value of future accumulated profits. More precisely, firm i is giving its best reply when it chooses a time path for $p_i(t)$ that maximizes the present value of the expected stream of future cash flows subject to the market shares' transition equation. Thus, the subscription price strategy $\widetilde{p}_i^*(D_i)$ is the one that solves the following optimal control problem:

$$\max_{p_i(t)} \int_0^\infty e^{-rt} \pi_i(t) dt \tag{30}$$

subject to

$$\dot{D}_i(t) = d_i(t) - \mu D_i(t) \tag{31}$$

$$0 \leq D_i(t) \leq 1 \tag{32}$$

where

$$d_i(t) \tag{33}$$

$$= \mu \left\{ \frac{1}{2} + \frac{\left[\widetilde{p}_j(1 - D_i(t)) - p_i(t)\right](r + \mu) + a f_i(D_i(t))(r + \mu) - \frac{a}{2}(1 + \phi)}{2\tau} \right\} \tag{34}$$

and

$$\pi_i(t) = a\left[(1 - \phi) D_i(t)^2 + \phi D_i(t)\right] + p_i(t) d_i(t). \tag{35}$$

Note that, due to the linearity of consumers' expectation rules and the linearity of firms' price strategies, the previous problem corresponds to a standard infinite horizon optimal control problem with a linear-quadratic structure[14].

In the following section, we derive the LPME of the game (when it exists) and we provide its complete characterization.

[14] For theorems stating necessary and sufficient conditions, see, for example, Long and Leonard(1992, Chapter 9).

4. LINEAR MARKOV PERFECT EQUILIBRIUM

Definition 1. A Markov perfect equilibrium is a quadruple of functions

$$(\widetilde{p}_1^*(.), \widetilde{p}_2^*(.), f_1(.), f_2(.))$$

such that:
(i) $\widetilde{p}_i^*(.)$ is firm i's best reply to both the pricing strategy $\widetilde{p}_j(.)$ and the expectation rule $(f_i(.), f_j(.))$, for $i, j = 1, 2, i \neq j$, and
(ii) expectations are rational in the sense that, for $i = 1, 2$,

$$f_i(D_i(t)) \equiv \int_t^{\infty} [D_i(v) + \phi(1 - D_i(v)] \, e^{-(r+\mu)(v-t)} dv$$

where

$$\frac{dD_i(t)}{dt} = d_i(t) - \mu D_i(t) \tag{36}$$

with $d_i(t)$ given by (33).

From definition 1, it follows that, in the LMPE of the game, firms optimally the path $p_i(t)$ (given the rival's price strategy and the consumers' expectation rules) and, simultaneously, consumers optimally choose which equipment to buy, perfectly anticipating the evolution of firms' market shares in the equipment market (i.e. the primary market).

The rest of the section is devoted to the identification of the LMPE of the game. We start with the analysis of firms' optimal price strategies. Then, we focus on the equilibrium expectation rules and, finally, we provide a full description of the LMPE of the game.

In the LPME, both firms choose the trajectories of equipment prices $\widetilde{p}_i^*(D_i), i = 1, 2$, which solve the optimal control problem (30)-(35), given consumers' expectation rules $(f_i(D_i), f_j(D_j))$ and the price strategy of the rival firm

$$\widetilde{p}_j(1 - D_i) = \widetilde{p}_j(D_j) = \eta_j + s_j D_j, j = 1, 2, i \neq j.$$

We introduce the current-value co-state variable $\lambda_i(t)$ and we define the current-value Hamiltonian for firm i as

$$H_i(t) = (p_i(t) + \lambda_i(t)) d_i(p_i(t), D_i(t)) + a \left[(1 - \phi) D_i(t)^2 + \phi D_i(t)\right] - \mu \lambda_i(t) D_i(t) \tag{37}$$

where

$$d_i(p_i(t), D_i(t)) = \mu \left\{ \frac{1}{2} + \frac{[\eta_j + s_j(1 - D_i(t)) - p_i(t)](r + \mu) + a[\delta_i + b D_i(t)](r + \mu) - \frac{a}{2}(1 + \phi)}{2\tau} \right\}$$

The necessary conditions to guarantee that firm i is optimally choosing the price of its equipment include:

$$\lambda_i(t) = \frac{2\tau - a(1+\phi)}{2(r+\mu)} + \eta_j + s_j(1 - D_i(t)) - 2p_i(t) + a[\delta_i + bD_i(t)], \quad (38)$$

and

$$\frac{d\lambda_i(t)}{dt} = (r+\mu)\lambda_i(t) - [p_i(t)+\lambda_i(t)][\frac{\mu(r+\mu)}{2\tau}(ab - s_j)] - (a\phi + 2a(1-\phi)D_i) \quad (39)$$

A similar set of equations is obtained from firm j's optimal control problem. From the two sets of equations, it is possible to determine the equilibrium linear Markov price strategies, conditional on consumers' expectations.

LEMMA 2. ***Equilibrium linear Markov price strategies conditional on consumers' expectations***
Given consumers' expectation rules (i.e., given the parameters b and $\delta_i, i = 1, 2$), any pair of equilibrium linear pricing strategies $\widetilde{p}_i(D_i) = \eta_i + s_i D_i$ must have the following properties
(i) $s_1 = s_2 = s$, with s satisfying the following condition

$$(r + 2\mu)(3s - ab) + \frac{2}{2\tau}\mu(r+\mu)(2s - ab)^2 - 2a(\phi - 1) = 0; \text{ and} \quad (40)$$

(ii) $\eta_i, i = 1, 2$ are such that

$$\frac{(r+\mu)}{\tau}\left((3\tau + 4s\mu - 2ab\mu)(\eta_j - \eta_i) + a(\tau + 2s\mu - ab\mu)(\delta_i - \delta_j)\right) = 0 \quad (41)$$

Proof. see the Appendix. ∎

Given the fact that consumers are forward looking agents, in the LMPE, they perfectly anticipate the equilibrium linear Markov price strategies pointed out in lemma 2. Therefore, in the equilibrium linear expectation rules, the constants δ_1, δ_2 and b in (25) must respect the restrictions pointed out in the following lemma.

LEMMA 3. ***Equilibrium expectation rules conditional on price strategies***
Given η_1, η_2 and s, the rational expectation requirement that $\Lambda_i^e(t) = \Lambda_i(t)$ can be satisfied by a linear expectation rule if and only if the constants δ_i and b satisfy the following conditions:
(i) for b:

$$\frac{1}{2\tau}[ab^2\mu(r+\mu) - 2b(s\mu(r+\mu) + \tau(r+2\mu)) + 2\tau(1-\phi)] = 0 \quad (42)$$

(ii) for δ_i :

$$\begin{aligned}\delta_i &= \frac{2b\mu(r+\mu)(\eta_i-\eta_j)+ab\mu(1+\phi)-2bs\mu(r+\mu)-2b\mu\tau-4\tau\phi}{2(r+\mu)(ab\mu-2\tau)} \\ i,j &= 1,2,\ i\neq j\end{aligned} \quad (43)$$

Proof. see the Appendix. ∎

Using equations (43) and subtracting δ_2 from δ_1 one obtains

$$\delta_1-\delta_2=\frac{2b\mu(\eta_1-\eta_2)}{(ab\mu-2\tau)} \quad (44)$$

Condition (44) shows that, in the LMPE of the game, the gap $\delta_1-\delta_2$ in the consumers' expectation rules is proportional to the gap $\eta_1-\eta_2$ in the firms' price strategies in the primary market.

In the LMPE of the game, consumers' expectation rules met the requirements in lemma 3 and, simultaneously, firms' price strategies in the primary market met the conditions in lemma 2. As a consequence, equilibrium price strategies and expectation rules will be perfectly symmetric as claimed in lemma 4, below.

LEMMA 4. ***Equilibrium price strategies and expectations rules*** *In the LMPE of the game:*
(i) consumers' expectation rules are perfectly symmetric, in the sense that $b_1=b_2=b$ and $\delta_1=\delta_2=\delta$;
(ii) firms' price strategies are perfectly symmetric, in the sense that $s_1=s_2=s$ and $\eta_1=\eta_2=\eta$.

Proof. see the Appendix. ∎

Lemma 4 illustrates the symmetric nature of the competitive LMPE of the game. In the LMPE, the responsiveness of equipment prices to changes in the instantaneous market shares in the primary (the "locked in" market) is the same for both firms ($s_1=s_2=s$). In addition, the price equilibrium strategies have the same intercept ($\eta_1=\eta_2=\eta$). Consequently, at equilibrium, any differences in the price of different equipment versions can only be explained by an initial asymmetry in the size of firms' "locked in" market. Similarly, consumers' expectations on the evolution of the firms' market shares are also symmetric: the expectation rules' intercepts are the same for the two firms ($\delta_1=\delta_2=\delta$) and the expected responsiveness of firms' future market shares with respect to current market shares is also the same for both firms ($b_1=b_2=b$).

4.1. A Complete Characterization of the LMPE

In the LMPE, consumers' expectations on the evolution of size of firms "locked in" market (i.e. firms' future market shares in the primary market) correspond to the equilibrium Markovian rules pointed out in equations (42) and (43), $i = 1, 2$. Similarly, firms post the prices of their equipment according to the equilibrium price strategies described by equations (40) and (41), $i = 1, 2$.

This provides us six equilibrium conditions, which can be used to solve for the LMPE and determine the equilibrium values of the six unknowns of the model: $b, s, \eta_1, \eta_2, \delta_1$ and δ_2. [15]

The following proposition and lemmas provide a complete characterization of the LMPE of the game (when it exists). In proposition 1, we identify the existence and uniqueness conditions. Then, when existence conditions are met, we solve for the LMPE of the game, which is pointed out in lemmas 5 to 7.

PROPOSITION 1. ***Existence and uniqueness of the LMPE***
(i) *A* ***unique competitive LMPE*** *exists if and only if the intensity of network effects* (a) *satisfies:*

$$a \leq \frac{\tau(6r + 4\mu)}{(5r + 6\mu)(1 - \phi)}. \tag{45}$$

(ii) Two competitive LMPE *exist if and only if* $r < \frac{2\mu}{\sqrt{5}}$ *and the intensity of network effects* (a) *satisfies:*

$$\frac{\tau(6r + 4\mu)}{(5r + 6\mu)(1 - \phi)} < a \leq \frac{\tau(r + 2\mu)^2(7\sqrt{5} - 15)}{\mu(r + \mu)(1 - \phi)(5 - \sqrt{5})(3 - \sqrt{5})^2}. \tag{46}$$

(iii) *When none of the above conditions is met,* ***no competitive LMPE*** *exists*[16]*.*

Proof. see the Appendix. ∎

Proposition 1 points out the conditions for the existence and uniqueness of the LMPE. These conditions bear on the intensity of network effects; on the discount rate and on the probability of exiting the industry. For weak network effects, the LMPE exists and it is unique. Conversely, for strong network effects, no competitive LMPE exists. In those circumstances, two scenarios are possible: either the game has no LMPE at all, or monopoly

[15] It turns out that the equations can be solved sequentially. First, the two equations (40) and (42) determine b^* and s^*, and their values are dependent on a . Second, η^* and δ^* are dependent on b^* but not on s^*.

[16] More precisely, the game does not have any LMPE when
(i) $r < \frac{2\mu}{\sqrt{5}}$ and $a > \frac{\tau(r+2\mu)^2(7\sqrt{5}-15)}{\mu(r+\mu)(1-\phi)(5-\sqrt{5})(3-\sqrt{5})^2}$; or
(ii) $r > \frac{2\mu}{\sqrt{5}}$ and $a > \frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}$.

arises in the LMPE. In the last case, network effects are so strong (and consequently, the primary market and aftermarkets are so closely related) that the smaller firm is forced to leave the market, after some periods of interaction. Finally, for intermediate network effects, there are two LMPE, as long as firms do not discount too much future earnings. In that case, exiting the market is not in the interest of the smaller network and, consequently, the LMPE exists and, due to the relative importance of network effects, two LMPE might arise depending on agents' decisions.

From now on, we assume that existence conditions are met and we provide a complete description of the LMPE of the game.

LEMMA 5. ***Responsiveness of consumers' expectations to the size of the "loked in" markets***
In equilibrium, the responsiveness of consumers' expectations to the size of the "loked in" markets (i.e. to current market shares), b^*, *corresponds to the solution of the cubic polynomial below, in the interval* $\left(0, \frac{1-\phi}{r+\mu}\right)$

$$\left\{ \begin{array}{c} b^3 a\mu\left(r+\mu\right)\left(r+2\mu\right)+ \\ b^2\left(4a\mu\left(\mu+r\right)\left(1-\phi\right)+2\tau(r+2\mu)^2\right)- \\ 10b\tau\left(\left(1-\phi\right)\left(r+2\mu\right)\right)+8\tau(1-\phi)^2 \end{array} \right\}=0. \tag{47}$$

In the presence of weak network effects (i.e., a satisfies (45)), $b^* \geq \frac{1}{r+2\mu}$ *is the unique solution in the interval* $\left(0, \frac{1-\phi}{r+\mu}\right)$. *In the presence of intermediate network effects (i.e. a satisfies (46)), as long as* $r < \frac{2\mu}{\sqrt{5}}$, *there are two solutions in the interval* $\left(0, \frac{1-\phi}{r+\mu}\right)$, b_1^* *and* b_2^*.

Proof. Follows directly from condition (72) in the proof of proposition 1. ∎

LEMMA 6. ***Responsiveness of subscription fees to the size of the "loked in" markets***
In equilibrium, the responsiveness of subscription fees to changes in the size of the "loked in" markets (i.e. to current market shares,s^*, *is given by*

$$s^*(b^*)=\frac{2\tau\left(r+2\mu\right)\left(1-\phi-b^*(r+2\mu)\right)}{\mu(r+\mu)\left(b^*\left(r+2\mu\right)+4\left(1-\phi\right)\right)}<0. \tag{48}$$

Proof. see the Appendix. ∎

LEMMA 7. ***Equilibrium intercepts***
The equilibrium rational expectation rules are $f_i(D_i)=\delta^*+b^*D_i$, $i=1,2$, *where* b^* *is given in lemma 5, and the equilibrium intercept,* δ^*, *is given by* $\delta(b^*)>0$, *with*

$$\delta(b)=\delta=\frac{1+\phi-b\left(r+\mu\right)}{2\left(r+\mu\right)}>0 \tag{49}$$

The equilibrium pricing strategies are $p_i = \eta^* + s^*\ D_i$, $i = 1, 2$, *where* $s^* < 0$ *is given in lemma 6, and the equilibrium intercept,* η^*, *is given by* $\eta(b^*)$, *with*

$$\eta(b) = \frac{\left\{ \begin{array}{c} b^3\tau(r+\mu)(r+2\mu)(r+3\mu)+ \\ +b^2\tau(r^2+5r\mu+2\mu^2+\phi(r^2+3r\mu+6\mu^2))+ \\ +b\tau(4\mu\phi(3+\phi)-16\mu-10r(1-\phi))+8\tau(1-\phi)^2 \end{array} \right\}}{b^2\mu(\delta+\mu)^2(4(1-\phi)+b(\delta+2\mu))} \tag{50}$$

Notice that η^* *might be positive as well as negative. In particular* $\eta\left(\frac{1-\phi}{r+2\mu}\right) = \frac{5\mu(r+\mu)(1-\phi)^3}{(r+2\mu)^2} > 0$ *and* $\eta\left(\frac{1-\phi}{r+\mu}\right) = -2\mu\phi\,(3r+2\mu)\,\frac{(\phi-1)^2}{(r+\mu)^2} < 0.$

Proof. see the Appendix. ∎

The previous lemmata provide us a complete description of the LMPE of the game. Lemma 5 together with lemma 7 elucidate us about the equilibrium expectation rules. Lemma 6 together with lemma 7 identifies the equilibrium price strategies to be adopted by firms when they strategically choose the price of their equipment. In the following section, the nature of firms' price competition is analyzed in detail.

5. PRICE COMPETITION IN THE PRIMARY MARKET

5.1. Does the absence of competition in markets fosters competition for the market?

Lemmas 6 and 7 describe the equilibrium path of equipment prices. From lemma 7, it follows that, at time t, the fixed component of the equipment price (η) can be either positive or negative, depending on the value of a, which constitutes a measure of network intensities for a given size of the aftermarket. When indirect network effects are weak (for low values of a), η will be positive: consumers do not significantly benefit from the consumption of other consumers[17] and, as a result, the fixed component of the equipment price is positive. This is equivalent to say that, at period t, dumping (i.e., charging p_i below marginal cost) is not an optimal strategy for any firm i who has just entered the industry ($D_i(t) = 0$). Conversely, when indirect network effects are strong enough (and LMPE exists), η is negative. In the last case, consumers significantly value from consumption of other consumers (for example, when the size of the firm entails strong learning and experience economies). Accordingly, when setting the price of their equipment, each firm will offer a fixed discount (η) in order to attract

[17] This might be the case in an industry where firms cannot exploit learning or experience economies.

new consumers, who will increase the value of firms' CGS (since indirect network effects are strong), stimulating consumption in the aftermarket.

The model also predicts (lemma 6) a negative correlation between the optimal equipment price charged by a given firm and its current (instantaneous) market shares in the primary market, i.e. the size of its "locked in" market (given that $s^* < 0$). In other words, *ceteris paribus,* an increase in the number of consumers who own certain equipment version, entails a decrease in price paid to acquire such version. This effect is stemmig from the interrelations between the primary market and aftermarkets. When the aftermarket is sufficiently profitable, these interrelations create endogenous and significant incentives to invest in cutting-price strategies.

More precisely, these interrelations might entail a negative relation between the pricing policy in the primary market and firms' market shares in that market, because when firms charge lower prices in the primary market, they increase their basis of consumers in the aftermarkets. The enlargement of the basis of (locked-in) consumers in the aftermarket leads to an increase in firms revenues obtained in the aftermarket, not only because more consumers buy complementary goods/services from this firm but also because this firm is able to charge higher prices for these services (see equation 5). Therefore, as long as the aftermarket is sufficiently profitable, firms with a larger basis of "locked in" consumers have even more incentives to adopt cutting-price policies in the primary market.

Notice that, in our setting, the aftermarket is always sufficiently profitable to entail this negative relation due to the shape of instantaneous profits in the aftermarket ((see equation (6)). This function is quadratic in market shares and this generates increasing returns in this variable: the enlargement of the existing basis of costumers of a given equipment i allows this firm to raise the price of CGS on a larger set of consumers. Consequently, today's benefit (in terms of changes in aftermarket profits) from gaining a new costumer, will be greater for larger networks.

It is also worthy to emphasize that the interrelations between primary market and aftermarkets are so deep that s is smaller than zero even in a static setting. As a matter of fact, when $r \to \infty$ and consumers do not care at all about future consumption:

$$\lim_{r \to \infty} s^*(b^*) = -2\frac{\tau}{\mu} < 0.$$

Therefore, even in a static context, larger firms have more incentives to compete for the market, giving up on revenues in the primary market in order to increase their revenues in the aftermarket. In a dynamic context, these effects would tend to be reinforced given that profits from the aftermarkets become even more significant (when a consumer buys a certain equipment, it is expectable that this consumer stays in the market for a certain number of periods. Such effect is conveyed by the co-state variable $\lambda_i(t)$, which can be interpreted as the expected total (current and future) extra-profits associated with a marginal increase in firm's instantaneous market shares (the shadow price of the state variable). From equation (38), it is easy enough to see that the equilibrium value of λ^* depends positively on $D_i(t)$, which in turn negatively affects the equipment prices for the reasons explained above. Therefore, when one accounts for the dynamic incentives underlying price competition in the primary market, we conclude that, in a dynamic setting, larger firms have even more incentives to sacrifice current revenues in the primary market in order to expand the basis of future "locked in" consumers and obtain greater revenues in the aftermarket (during several periods).

Therefore, one of our main results is that, in a static setting as well as in a dynamic framework, larger firms have incentives to compete more fiercely for the market (i.e. charge lower prices in the primary market) in order to exploit the rents associated from the lack of competition in the aftermarket.

5.2. Trajectories of market shares

In the previous section, we concluded that larger firms charge lower prices for their equipment. Furthermore, from equation (10) follows that the instantaneous utilities with CGS purchased by a consumer who owns a certain equipment version are also positively affected by the number of total costumers of that version. For these reasons, one could think that an increasing proportion of new consumers would prefer to buy the equipment offered by the larger manufacturer and, sooner or later, the smaller network would be forced to leave the market. However, that is not necessarily the case because despite the attractiveness of the equipment version offered by the larger manufacturer to new consumers, this firm is faced with higher exit rates ($\mu D_i(t)$). Therefore, if indirect network effects are not too strong, after some periods of interaction, the smaller firm will be able to overcome its disadvantage and, in the steady state equilibrium, the firms divide the market evenly.

Indeed, the equilibrium trajectories of firms' market shares in the primary markets can be easily recovered from the market shares' law of motion

$$\dot{D}_i(t) = \frac{dD_i(t)}{dt} = d_i^*(t) - \mu D_i(t). \tag{51}$$

where $d_i^*(t)$ is the instantaneous demand for equipment version i when firms are setting the LMPE equipment prices (the ones which follow from lemmas 5-7). This leaves us with a first order differential equation, which allows us to recover the equilibrium market shares' trajectories:

$$D_i(t) = \frac{1}{2} + \left(D_i(0) - \frac{1}{2}\right) e^{\left(r+\mu-\frac{1-\phi}{b^*}\right)t} \tag{52}$$

PROPOSITION 2. ***Market shares' trajectories: rate of convergence***

In the LMPE, market shares converge to the steady state equilibrium $\left(\widehat{D}_i = \frac{1}{2}\right)$ *at a rate given by*

$$\frac{1-\phi}{b^*} - (r+\mu) \tag{53}$$

The rate of convergence to the steady state market shares is decreasing in the equilibrium responsiveness of consumers' expectations to the relative size of firms' "locked in" market (b^*)*. This means that convergence to the steady state is slower, the stronger are the network effects(a is larger) and the less differentiated are the CGS offered in the aftermarket* (τ *is smaller).*

Proof. From equation (53), it is straightforward to see that the rate of convergence is decreasing in b^*. Furthermore, from equation (47) it is straightforward as well to see that b^* is increasing in $\frac{a}{\tau}$ whenever $a \leq \frac{\tau(6\delta+4\mu)}{(5\delta+6\mu)(1-\phi)}$. ∎

The former proposition shows that when the indirect network effect is stronger and consumers attribute an higher value to the total number of costumers who own a similar equipment, an entrant's market share increases more slowly than in a less profitable market. This parallels Doganoglu's (2000) results according to which positive consumption externalities slow down convergence towards the steady state.

5.3. Steady state equipment prices

PROPOSITION 3. *The steady state prices in a LMPE are given by*

$$\widehat{p}(b^*) = \frac{\tau}{(r+\mu)} + \frac{2\tau\left(b^{*2}(r+2\mu)(r+2\mu\phi) - (1-\phi)(5r+2\mu(4+\phi))b^* + 4(\phi-1)^2\right)}{b^{*2}\mu(r+\mu)^2(4(1-\phi)+b^*(r+2\mu))} \tag{54}$$

and $\widehat{p}_i < \frac{\tau}{r+\mu}$.
Furthermore, $\widehat{p}\left(\frac{1-\phi}{r+2\mu}\right) = \frac{\tau}{\delta+\mu} > 0$ *and* $\widehat{p}\left(\frac{1-\phi}{\delta+\mu}\right) = -\frac{\tau(r(1+5\phi)+2\mu(1+\phi))}{(5r+6\mu)(1-\phi)(r+\mu)} < 0$.

Proof. see the Appendix. ∎

The previous proposition shows that, in the steady state LMPE, as long as indirect network effects are relatively strong (and consequently b^* is higher), the steady state equipment prices are negative. This means that, for a given size of the aftermarket, when network indirect effects are relatively strong, firms might be willing to pay to new costumers, in order to convince them to buy their equipment versions. Conversely, when network effects are relatively weak, the steady state equipment prices are positive.

In any case, it should be noticed that, in the steady state of our game, firms are charging lower equipment prices than the ones they would charge in the absence of network effects. In the absence of network effects, $a \to 0$, and it is easy to check that $\widehat{p_i} \to \frac{\tau}{r+\mu}$[18]. This means that firms would charge steady-state equipment prices equal to four times the present value of the travel cost supported by the marginal consumer[19]:

$$\widehat{p_i} = (4)\,\frac{\tau(1/2)^2}{r+\mu}.$$

According to proposition 3, the reduction in price due to the existence of network effects depends on the parameters of the model, such as the effective interest rate $(r+\mu)$ and the degree of compatibility between networks. From proposition 3, it follows that as $\phi \to 0$,

$$\widehat{p}(b^*)_{\rceil\phi\to 0} \to \frac{\tau}{(r+\mu)} + \frac{2\tau\left(b^{*2}r\,(r+2\mu) - (5r+8\mu)b^* + 4\right)}{b^{*2}\mu(r+\mu)^2(4+b^*(r+2\mu))}$$

where $\widehat{p}(b^*)_{\rceil\phi\to 0} < \frac{\tau}{(r+\mu)} \forall b^* \in \left(0, \frac{1-\phi}{r+\mu}\right)$. Consequently, the degree of incompatibility between networks enhances the importance of network effects. When $\phi \to 0$, consumers cannot benefit at all from network indirect effects created by the rival's network and, accordingly, when networks are fully incompatible, the importance of investing in the creation of a "locked in" market is even larger and firms will always charge lower steady state equipment prices than the ones that would be charged in the absence of indirect network effects.

Conversely, when $\phi \to 1$:

$$\widehat{p}(b^*)_{\rceil\phi\to 1} \to \frac{\tau}{(r+\mu)} + \frac{2\tau\,((r+2\mu))}{b^*\mu(r+\mu)^2} > \frac{\tau}{(r+\mu)}.$$

When networks are fully compatible, firms' specific network effects are no longer relevant (because consumers benefit as well from the rivals' network). As a consequence, once consumers are locked in to equipment manufacturers, firms do not have any incentives to expand their own networks (because they can benefit as well from rival's indirect network benefits). Consequently, when choosing prices in the primary market, firm will set higher equipment prices (in comparison with the ones that would prevail in the absence of indirect network effects). The justification for this is the

[18] From equation (47) $b^* \to \frac{1-\phi}{r+2\mu}$ as $a \to 0$.

[19] In the steady state equilibrium, the marginal consumer is located at a distance of 1/2 of both firms.

fact that, under perfect compatibility, firms can appropriate all the (potential) *"network benefits"* obtained by consumers. Thus, in our framework, the degree of compatibility between networks (ϕ) can be interpreted as a kind of "*network effects multiplier*": when ϕ is low, indirect network effects are reinforced and, conversely, when ϕ is high, indirect network effects are weakened and there is a *competition dampening effect*.

6. CONCLUSION

In this paper, we investigate dynamic price competition between firms, who strategically interact in two distinct but interrelated markets: primary markets and aftermarkets. In the primary market firms sell two distinct versions of a certain original equipment and, afterwards, firms provide CGS to their costumers (in the aftermarket). In our setting these two markets are assumed to be closely related, not only because consumers are considered to be totally locked in to the equipment they buy in the primary market (i.e. CGS can only be purchased from the equipment manufacturer) but also because we consider that consumption of CGS exhibit (partially compatible) indirect network externalities (i.e. consumers benefit from the total number of equipment owners, but the benefit exerted by consumers who own a similar equipment version is greater than the benefit exerted by consumers who own a different version of the same original equipment).

In order to analyze the nature of competition under these circumstances, we propose a linear-quadratic differential game of two-dimensional price competition (the price of equipment and the price of CGS) and we analyze the predictions of our model with respect to the trade-off between *competition in the market* (competition in the aftermarket, exploiting the existent basis of (locked in) consumers) and *competition for the market* (competition in the primary market in order to build a durable basis of (locked in) consumers).

Within the setting previously described, our first main result is that the absence of competition in the market (due to consumers' lock in) effectively fosters competition for the market.

In line with the arguments pointed out in competition policy cases such as *Kodak, Xerox or Chrysler*, we show that when costumers are totally locked in to the original equipment' manufacturers, these are monopolists in the provision of CGS to their costumers (and consequently competition in the market is substantially damaged). However, we argue that this situation is not necessarily welfare detrimental and, we analytically show that the expectation of monopoly rents in the aftermarket induces firms to compete more fiercely in the primary market (in order to create a durable basis of costumers and increase the future size of the aftermarket).

Concerning the analysis of the determinants of the intensity of competition for the market, within our setting, several factors might foster or dampen the intensity of competition for uncolonized consumers in the primary market. More precisely: the relative size of the aftermarket, the intensity of indirect network effects, the degree of compatibility between networks and the effective discount rate. Actually, we show that, in some circumstances (when the aftermarket is large enough, the network effects are sufficiently strong, the compatibility between networks is sufficiently weak and the effective discount rate is sufficiently low), competition for new consumers might become so tough that firms might even have incentives to charge negative prices.

In addition, we have also demonstrated that the incentives to compete for the market will be larger for firms with a larger basis of costumers in the primary market (i.e. a larger "locked in" market. Such effect is entailed by the strong relations between primary markets and aftermarkets: the larger the existent basis of costumers in the primary market, the greater will be the monopolist rents obtained in the aftermarket. Our analysis reveals that static incentives are enough to generate this negative correlation. However, when dynamic effects are also considered, this correlation tends to be reinforced.

In any case, when network effects are not too strong, the increasing returns effect just described is not sufficient to force the smaller firm to leave the market and, in the steady state LMPE, both firms survive, sharing the market evenly.

When network effects are very intense (for a given size of the aftermarket), the game has no competitive LMPE. In our future research, we intend to investigate whether a monopolistic equilibrium could arise in such circumstances, pointing out the conditions under which the smaller firm would be evicted from the market. Other extensions of the model are worthwhile as well. A natural extension of the model would be to consider finite switching costs, allowing consumers to switch between brands in the aftermarket. A second possible extension would be to consider that original equipment versions are (horizontally) differentiated at the eyes of consumers and, in that case, it would be worthy to investigate whether consumers have incentives to buy both original equipment *(multi-homing).* A third possible extension would be to introduce the possibility of price commitment in the aftermarket (while such commitment doesn't seem to be relevant in industries such as printers, software,... in other industries this commitment is very important, e.g. in the case of mobile telecommunications).

Appendix

Proof of Lemma 1:

Since the actual values $\Lambda_i(t)$ are defined by (14), for rational expectations to hold, functions $F_i(.)$ must satisfy the following properties

$$\frac{\phi}{r+\mu} \leq F_i(D_1(t), D_2(t)) \leq \frac{1}{r+\mu} \tag{55}$$

$$F_1(D_1(t), D_2(t)) + F_2(D_1(t), D_2(t)) = \frac{1+\phi}{r+\mu} \tag{56}$$

for all $(D_1(t), D_2(t)) \in \Delta$.

Since there are only two firms, condition (56) implies that once specified the function $F_1(.)$, it is possible to infer immediately the function $F_2(.)$ because of such functional dependence. Also, given that $D_2 = 1 - D_1$, we can write

$$\Lambda_1^e(t) = F_1(D_1(t), D_2(t)) = F_1(D_1(t), 1 - D_1(t)) \equiv f_1(D_1(t))$$

and

$$\begin{aligned} \Lambda_2^e(t) &= \frac{1+\phi}{r+\mu} - F_1(D_1(t), D_2(t)) = \frac{1+\phi}{r+\mu} - f_1(D_1(t)) \\ &= \frac{1+\phi}{r+\mu} - f_1(1 - D_2(t)) \equiv f_2(D_2(t)). \end{aligned}$$

Since agents are assumed to be forward looking, $\Lambda_i^e(t)$ must be equal to $\Lambda_i(t)$, i.e., for all t, the following condition must be observed:

$$\Lambda_i^e(t) = f_i(D_i(t)) = \int_t^\infty \left[D_i(v) + \phi(1 - D_i(v))\right] e^{-(r+\mu)(v-t)} dv \equiv \Lambda_i(t). \tag{57}$$

A special case of interest is the linear expectation rule, where, for $D_1 \in [0, 1]$

$$\frac{\phi}{r+\mu} \leq f_1(D_1) = \delta_1 + b_1 D_1 \leq \frac{1}{r+\mu} \tag{58}$$

and, for $D_2 \in [0, 1]$

$$f_2(D_2) = \frac{1+\phi}{r+\mu} - f_1(1 - D_2(t)) = \frac{1+\phi}{r+\mu} - [\delta_1 + b_1(1 - D_2)] \equiv \delta_2 + b_2 D_2. \tag{59}$$

Comparing (59) with (58), it follows that

$$b_2 = b_1 \equiv b \tag{60}$$

and

$$\delta_2 = \frac{1+\phi}{r+\mu} - b - \delta_1, \tag{61}$$

which correspond, respectively, to equations (27) and (28) in lemma 1.

In addition, notice that if one sets $D_1 = 0$ in (58), we obtain the restriction

$$\frac{\phi}{r+\mu} \leq \delta_1 \leq \frac{1}{r+\mu}. \tag{62}$$

Similarly, setting $D_1 = 1$ in (58), we obtain

$$\frac{\phi}{r+\mu} \leq \delta_1 + b \leq \frac{1}{r+\mu}$$

which, together with (62) yields the restriction (26) in lemma 1.

Furthermore, the rational expectation requirement that $\Lambda_i^e(t) = \Lambda_i(t)$ implies the restriction that $b > 0$. To see this, note that $\Lambda_i^e(t) = \Lambda_i(t)$ implies

$$\dot{\Lambda}_i^e(t) = \dot{\Lambda}_i(t)$$

which implies, from differentiating (57) with respect to time,

$$\begin{aligned} b\dot{D}_i(t) &= (r+\mu)\Lambda_i(t) - [D_i(t) + \phi(1 - D_i(t))] \\ &= (r+\mu)\Lambda_i^e(t) - [D_i(t) + \phi(1 - D_i(t))] \\ &= (r+\mu)\left[\delta_i + bD_i(t)\right] - [D_i(t) + \phi(1 - D_i(t))] \end{aligned}$$

This gives us the differential equation

$$b\dot{D}_i(t) = (r+\mu)\left[\delta_i + bD_i(t)\right] - [D_i(t) + \phi(1 - D_i(t))] \tag{63}$$

Upon integration, we get

$$D_i(v) = \frac{\delta_i(r+\mu) - \phi}{(1-\phi) - b(r+\mu)} + \left[D_i(t) - \frac{\delta_i(r+\mu) - \phi}{(1-\phi) - b(r+\mu)}\right] e^{\frac{(r+\mu)b-(1-\phi)}{b}(v-t)}.$$

Substituting this $D_i(v)$ into (14) we get

$$\Lambda_i(t) = \frac{\delta_i(1-\phi) - b\phi}{(1-\phi) - b(r+\mu)} + (1-\phi)\left(D_i(t) - \frac{\delta_i(r+\mu) - \phi}{(1-\phi) - b(r+\mu)}\right)\int_t^{\infty} e^{-\frac{(1-\phi)}{b}(v-t)} dv$$

which is equal to $\Lambda_i^e(t) = \delta_i + bD_i(t)$ if and only if b is strictly positive.■

Proof of Lemma 2:

Substituting for $\lambda_i(t)$ in equation (39) using the right-hand side of (38), replacing $p_i(t)$ by $\eta_i + s_i D_i(t)$ and rearranging we obtain

$$\frac{d\lambda_i(t)}{dt} = A_i + B_i\ D_i(t) \tag{64}$$

where the values of A_i and B_i are easy to compute.

In equation (38), replace $p_i(t)$ by $\eta_i + s_i D_i(t)$, then differentiate with respect to t and get the following condition:

$$\frac{d\lambda_i(t)}{dt} = [-s_j - 2s_i + ab]\frac{dD_i(t)}{dt} \tag{65}$$

In the equation (36) for $\frac{dD_i(t)}{dt}$ let us substitute $\eta_i + s_i D_i(t)$ for $p_i(t)$. Then substitute the resulting expression for $\frac{dD_i(t)}{dt}$ in equation (65) and rearrange in order to obtain

$$\frac{d\lambda_i(t)}{dt} = F_i + G_i\ D_i(t) \tag{66}$$

where the values of F_i and G_i are easy to compute.

In the LMPE of the game it must be that, at each $t \geq 0$, and for $i = 1, 2$, $A_i + B_i\ D_i(t) = F_i + G_i\ D_i(t)$. It follows that $A_i = F_i$ and $B_i = G_i$.

Equating B_i and G_i for $i = 1, 2$, yields two equations in s_1 and s_2 :

$$(r + 2\mu)(2s_i + s_j - ab) + \frac{2}{2\tau}\mu (r + \mu)(s_i + s_j - ab)^2 - 2a(\phi - 1) = 0, \tag{67}$$

for $i = 1, 2$, $j \neq i$.

Subtracting the second equation from the first we obtain

$$(r + 2\mu)(s_1 - s_2) = 0$$

Not surprisingly it turns out that $s_1 = s_2 = s$: the impact of a firm' s current market share on the price it sets is the same for both firms. Now from equations (67) we obtain the following equation for s :

$$(r + 2\mu)(3s - ab) + \frac{2}{2\tau}\mu (r + \mu)(2s - ab)^2 - 2a(\phi - 1) = 0, \tag{68}$$

which corresponds to equation (40) in lemma 2.

Equating A_i and F_i for $i = 1, 2$, yields two equations in η_1 and η_2. Subtracting the second from the first, we obtain

$$\frac{(r + \mu)}{\tau}\left((3\tau + 4s\mu - 2ab\mu)(\eta_2 - \eta_1) + a(\tau + 2s\mu - ab\mu)(\delta_1 - \delta_2)\right) = 0, \tag{69}$$

which corresponds to equation (41) in lemma 2.■

Proof of Lemma 3:

Replace the left-hand side of equation (63) by $b\,(d_i(D_i(t)) - \mu D_i(t))$ where

$$d_i(D_i(t)) = \mu \left[\frac{1}{2} + \frac{(\eta_j - \eta_i) + s(1 - 2D_i(t))(r + \mu) + a(\delta_i + bD_i(t))(r + \mu) - \frac{a}{2}(1 + \phi)}{2\ \tau}\right]$$

The resulting equation is of the form $MD_i + N_i = 0$. Since this must hold for all values of $D_i \in [0, 1]$, it follows that $M = 0$, i.e.,

$$\frac{1}{2\tau}[ab^2\mu(r + \mu) - 2b(s\mu(r + \mu) + \tau(r + 2\mu)) + 2\tau(1 - \phi)] = 0 \tag{70}$$

and $N_i = 0$, i.e.,

$$\delta_i = \frac{2b\mu(r + \mu)(\eta_i - \eta_j) + ab\mu(1 + \phi) - 2bs\mu(r + \mu) - 2b\mu\tau - 4\tau\phi}{2(r + \mu)(ab\mu - 2\tau)}$$

■

Proof of Lemma 4:

The fact that $b_1 = b_2 = b$, stems directly from lemma 1. Similarly, the condition $s_1 = s_2 = s$, follows directly from lemma 2.

Concerning the equilibrium intercepts of consumers' expectation rules and firms' price strategies, from equations (69) and (44), one has

$$\frac{(r+\mu)}{(ab\mu - 2\tau)}(\eta_1 - \eta_2)(6\tau + 8s\mu - 5ab\mu) = 0.$$

From the previous condition, it follows that $\eta_1 = \eta_2 = \eta$. Introducing this in equation (44), one obtains that $\delta_1 = \delta_2 = \delta$. ■

Proof of Proposition 1:

From equation (70) one obtains

$$s(b) = \frac{ab^2\mu(r+\mu) + 2\tau(1-\phi-b(r+2\mu))}{2b\mu(r+\mu)} \tag{71}$$

Substituting this value for s in equation (68) one obtains

$$X(b) = \left\{ \begin{array}{c} b^3 a\mu\,(r+\mu)\,(r+2\mu) + \\ b^2\left(4a\mu\,(\mu+r)\,(1-\phi) + 2\tau(r+2\mu)^2\right) - \\ 10b\tau\,((1-\phi)\,(r+2\mu)) + 8\tau(1-\phi)^2 \end{array} \right\} = 0, \tag{72}$$

When the LMPE exists, the optimal value(s) of b^* must be such that condition (72) holds and

$$0 < b^* \leq \frac{1-\phi}{r+\mu}\text{[20]},$$

where $X(b)$ is a third-order polynomial in b and:

(i) $X(-\infty) = -\infty$;

(ii) $X(+\infty) = +\infty$;

(iii) $X(0) = 8(1-\phi)^2 > 0$[21].

In the absence of network effects, $a = 0$, and $X(b)$ would become a second-order polynomial with two roots

$$b^* = \left\{ \frac{1-\phi}{r+2\mu}; 4\frac{1-\phi}{r+2\mu} \right\}.$$

Since $X(b)$ is strictly increasing in a for all the values of $a > 0$, by continuity, one has that $X(b)$ in (72) has two positive roots in the interval $\left[\frac{1-\phi}{r+2\mu}, 4\frac{1-\phi}{r+2\mu}\right]$, as long as a is relatively small.

[20] These thresholds follow directly from lemma 1.

[21] From (i) and (iii), it follows that $X(b)$ has necessarily one negative root.

Furthermore, note that

$$X\left(\frac{1-\phi}{r+2\mu}\right)=\frac{5a\mu(r+\mu)(1-\phi)^3}{(r+2\mu)^2}>0, \tag{73}$$

which is is positive as long as $a>0$ and $\phi<1$.

Moreover, at the upper threshold for b^*, one has

$$X\left(\frac{1-\phi}{r+\mu}\right)=\left(\frac{1-\phi}{r+\mu}\right)^2\mu\left(a\left(1-\phi\right)\left(5r+6\mu\right)-\tau\left(6r+4\mu\right)\right)\right),$$

which is negative iif

$$a\leq\frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}, \tag{74}$$

and positive otherwise.

Since $X\left(\frac{1-\phi}{r+2\mu}\right)>0$, straightforwardly we can conclude that condition (74) is sufficient to ensure that:

1. the polynomial (72) has two positive roots;
2. only one of those roots satisfies the equilibrium requirement $b^*\leq\frac{1-\phi}{r+\mu}$ and, consequently, when condition (74) holds, the LMPE of the game exists and, moreover, it is unique. This corresponds to case (i) in proposition 1.

When condition (74) doesn't hold, two situations might arise:

1. there are two positive roots such that $0<b^*\leq\frac{1-\phi}{r+\mu}$ and case (ii) in proposition 1 is observed; or
2. no positive roots satisfy the conditions $0<b^*\leq\frac{1-\phi}{r+\mu}$ and case (iii) in proposition 1 arises.

Note that, when $X'\left(\frac{1-\phi}{r+\mu}\right)<0$, we are necessarily in case 2. Conversely, when $X'\left(\frac{1-\phi}{r+\mu}\right)>0$, both cases are possible.

Differentiating $X(b)$ with respect to b we obtain

$$X^{'}(b)=-10\tau(r+2\mu)(1-\phi)+2b(4a\mu(r+\mu)(1-\phi)+2\tau(r+2\mu)^2)+3b^2a\mu(r+\mu)(r+2\mu),$$

with

$$\frac{\partial X^{'}(b)}{\partial a}=b\mu\left(r+\mu\right)\left(6b\mu+3br+8(1-\phi)\right)>0.$$

Note that, when $a=\frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}$ we obtain

$$X'\left(\frac{1-\phi}{r+\mu}\right)=\frac{-2\tau(1-\phi)(15r^3+20r^2\mu-12r\mu^2-16\mu^3)}{(r+\mu)(5r+6\mu)},$$

with $X'(\frac{1-\phi}{r+\mu})$ positive whenever $r \in \left[0, \frac{2\mu}{\sqrt{5}}\right[$ and negative when $r \in \left]\frac{2\mu}{\sqrt{5}}, +\infty\right[$.

Consequently, it is immediately possible to conclude that, when $r \in \left]\frac{2\mu}{\sqrt{5}}, +\infty\right[$, there is no LMPE when $a > \frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}$ since either $X(b)$ has two positive roots strictly larger than $\frac{1-\phi}{r+\mu}$ or it has no positive roots[22]. In those circumstances, situation (iii) in proposition 1 arises.

When $r \in \left[0, \frac{2\mu}{\sqrt{5}}\right[$, there is a range of values $a \in (\frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}, \overline{a}]$ for which there exist two LMPE since $X(b)$ has two positive roots lower than $\frac{1-\phi}{r+\mu}$. In order to determine the value of $\overline{a}$, one only has to find the positive values of a and b, which solve the two equations $X(b) = 0$ and $X'(b) = 0$. We find.

$$\overline{a} = \frac{\tau(r+2\mu)^2(7\sqrt{5}-15)}{\mu(r+\mu)(1-\phi)(5-\sqrt{5})(3-\sqrt{5})^2}$$

and it is easy to show that $\overline{a} > \frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)} \Leftrightarrow r < \frac{2\mu}{\sqrt{5}}$. Therefore, when $r \in \left[0, \frac{2\mu}{\sqrt{5}}\right[$, two LMPE exist when $a \in (\frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}, \frac{\tau(r+2\mu)^2(7\sqrt{5}-15)}{\mu(r+\mu)(1-\phi)(5-\sqrt{5})(3-\sqrt{5})^2}]$ and, situation (ii) in proposition 1 occurs.

When $a > \frac{\tau(r+2\mu)^2(7\sqrt{5}-15)}{\mu(r+\mu)(1-\phi)(5-\sqrt{5})(3-\sqrt{5})^2}$ or $a > \frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)} \vee r \in \left[0, \frac{2\mu}{\sqrt{5}}\right[$, no LMPE exists and the situation (iii) in proposition 1 takes place.■

Proof of Lemma 6:

Solving (72) with respect to a, one gets

$$a = \frac{10b\tau\left(\left(1-\phi\right)\left(r+2\mu\right)\right) - 8\tau(1-\phi)^2}{b^3\mu\left(r+\mu\right)\left(r+2\mu\right) + b^2\left(4\mu\left(\mu+r\right)\left(1-\phi\right) + 2\tau(r+2\mu)^2\right)} \tag{75}$$

Introducing condition (75) into equation (71), one gets

$$s^*(b^*) = \frac{2\tau\left(r+2\mu\right)\left(1-\phi-b^*(r+2\mu)\right)}{\mu(r+\mu)\left(b^*\left(r+2\mu\right)+4\left(1-\phi\right)\right)} < 0,$$

which is necessarily negative given that

$$b^* > \frac{1-\phi}{r+2\mu},$$

from lemma 5.■

$$0 < b^* \leq \frac{1-\phi}{r+\mu}{}^{23},$$

where $X(b)$ is a third-order polynomial in b and:

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[22]This amounts to say that, when $r \in \left]\frac{2\mu}{\sqrt{5}}, +\infty\right[$, the condition $a \leq \frac{\tau(6r+4\mu)}{(5r+6\mu)(1-\phi)}$ is not only sufficient but also necessary for the existence of a (in this case unique) LMPE.

[23]These thresholds follow directly from lemma 1.

(i) $X(-\infty) = -\infty$;

(ii) $X(+\infty) = +\infty$;

(iii) $X(0) = 8(1-\phi)^2 > 0$[24].

Proof of Lemma 7:

Following from Lemma 4, the equilibrium values of δ_i are such that $\delta_1 = \delta_2 = \delta$. Introducing such condition in equation (28) from Lemma 1 and solving for δ, one gets condition (49)

$$\delta(b) = \delta = \frac{1+\phi-b\,(r+\mu)}{2\,(r+\mu)}.$$

The equilibrium values of η_i as functions of b (equations (50) in lemma 7), $i = 1, 2$, are obtained by solving the system of equations (41) for η_1 and η_2, then substituting s for its value from equation (71) and finally δ_1 and δ_2 for their values from equations (49). ■

Proof of Proposition 3

The steady-state subscription prices are given by

$$\widehat{p}_i = \eta_i(b^*) + s(b^*)\widehat{D}_i \tag{76}$$

where $\widehat{D}_i$ is the steady-state market share of firm i, which is equal to $\frac{1}{2}$. Introducing this into (76) and considering the equilibrium values of $\eta(b^*)$ and $s(b^*)$, the steady-state subscription prices are

$$\widehat{p}(b^*) = \frac{\tau}{(r+\mu)} + \frac{2\tau\left(b^{*2}\,(r+2\mu)\,(r+2\mu\phi) - b^*(1-\phi)(5r+2\mu(4+\phi)) + 4\,(\phi-1)^2\right)}{b^{*2}\mu(r+\mu)^2(4(1-\phi)+b^*(r+2\mu))},$$

which correspond to the values of $\widehat{p}(b^*)$ pointed out in proposition 3.

Moreover, it is easy to demonstrate that

$$\frac{2\tau\left(b^{*2}\,(r+2\mu)\,(r+2\mu\phi) - b^*(1-\phi)(5r+2\mu(4+\phi)) + 4\,(\phi-1)^2\right)}{b^{*2}\mu(r+\mu)^2(4(1-\phi)+b^*(r+2\mu))} < 0,$$

in the interval $b \in \left(\frac{1-\phi}{r+2\mu}, 4\frac{1-\phi}{r+2\mu}\right)$, which is always the case, given that $\frac{1-\phi}{r+2\mu} < b < \frac{1-\phi}{r+\mu}$.As a consequence, in the LMPE $\widehat{p}(b^*)$ is necessarily smaller than $\frac{\tau}{(r+\mu)}$ ■

[24] From (i) and (iii), it follows that $X(b)$ has necessarily one negative root.

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