The financing of innovation, learning and renegotiation[*]

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Abstract

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1 Introduction

The value of a new investment project is most often subject to two different kinds of uncertainty. The first kind is the uncertainty about the time of the successful completion of the investment project. The second kind is the future value of the project conditional upon its completion. The research and development process for a new pharmaceutical product illustrates both aspects of uncertainty. The idea for a new drug is most likely based on some initial and very preliminary research. The development project itself requires substantial further work and experiments before the value of the initial idea can be assessed with sufficient confidence. This process requires substantial investments and produces over time more information as to whether the project is successful or should be abandoned due to poor results. However even if the new drug proves to be an innovation, market conditions may have changed, favorably or unfavorably during the development time, and thereby affecting the true value of the project. This paper examines the financing of a project in the presence of uncertainty about the successful completion of the project.

The basic model of this paper considers an entrepreneur who proposes a project, essentially an idea, to outside investors. The realization of the project requires investment funds and as the entrepreneur is wealth constrained she is required to seek outside financing. It is initially unknown, both to the entrepreneur as well as to the investor whether the project is a potential success or failure. The uncertainty about the outcome is represented by a simple stochastic process. If the project is a potential success, then there is a positive probability in every period that the project will be completed successfully. The probability is proportional to the funds allocated to the project in this period and represents the research intensity. If the project is a failure, then the probability of completion is equal to zero at all funding levels. The development of the project initiates a Bayesian learning process as continued failure to generate success will lead the participants to update their beliefs of the likelihood of eventual success. The funding of the project ends either with the success or with the stopping of the project in the light of persistent negative news.

The paper analyzes the financing of the project when the entrepreneur controls the allocation of the funds. It examines whether the funds are released at the efficient rate by the investor and whether the project is abandoned at the efficient stopping point. It further investigates how entrepreneur and investor share the proceeds of the project as a function of the elapsed time and received funding until the completion of the project.

The basic features of the model may be associated most directly with the financing of high-technology ventures and start-ups. However the optimal financing of research is also a concern for the intertemporal capital budgeting process *within* a firm or an organization, where the timing in the release of the funds for a given project are to be determined.

Next, we briefly discuss some of the basic aspects of the model and then provide a general overview over the results and the related literature. For concreteness consider a firm which intends to develop a new product. It needs to hire a specialized scientist to oversee the development process. The speed of the development process depends on the level of funding which the company offers the scientist for her research. The scientist can allocate these funds either efficiently towards the development of the product or divert the funds for private ends, say fundamental research for which the scientist has a preference. The firm therefore offers the scientist a share in the proceeds of the new product in the event of its realization to provide the proper incentives. Suppose first that the firm simply offers a constant share to the scientist and commits itself to finance the project for a fixed number of periods. Can we expect the scientist to allocate the funds to the product development or could she be tempted to use the funds differently? In a second step, we then ask how the incentives can be improved by modifying sharing rule and funding allocations over time.

The scientist could reason as follows. With a sufficiently long funding horizon, the chances to develop the product in the future are sufficiently high and thus she might initially direct the resources towards fundamental research. She is however aware of the fact that development takes time and thus at some point in the future she will direct the funds to the development of the product. As she is applying herself to the development of the product but without being successful, she becomes gradually more pessimistic about the development prospects. The expected value of the prize thus decreases and so she might turn towards the end of the funding horizon again to fundamental research as the incentives provided by the firm become gradually weaker. The incentive problems therefore arise most pronounced at the beginning and the end of the funding horizon. How could the firm respond to this situation and improve the incentives? Clearly, the firm could increase the share of the scientist towards the end to maintain the incentives. But as the expected returns decrease towards the end, it might do so only to a limited extent and rather stop the process completely. It is then to be expected that the project is stopped too early relative to its efficient stopping point. How about the incentives in the beginning of the process? An immediate answer would be to simply decrease the funding horizon and thereby cut out the initial phase. But, the scientist realizes that this would imply to stop the project too early, and the firm would eventually offer to provide additional funding to continue the development. She would anticipate the future behavior of the firm, and no matter how long the firm would promise initial funding she would be able to count on future funding and hence pursue her policy unchanged. The option of a shorter funding horizon is thus limited by renegotiation and the inability of the firm to commit to inefficient policies in the future. As the firm cannot credibly commit to lower the funding level in the future, the firm can still reduce current funding. This lowers the likelihood of success in every period and thus decreases the value of future discovery with positive discounting. The value of postponing the product development is thus decreasing and the scientist has larger incentives to start the applied research immediately. The funding volume is then set below the efficient level to lower the option value of future research in favor of immediate research. A final observation in this context pertains to the role of the discount factor. As the scientist values the future more, she has less incentive for obtain immediate success and hence the incentive problem becomes more severe as the discount factor δ increases.

The model is formally presented in Section 2. The equilibrium analysis begins in Section 3 by considering observable actions by the entrepreneur. The information about the project is symmetric in this environment and renegotiation can occur under symmetric information. First we consider financial contracts in which the investor breaks even in expectations in every period. The unique renegotiation-proof equilibrium in this setting displays the properties just described. Funding occurs at a slower than efficient rate and ceases altogether too early relative to the efficient stopping point. The divergence between the efficient rate and the equilibrium rate of funding is monotonically increasing over time. We then remove the short-term financing constraint and consider contracts in which the investor only requires to break even over an arbitrarily time interval. Long-term contracts permit intertemporal risk sharing. We find that long-term contracts can strictly improve upon short-term contracts if and only if short-term contracts would provide efficient financing at the start of the project. The role of long-term contracts is then to extend the time horizon over which funding can be provided at the efficient level. If short-term contracting already provides less than efficient funding at the start, which is bound to happen for large discount factors δ, then long-term contracts yield equivalent results to short-term contracts.

Section 4 examines equilibrium financing when the allocation decision of the entrepreneur is unobservable to the investor. The private beliefs of entrepreneur and investor about the project can now diverge over the funding horizon as the investor can't observe whether the entrepreneur allocated the funds to the development of the project as desired. The allocation decision by the entrepreneur is thus subject to a classical moral hazard problem. But the moral hazard problem in the current period turns into an adverse selection problem in the future periods. As the investor doesn't know whether the entrepreneur did or did not exert effort on the project in the past period, he has only imperfect information about the true probability of future success. From the point of view of the entrepreneur, the control over the funding flow then translate into control over the information flow as well. The asymmetry in the information surprisingly reduces the contractual inefficiencies with observable actions. With observable actions, the entrepreneur could convince the investor to simply renew the funding proposal of the previous period if she diverted funds in the preceding period. As the investor observes the diversion, he has no reason to evaluate the project any different than in the previous period. With unobservable actions of the entrepreneur, the investor needs evidence to accept a proposal which he would have accepted yesterday but would accept today only if he knew that in the interim no new information about the prospects of the projects surfaced. The proposal by the entrepreneur therefore needs to have signalling character and separate between projects which generated information in the preceding period and those which didn't. However we show that the unique perfect Bayesian equilibrium is a pooling equilibrium where all types offer the same contract. In particular, along the equilibrium path the investor always believes that the entrepreneur invested the funds into the project in the past. The renegotiation of any contract thus occurs in equilibrium always under the assumption that the project received the funds but didn't generate yet the desired success. The equilibrium belief by the investor lessens the value of the diversion option for the entrepreneur. In consequence, the funding in the asymmetric information model is uniformly higher than in the symmetric model. While we spell out the consequences for short and long-term contracts, it might suffice here to point out that the informational constraints on the investor are particularly important for the funding just before the project is stopped. In a variety of situations the funding now occurs at the efficient rate. If the investor could therefore make an initial choice whether or not he would like to track the decisions by the entrepreneur, he would prefer to stay at a distance and not observe the decisions of the entrepreneur, giving rise to informationally arm's length contracts.

The funding of the project was considered so far in an environment where the entrepreneur and the investor only control flow investments and there are no competing projects. Section 5 considers the implications of competing projects and fixed costs to start the project. Briefly, we find that competition generally enhances the efficiency of the financing as a possibility of preemption decreases the option value of postponing the investment today. If the parties to the project have to make initially a fixed investment which determines the scale of the development process, then we find that the scale is chosen smaller than efficient to slow down the future volume of financing and again depress the option value of waiting. Section 6 presents some concluding remarks.

?) consider a dynamic model of debt with renegotiation. In contrast to our model, the investor in **?**) faces no moral hazard problem and it is the cash flow which can be diverted rather than the investment flow as in our model. Moreover, in our model there are no assets beyond the human capital of the entrepreneur. The distinction between observable and unobservable actions has first been introduced in the finance literature by **?**).

2 Model

The project and the investment technology is presented in Subsection 2.1. The successful realization of the project is positively correlated with the received funding. The investment flow induces a Bayesian learning process about the prospects of the project. The evolution of the posterior beliefs and the efficient stopping time are presented in Subsection 2.2. The time structure of the contracting game between entrepreneur and investor is described in Subsection 2.3.

2.1 Project with Unknown Returns

The entrepreneur owns a project with unknown returns. The project is either "good" with prior probability α_0 or "bad" with prior probability $1 \Leftrightarrow \alpha_0$. If the project is "good", then in every period t, there is a certain probability that the project is successfully completed and yields a fixed payoff R. The probability of success in period t, *conditional* on the project being good, is denoted by λ_t. The probability λ_t is an increasing function of the investment flow in period t. More precisely, a success probability λ_t requires an investment flow of $c(\lambda_t)$ in period t. We assume that $c(\lambda_t)$ is a linear function of λ_t:

$$c(\lambda_t) = c\,\lambda_t,\ c > 0.$$

The maximal conditional probability of success is denoted by λ (without subscript), with $0 < \lambda < 1$ and any probability $\lambda_t \in [0, \lambda]$ is feasible in every period. Any investment beyond $c(\lambda)$ does not increase the probability of success. If the project is "bad", then it will never yield a positive return and the probability of success is zero independent of the investment flow. The project can receive funding over any number of periods and time is discrete with $t = 0, 1, \ldots$. Entrepreneur and investor are both risk-neutral and have a common discount factor $\delta \in (0, 1)$.

2.2 Learning and Efficient Stopping

The uncertainty about the project is resolved over time as the inflow of funds either produce a success or lead to the stopping of the project. The investment process is like an experiment which produces information about the future likelihood of success. The current information is represented by the posterior belief α_t that the project is good. The evolution of the posterior belief α_{t+1}, conditional on no success in period t, is given by Bayes' rule as a function of the prior belief α_t and the investment flow $c\lambda_t$ as:

$$\alpha_{t+1} = \frac{\alpha_t (1 \Leftrightarrow \lambda_t)}{\alpha_t (1 \Leftrightarrow \lambda_t) + 1 \Leftrightarrow \alpha_t}. \tag{1}$$

The posterior belief α_{t+1} thus decreases over time if success hasn't materialized yet. The decline in the posterior belief is stronger for larger investments, as the participants become more pessimistic about the likelihood of future success. The posterior belief changes only slowly for very precise beliefs about the nature of the project, i.e. if α_t is either close to 0 or 1. Correspondingly the event of no success is most informative with very diffuse beliefs and α_t is close to $\frac{1}{2}$.

The project should receive funds as long as the expected return from the investment exceeds the costs, or

$$\alpha_t \lambda_t R \Leftrightarrow c\lambda_t \geq 0, \tag{2}$$

for some $\lambda_t \in [0, \lambda]$. It follows that the efficient stopping point α^* is given by:

$$\alpha^* \lambda_t R \Leftrightarrow c\lambda_t = 0 \;\; \Leftrightarrow \;\; \alpha^* = \frac{c}{R}. \tag{3}$$

The posterior belief α^* at which stopping is efficient is decreasing in the return R and increasing in the marginal cost c of generating success probabilities. Due the linear structure of the investment problem indicated in (2) it is optimal to choose $\lambda_T = \lambda$ in the final period, denoted by T. The social value $V(\alpha_T)$ of the project in the terminal period T is then:

$$V(\alpha_T) = \alpha_T \lambda R \Leftrightarrow c\lambda.$$

and in general the value of the project is given by a familiar dynamic programming equation:

$$V(\alpha_t) = \max_{\lambda_t} \{\alpha_t \lambda_t R \Leftrightarrow c\lambda_t + (1 \Leftrightarrow \alpha_t \lambda_t)\, \delta V(\alpha_{t+1})\} \tag{4}$$

where the posterior belief α_{t+1} is determined by Bayes' rule as in (1). A conditional probability of success λ_t in period t yields an expected return $\alpha_t \lambda_t R$.and costs $c\lambda_t$. With probability $1 \Leftrightarrow \alpha_t \lambda_t$ no success is observed in period t and the project is continued under the new assessment α_{t+1} in the next period. The value function (4) indicates again the linearity in λ_t and it can be shown that it is optimal to invest $\lambda_t = \lambda$ in every period in which the project is operated.

The stopping point is characterized by the posterior belief in (3), but for any given prior belief α_0 there is a one-to-one relationship between the stopping point α^* and the stopping time T^* which expresses the same policy in terms of *real time*:

$$T^* \triangleq \max \left\{ T \;\middle|\; \frac{\alpha_0 \left(1 \Leftrightarrow \lambda\right)^T}{\alpha_0 \left(1 \Leftrightarrow \lambda\right)^T + 1 \Leftrightarrow \alpha_0} \geq \alpha^* \right\}$$

Evidently, the optimal stopping time T^* depends on the prior belief α_0 at which the project is started. The dependence on α_0 is suppressed for notational convenience. The stopping time T^* represents the time elapsed between starting at α_0 and arriving at the last posterior belief exceeding α^*.

Proposition 1 (Optimal Investment Policy)

1. *The optimal policy is to invest maximally $c\lambda$ until T^*.*

2. *The social value of the project is:*

$$V(\alpha_0) = \alpha_0 \lambda \left(R \Leftrightarrow c\right) \frac{1 \Leftrightarrow \delta^{T^*} \left(1 \Leftrightarrow \lambda\right)^{T^*}}{1 \Leftrightarrow \delta(1 \Leftrightarrow \lambda)} \Leftrightarrow (1 \Leftrightarrow \alpha_0)\, c\lambda \frac{1 \Leftrightarrow \delta^{T^*}}{1 \Leftrightarrow \delta}. \qquad (5)$$

The results follow from standard dynamic programming arguments and the proofs are omitted. The value function $V\left(\alpha_0\right)$ presents an intuitive decomposition of the value of the project. The first term in (5) is the expected value of the project conditional on the project being good. Notice that the value of the project is discounted at a rate which compounds the pure discount rate δ and the probability of *no* discovery $(1 \Leftrightarrow \lambda)$ which results in the factor $\delta(1 \Leftrightarrow \lambda)$. The second term captures the case that the project is bad which occurs with probability $(1 \Leftrightarrow \alpha_0)$. In this case, costly experimentation will continue with probability 1 until the stopping time T^* is reached.

2.3 Contracting

The entrepreneur has initially no wealth and seeks to obtain external funds to realize the project. Financing is available from a competitive market of investors. Entrepreneur and investors share initially the same assessment about the likelihood of success represented by the prior belief α_0. The funds are supplied by the investor and the entrepreneur controls the allocation of the funds. She can either invest the funds into the project or divert the capital flow to her private ends. In the case she chooses to divert the funds she can either consume the funds or save the funds for future consumption or investment at the interest rate $r = (1 - \delta)/\delta$. The model permits an equivalent and more standard formulation of the agency problem: the efficient application of the investment requires effort, which is costly for the entrepreneur. By reducing the effort, the entrepreneur also reduces the probability of success and hence the efficiency of the invested capital. In both cases, a conflict of interest arises about the use of the funds.

We distinguish between observable and unobservable actions by the entrepreneur. If the action is observable, then entrepreneur and investor will always have the same posterior belief about the project. In any case, the allocation decision of the entrepreneur is not enforceable by the investor.

In every period the entrepreneur can offer the investor a contract of arbitrary length. A contract $\mathcal{S}_t$ offered in period t is a sequence of shares and funding levels:

$$\mathcal{S}_t = \{S_\tau, \lambda_\tau\}_{\tau=t}^{T},$$

where S_t denotes the share of the entrepreneur if the project succeeds in period t. The investor receives the remaining share $1 - S_t$. The restriction to share contracts is without loss of generality due to the binary nature of the project, success or failure. Denote by h_t a particular history including all events up to $t - 1$ and let $h_t \in H_t$ be the set of possible histories in period t. The investor responds with an acceptance or rejection of the contractual offer,

$$d_t : H_t \times \mathcal{S}_t \to \{0, 1\}.$$

The entrepreneur finally decides about the investment of the capital:

$$i_t : H_t \times \mathcal{S}_t \rightarrow [0, \lambda_t] .$$

Entrepreneur and investor can commit to long-term contracts, but they are free to renegotiate the current contract at any point in time. In this context acceptance of the new contract by the investor implies the discarding of the previous contract. Symmetrically, the rejection of a new contract implies the continued validity of the current contract. If entrepreneur and investor can't agree on a new contract and the previous contract has expired, then no funding is provided and the investor maintains the share in the project which he had in the last period in which they had a non-trivial agreement.

Finally, we wish to emphasize that while there is no initial asymmetry in the information between entrepreneur and investor, the asymmetry may arise over time as the project receives funding. The source of the asymmetry is the unobservability of the fund allocation. If entrepreneur and investor have different assessments over how the funds have been employed, then they will have different posteriors over the likelihood of success.

3 Observability

The main issues of this section are introduced with a simple finite horizon model in Subsection 3.1. The equilibrium concepts are then formally defined in Subsection 3.2. The equilibrium funding with short-term contracts is explored in Subsection 3.3. Finally Subsection 3.4 examines how long-term contracts may affect the volume and speed of financing.

3.1 A Finite Horizon Model

Consider first a project with an *efficient* financing horizon of a single period. We then investigate how the incentives of the entrepreneur depend on the length of the time horizon over which the project could receive financing before the project simply vanishes. This simple example will convey the difficulty of providing incentives for the efficient completion of the a given project.

Consider a project (R, c) with an ex-ante probability α of being successful. Suppose initially that the project can only be completed in period 0, and any investment in future periods would necessarily yield no returns. The entrepreneur and investor have common prior beliefs α in period 0. The entrepreneur offers the investor a share of $(1 \Leftrightarrow S_0)$ against funding at the level of λ_0. With a one period contract the investor accepts the offer if he can break even in expectations:

$$\alpha\lambda_0 (1 \Leftrightarrow S_0) R \geq \lambda_0 c, \tag{6}$$

and if he expects the entrepreneur to apply the funds to the project. The incentive compatibility constraint for the entrepreneur is simply

$$\alpha\lambda_0 S_0 R \geq \lambda_0 c. \tag{7}$$

The incentive constraint (7) explicitly allows the entrepreneur only two choices: investing or diverting. But as the objective function under both alternatives is linear in λ_0, this is without loss of generality. The project receives financing if individual rationality of the investor and incentive compatibility of the entrepreneur are jointly satisfied, or

$$\alpha\lambda_0 R \geq 2\lambda_0 c.$$

It follows immediately from (6) and (7) that the project can only be financed if

$$\alpha \geq \frac{2c}{R}.$$

In consequence the project ceases to receive financing too early relative to the first best as $\alpha^* = c/R$. The inefficient early stopping is due to the team-like incentive problem. The investor would need to receive all the expected returns if he were to invest, but the entrepreneur could divert the funds and hence would need to receive at least $c\lambda_0$ in expectations to allocate the funds. Both claims are in conflict at any $\alpha < 2c/R$. Define

$$\alpha_S \triangleq \frac{2c}{R} \tag{8}$$

as the stopping point under short-term contracts and suppose that $\alpha_0 \geq \alpha_S$. As the entrepreneur has all the bargaining power with take-it-or-leave-it offers, she offers the efficient financing volume $\lambda_0 = \lambda$ and receives a share

$$S_0 = 1 \Leftrightarrow \frac{c}{\alpha R}$$

and the remainder allows the investor to break even and satisfy the individual rationality constraint with equality. The value of the contract for the entrepreneur is:

$$E(\alpha) = \alpha \lambda R \Leftrightarrow c\lambda$$

Notice that due to the binary structure of the signal, a share contract forms the optimal incentive contract. The earlier assumption that it is efficient to fund the project only once simply says that the posterior belief falls below the efficient stopping point:

$$\frac{(1 \Leftrightarrow \lambda)\,\alpha}{1 \Leftrightarrow \lambda\alpha} \leq \alpha^*.$$

Suppose now that the entrepreneur will have a second chance to realize the project if either she didn't receive funding or diverted the funds in the initial period. We thus extend the time horizon T of the game and ask what happens to funding and incentives as the time horizon for possible completion is extended. We wish to emphasize that the efficient financing program remains unaffected by the longer time horizon. Again, any contract (S_0, λ_0) signed in the initial period must allow the investor to break even:

$$\alpha\lambda_0 (1 \Leftrightarrow S_0) R \geq \lambda_0 c.$$

The incentive constraint for the entrepreneur is however complicated by intertemporal considerations. The incentive constraint is now given by

$$\alpha\lambda_0 S_0 R \geq \lambda_0 c + \delta\lambda (\alpha R \Leftrightarrow c),$$

as diverting today allows the investor to seek new funding tomorrow. Hence the share accorded to the entrepreneur has to increase to

$$S_0 \geq \frac{c}{\alpha R} + \delta \left(1 \Leftrightarrow \frac{c}{\alpha R}\right), \tag{9}$$

which gives her in expected terms the value of the diversion plus the option of developing the project tomorrow. By extending the argument to an arbitrary finite time horizon, one can easily infer that the share in the initial period has to exceed

$$S_0 \geq \frac{c}{\alpha R} \sum_{t=0}^{T-1} \delta^t + \delta^T \left(1 \Leftrightarrow \frac{c}{\alpha R}\right). \tag{10}$$

Thus as the time horizon increases, the incentives offered to the entrepreneur in the initial period have to increase as well. The option to divert the funds and postpone the completion of the project to the future period increases in value in T as well as in the discounting factor δ. The strength of the incentives are inversely related to the expected return αR of the project. Thus rich projects can offer weaker incentives than poor projects.

As the participation constraint (6) of the investor still has to hold, there will be values for (δ, T) such that incentive constraint of the entrepreneur and participation constraint of the investor will be become incompatible. In consequence an inefficient delay arises in the completion of the project. To put it differently, the entrepreneur will not be able to receive financing in the initial period, but only in some later period, when the option value of diverting the funds is sufficiently low. Fix the minimal time horizon such that the project will not receive financing in the initial period as T. Suppose we then lengthen the time horizon by one additional period to $T+1$. Can funding now be provided again in the initial period? By the recursive argument, we just established that the entrepreneur will under no circumstances receive financing in the following period. But this implies that the value of a diversion is lower in the initial period, then in the subsequent period. In turn, the incentive constraint of the entrepreneur is easier to satisfy.. While it may still not be enough to ensure financing, any further lengthening of the time horizon will ease the incentive problem even more. But as soon as funding is possible again, the value of diverting the funds in the preceding period increases as well. Thus with a sufficiently long-time horizon, periods in which funding is available alter with periods in which funding is not available. The cyclic nature of the unique equilibrium just described is naturally due to the existence of a final period, which anchors the construction of the equilibrium. The corresponding equilibrium in the infinite horizon model to be examined next is a mixed strategy equilibrium in which financing only occurs with probability less than one. The ratio of periods in which financing is offered to the total length of the time horizon is simply replaced by the probability of receiving funding in any given period. The implications are very similar, as a probability of funding less than one means that the equilibrium volume and speed of financing is below the efficient level.

3.2 Equilibrium

In the environment with observable actions, the information of entrepreneur and investor is symmetric in every period. For a given triple $\{\mathcal{S}_t, d_t, i_t\}_{t=0}^{\infty}$ of strategies, denote the value function of the entrepreneur in period t by $E(\alpha_t, h_t)$ and the one of the investor by $I(\alpha_t, h_t)$. We hasten to add that α_t is of course part of h_t, but it will be convenient to set α_t apart from the general history of the game. The continuation value in any period after offering a contract $\mathcal{S}_t$ and/or responding with a funding decision d_t is denoted by $E(\mathcal{S}_t | \alpha_t, h_t)$ or $E(\mathcal{S}_t, d_t | \alpha_t, h_t)$, and likewise for $I(\mathcal{S}_t | \alpha_t, h_t)$ or $E(\mathcal{S}_t, d_t | \alpha_t, h_t)$.

Definition 1 (Subgame perfect equilibrium)
A subgame perfect equilibrium (SPE) is a sequence of policies

$$\{\mathcal{S}_t^*, d_t^*, i_t^*\}_{t=0}^{\infty}$$

such that for all (α_t, h_t) the following inequalities hold:

$$\begin{aligned}
E(\mathcal{S}_t^* | \alpha_t, h_t) &\geq E(\mathcal{S}_t | \alpha_t, h_t), && \textit{for all } \mathcal{S}_t; \\
I(\mathcal{S}_t, d_t^* | \alpha_t, h_t) &\geq I(\mathcal{S}_t, d_t | \alpha_t, h_t), && \textit{for all } \mathcal{S}_t \textit{ and } d_t; \\
E(\mathcal{S}_t, d_t, i_t^* | \alpha_t, h_t) &\geq E(\mathcal{S}_t, d_t, i_t | \alpha_t, h_t) && \textit{for all } S_t, d_t \textit{ and } i_t.
\end{aligned}$$

In general, the equilibrium set of a stage game repeated infinitely often is distinct from the equilibrium set of the same stage game repeated only finitely many times. The game considered here is no exception to the rule. But the sequential move structure of the game together with the ability of the players to renegotiate their contract at any point, suggest a narrowing of the equilibrium analysis to renegotiation-proof equilibria. Here we adopt the notion of weakly renegotiation-proofness first suggested by **?**) for repeated games with simultaneous move stage games.

Definition 2 (Weakly renegotiation-proof)
A subgame perfect equilibrium $\{\mathcal{S}_t^*, d_t^*, i_t^*\}_{t=0}^{\infty}$ *is weakly renegotiation-proof if there do not exist continuation equilibria at some* (α, h) *and* (α, h') *with* $h \neq h'$ *such that* $(E(\alpha, h), I(\alpha, h)) \geq (E(\alpha, h'), I(\alpha, h'))$, *with at least one strict inequality.*

The renegotiation considered here occurs between time periods. It is conceptually different from renegotiation in static principal-agent models as considered by **?**) and **?**) where renegotiation occurs after the agent has chosen her effort level but before the outcome has been revealed. The role of observable but non-verifiable information in the renegotiation of agency relationships has been studied by **?**).

The notion of weakly renegotiation-proof is often interpreted as an internal consistency requirement. Indeed, **?**) suggested a strengthening of the notion by defining as strongly renegotiation-proof any weakly renegotiation-proof profile with none of its continuation equilibria being strictly Pareto dominated by another weakly renegotiation-proof profile. This distinction is immaterial to our argument, as they all coincide in this sequential move game with symmetric information. In fact, we shall towards the end of this section that every renegotiation-proof equilibrium is equivalent to a Markov perfect equilibrium as defined by **?**).

Definition 3 (Markov perfect equilibrium)
A Markov perfect equilibrium (MPE) is a subgame perfect equilibrium

$$\{\mathcal{S}_t^*, d_t^*, i_t^*\}_{t=0}^{\infty}$$

such that the sequence of policies satisfy $\forall \alpha_t$, $\forall h_t$, $\mathcal{S}_t^*(\alpha_t, h_t) = \mathcal{S}_t^*(\alpha_t)$, $d_t^*(\alpha_t, h_t, \mathcal{S}_t) = d_t^*(\alpha_t, \mathcal{S}_t)$, *and* $i_t^*(\alpha_t, h_t, \mathcal{S}_t) = i_t^*(\alpha_t, \mathcal{S}_t)$.

3.3 Short-Term Financing

In this section we restrict our attention to short-term contracts. The equilibrium analysis is restricted first to the class of Markov perfect equilibria. The equivalence between Markov and weakly renegotiation-proof equilibria is established in the next subsection. A short-term contract is a break-even contracts if

$$\alpha_t \lambda_t R \left(1 \Leftrightarrow S_t\right) = \lambda_t c$$

and the share of the entrepreneur in a break-even contract is

$$S_t = 1 \Leftrightarrow \frac{c}{\alpha_t R}.$$

A contract is said to provide full funding if $\lambda_t = \lambda$. The probability by which a funding proposal is accepted by the investor in α_t is denoted by $p\left(\alpha_t\right)$.

Theorem 1 (Short-Term Financing)

1. *There is a unique MPE with short-term contracts. It consists of a sequence of break-even contracts with full funding.*

2. *The probability of financing, $p\left(\alpha\right)$, is zero for all $\alpha \leq \alpha_S$. Otherwise it is strictly positive and weakly increasing in α.*

3. *For every $\left(R, c, \lambda\right)$ there exists $\hat{\delta}$ with*

$$\hat{\delta} \triangleq \frac{R \Leftrightarrow 2c}{R \Leftrightarrow 2c + \lambda c} \tag{11}$$

   such that

   (a) for $\delta \leq \hat{\delta}$, $\lim_{\alpha \to 1} p\left(\alpha\right) = 1$,

   (b) for $\delta > \hat{\delta}$, $\lim_{\alpha \to 1} p\left(\alpha\right) < 1$.

Proof. See Appendix. ■

The characterization of the equilibrium financing is particularly transparent in the case of the certain project, where $\alpha_0 = 1$, and the successful completion of the project is certain and only a matter of time. The following results are obtained immediately as limit results as $\alpha \to 1$.

Corollary 1 (Certain Project)
The certain project $\left(\alpha = 1\right)$ is financed efficiently if and only if $\delta \leq \hat{\delta}$. For $\delta > \hat{\delta}$, the value of the entrepreneur is given by

$$E\left(1\right) = \left(R \Leftrightarrow 2c\right) / \delta,$$

and the probability of funding is

$$p(1) = \frac{(R - 2c)(1 - \delta)}{c\delta\lambda}.$$

Observe first that the probability of financing is decreasing in δ and λ. The discount factor increases the value of the option to divert and hence the investor responds in equilibrium by a slowdown in the funding. Similarly, a large efficient financing volume λ allows the entrepreneur to divert in the current period and still expect a successful completion of the project in the next period with a sufficiently high probability. The extent to which the open-endedness of the investment problem hurts the entrepreneur is most clearly expressed in the value function which is decreasing in δ for all δ exceeding the threshold described in (11).

Finally, the following example of a certain project points out that the focus on renegotiation-proof equilibria imposes indeed restrictions on the equilibrium set. Consider the following two strategy profiles: (i) the entrepreneur offers in each period break-even contracts with full financing and the investor accepts these contracts if he always accepted them in the past, and if the entrepreneur always invested the funds in the project. Otherwise he rejects all contract proposals; and (ii) the entrepreneur pursues the same strategy as before, but the investor accepts contracts in the future only if he is observing a diversion of funds today. The second strategy profile leads the investor to refuse any contract today, and hence also in the future and the project will never receive funding. The continuation equilibrium established with the second strategy profile sustains the first strategy profile as a subgame perfect equilibrium which provides full funding forever The strategy profiles rely in an obvious way on continuation plays which are not renegotiation-proof. The second profile offers intertemporal incentives through continuation play which destroy the possibility of current financing.

3.4 Long-Term Financing

The short-term contracts restricted the intertemporal distribution of the incentives. The entrepreneur was required to offer the investor a sequence of contracts in which the investor breaks even in every period. In this section the entrepreneur is allowed to make proposals in which only the intertemporal participation constraint of the investor is satisfied. This may allow entrepreneur and investor to re-allocate some of the expected surplus across periods and therefore improve the efficiency of the funding. While the players can commit to intertemporal plans, they cannot commit to avoid renegotiation in future periods.

The short-term contracts gave the entrepreneur in every period a share which was determined exclusively by the break-even condition of the investor. Once long-term contracts are considered, the shares of the entrepreneur can be redistributed to arrange the incentives more efficiently over time. Consider first the allocation with short-term contracts under $\delta \leq \hat{\delta}$. Here S_t is sufficiently large in the beginning to support full funding. It might thus be possible to weaken the incentives in the early phase of the project to increase them in the later phase where funding with probability one is impossible to maintain with short-term contracts. If there is such a surplus to redistribute, the question is then how. In any period t in which the share S_t of the entrepreneur increases relative to short-term financing, the volume of financing can be increased as well as the incentive constraint becomes less restrictive. Any increase in the share of the entrepreneur therefore allows a contemporaneous increase in the funding volume. As renegotiation always leads to the selection of the efficient outcome relative to all renegotiation-proof allocations the surplus will be used in equilibrium as early as possible, and thus long-term contracts lead to an extension of the interval over which full funding is possible. In the case of $\delta > \hat{\delta}$, short-term financing never allowed for full funding in any phase of the project's development. Could long-term contracts in this environment bias the funding towards a certain phase of the project to increase the value of the partnership between entrepreneur and investor? Efficiency considerations would again suggest to accelerate funding in the early stages of the project. In terms of incentives for the entrepreneur, it would mean to increase the incentives early on and decrease them in the later stage, again relative to the incentives provided in the short-term regime. Ideally, entrepreneur and investor would therefore like to commit themselves to stop funding in future periods altogether. But as we showed earlier, this is not renegotiation-proof as the participants would renegotiate and restart funding, even if they do so at a moderate level. As the short-term financing equilibrium presents the minimal level of future financing, it follows that for $\delta > \hat{\delta}$, the project never provides any surplus to re-allocate the incentives, and long-term contracts which are renegotiation proof cannot improve upon short-term financing.

Theorem 2 (Long-Term Financing)

1. *There is a unique MPE in long-term financing contracts.*

2. *If $\delta > \hat{\delta}$, then long-term financing can't improve on short-term financing.*

3. *If $\delta \leq \hat{\delta}$, then long-term financing increases the number of periods where funding is provided with probability* 1. *The unique equilibrium is then characterized by a single long-term contract until T with $p(\alpha_t) = 1$ for all $t \leq T$, followed by a sequence of short-term contracts with $p(\alpha_{t'}) < 1$ for $t' > T$.*

Proof. See Appendix. ■

We conclude this section by formally stating the equivalence result which we mentioned in the introduction to the definition of the weakly renegotiation-proof and the Markov perfect equilibrium.

Theorem 3 (Equivalence)
The set of Markov perfect equilibria (with short or long-term financing) is equivalent to the set of weakly renegotiation-proof equilibria.

Proof. See Appendix. ■

4 Non-Observability

The principal themes of this section are first developed in a two period example in Subsection 4.1. The formal definition of the Perfect Bayesian Equilibrium is given in Subsection 4.2. The complete results for short-term financing are presented in Subsection 4.3. The extent to which long-term contracts can improve the efficiency of short-term financing is discussed in Subsection 4.4.

4.1 A Two Period Model

The project has a prior probability of α_0 of being successful.. By assumption, the project can receive financing only over two periods, $t \in \{0, 1\}$, after which the value of the project simply expires. We shall restrict ourselves in this example to contracts in which the investor breaks even in every period (short-term financing). The structure of the equilibrium financing becomes most transparent when analyzed recursively. Suppose that entrepreneur and investor have common posterior beliefs α_1 in period 1. The entrepreneur offers the investor a share of $(1 \Leftrightarrow S_1)$ against funding at the level of λ_t. With a one period contract the investor accepts the offer if he can break even in expectations:

$$\alpha_1 \lambda_1 (1 \Leftrightarrow S_1) R \geq \lambda_1 c, \tag{12}$$

and if he expects the entrepreneur to apply the funds to the project:

$$\alpha_1 \lambda_1 S_1 R \geq \lambda_1 c. \tag{13}$$

As before, the project receives funding in the last period if and only if

$$\alpha_1 \geq \frac{2c}{R}.$$

Suppose then that $\alpha_1 \geq 2c/R$ holds, the project receives full funding $\lambda_1 = \lambda$ and the entrepreneur receives

$$S_1 = 1 \Leftrightarrow \frac{c}{\alpha_1 R}.$$

Next we allow entrepreneur and investor to hold different beliefs at the beginning of period 1. The belief $\hat{\alpha}_1$ of the investor can be anywhere between the minimal posterior belief which is obtained by conditioning the prior belief α_0 on the event that the entrepreneur invested the funds in period 0 and his prior belief $\hat{\alpha}_0 = \alpha_0$:

$$\hat{\alpha}_1 \in \left[\frac{(1 \Leftrightarrow \lambda_0) \alpha_0}{1 \Leftrightarrow \lambda_0 \alpha_0}, \alpha_0 \right].$$

The belief of the investor in period 1 is based on two considerations: (i) did the entrepreneur invest the funds in period 0? and (ii) does the offer $\{S_1, \lambda_1\}$ convey some information about the investment decision in period 0? Consider the informativeness of the offer $\{S_1, \lambda_1\}$ first, and we argue that the offer cannot contain any additional information. Suppose two different types in terms of the posterior belief α_1 and α_1', with $\alpha_1 > \alpha_1'$, would offer two different contracts, $\{S_1, \lambda_1\}$ and $\{S_1', \lambda_1'\}$ respectively. Suppose further that the contracts satisfy (12) and (13) for α_1 and α_1' respectively and hence would be accepted by the investor if he knew the type for sure. If not, the respective contract would be rejected by the investor, and hence the entrepreneur would in any case propose another contract. But if either contract is accepted and the entrepreneur plans to invest the funds, then her expected payoff is $\alpha_1 \lambda_1 S_1 R$ and hence only the expected share $\lambda_1 S_1$ matters. In consequence separating cannot be achieved through contracts with different expected shares: $\lambda_1 S_1 \neq \lambda_1' S_1'$. But any two contracts which offer the same expected share can only have different values to the entrepreneur if she were to plan to divert the funds. However in equilibrium contracts (S_1, λ_1) which imply fund diversion are not accepted and hence there can be no separating offers in equilibrium.

This brief discussion shows that the belief of the investor in period 1 is formed exclusively by evaluating whether or not the entrepreneur applied the funds in period 0. Conditional on being convinced that the entrepreneur did the right thing in period 0, the investor has the belief

$$\hat{\alpha}_1 = \frac{(1 \Leftrightarrow \lambda_0)\, \alpha_0}{1 \Leftrightarrow \lambda_0 \alpha_0}$$

and funding occurs under the unique contract:

$$\left\{ S_1 = 1 \Leftrightarrow \frac{c}{\hat{\alpha}_1 R},\ \lambda_1 = \lambda, \right\}.$$

The value of the project in period 1 for the entrepreneur is thus given by

$$E(\alpha_1, \hat{\alpha}_1) = \alpha_1 \lambda R \Leftrightarrow \frac{\alpha_1 \lambda c}{\hat{\alpha}_1}, \tag{14}$$

where α_1 is the probability of operating a successful project in period 1 and $\hat{\alpha}_1$ is the belief of the investor. In equilibrium $\hat{\alpha}_1$ and α_1 coincide and in this case we may simple write

$$E(\alpha_1) \triangleq E(\alpha_1, \hat{\alpha}_1),$$

with

$$E(\alpha_1) = \alpha_1 \lambda R \Leftrightarrow \lambda c. \tag{15}$$

We can now proceed to the initial period. Again, any contract $\{S_0, \lambda_0\}$ must allow the investor to break even:

$$\alpha_0 \lambda_0 \left(1 - S_0\right) R \geq \lambda_0 c.$$

The incentive constraint of the entrepreneur contains again intertemporal considerations:

$$\alpha_0 \lambda_0 S_0 R + \delta \left(1 - \alpha_0 \lambda_0\right) E\left(\alpha_1\right) \geq \lambda_0 c + \delta E\left(\alpha_0, \alpha_1\right), \tag{16}$$

where the belief of the investor is $\hat{\alpha}_1 = \alpha_1$, conditional on the inequality (16) being satisfied. Again, the incentive constraints explicitly allows only two options for the entrepreneur, investing or diverting, but we shall see shortly that this is without loss of generality. After inserting (14) and (15) into (16) we find

$$\alpha_0 \lambda_0 S_0 R + \delta \left(1 - \alpha_0 \lambda_0\right) \left(\alpha_1 \lambda R - \lambda c\right) \geq \lambda_0 c + \delta \left(\alpha_0 \lambda R - \frac{\alpha_0 \lambda c}{\alpha_1}\right) \tag{17}$$

When comparing the alternatives for the entrepreneur in (17) with (13) two new elements appear. First, by investing in the project today, the entrepreneur forecloses with positive probability continued funding in the future. Second, by diverting the funds today, the entrepreneur maintains her current belief about the value of the project as no new information is produced. But as the investor believes the project received funding, his belief in the following period is indeed α_1. In terms of conditional expectations where the conditioning is based on the information of the entrepreneur, the investor requires in the following period a compensation which exceeds the cost of the funds. This explains the rhs of (17). We simplify (17) to

$$\alpha_0 \lambda_0 S_0 R \geq c \lambda_0 + \delta \alpha_0 \lambda \left(R - \frac{c}{\alpha_1}\right), \tag{18}$$

which is to be compared with the equivalent condition (9) in the case of observable actions. The value of the deviation is now diminished by $\frac{\alpha_0}{\alpha_1} c > c$, and is thus smaller in the case of asymmetric information. The inability of the entrepreneur to separate ex-post leads the investor to always assess the project as if the investment occurred in all previous periods. This reduces the value of a deviation and eases the incentive compatibility constraint. We may recall in this context that the incentive constraint of the entrepreneur becomes harder to satisfy the weaker the prospect of the project seemed in the observable case. The entrepreneur when faced with the termination of the project had strong incentives to divert the funds in order to reapply for more funds in the following period. In the environment with asymmetric information, this becomes obviously less of a constraint as the investor maintains his belief that the project received financing throughout, and thus is less prone to a renew funding if in his opinion the project generated sufficient evidence to rationally stop funding.

4.2 Equilibrium

The evolution of the posterior belief α_t is subject to moral hazard as the investor cannot observe the allocative decision by the entrepreneur. Entrepreneur and investor can therefore hold different beliefs during the game. As the entrepreneur observes her own decision, her belief always follows the 'true' Bayes' law, and thus coincides with the 'true' posterior belief α_t. The posterior belief of the investor is denoted by $\hat{\alpha}_t$. The posterior belief α_t is the private information of the entrepreneur and $\hat{\alpha}_t$ is the public state of the game.

Definition 4 (Perfect Bayesian Equilibrium)
A Perfect Bayesian Equilibrium (PBE) is a sequence of policies

$$\{\mathcal{S}_t^*, d_t^*, i_t^*\}_{t=0}^{\infty}$$

such that for all $(\alpha_t, \hat{\alpha}_t, h_t)$ *the following conditions hold:*

$$\begin{array}{lcll} E\left(\mathcal{S}_t^* \,|\alpha_t, \hat{\alpha}_t, h_t\right) & \geq & E\left(\mathcal{S}_t \,|\alpha_t, \hat{\alpha}_t, h_t\right), & \textit{for all } \mathcal{S}_t; \\ I\left(\mathcal{S}_t, d_t^* \,|\hat{\alpha}_t, h_t\right) & \geq & I\left(\mathcal{S}_t, d_t \,|\hat{\alpha}_t, h_t\right), & \textit{for all } \mathcal{S}_t \textit{ and } d_t; \\ E\left(\mathcal{S}_t, d_t, i_t^* \,|\alpha_t, \hat{\alpha}_t, h_t\right) & \geq & E\left(\mathcal{S}_t, d_t, i_t \,|\alpha_t, \hat{\alpha}_t, h_t\right) & \textit{for all } S_t, d_t \textit{ and } i_t. \end{array}$$

and Bayes' rule is applied to α_t *and* $\hat{\alpha}_t$ *whenever possible.*

The application of Bayes' rule for the entrepreneur yields α_{t+1} as a function of α_t and λ_t as expressed in (1). The evolution of the investor's beliefs has to take into account the incentives provided in the last period through $\mathcal{S}_t$ and the information possibly contained in the current contract offer $\mathcal{S}_{t+1}$:

$$\hat{\alpha}_{t+1} = \mathbb{E}\left[\frac{\hat{\alpha}_t\left(1 \Leftrightarrow i_t\right)}{1 \Leftrightarrow i_t \hat{\alpha}_t} \,|\mathcal{S}_t, \mathcal{S}_{t+1}\right]. \tag{19}$$

The belief $\hat{\alpha}_{t+1}$ of the investor thus depends on his estimate about the investment i_t the project received in period t. The posterior belief of the investor is represented in (19) directly as a point belief as we shall shortly see that the investor will never have a non-degenerate distribution of beliefs.

4.3 Short-Term Financing

Perhaps it is best to start with the features of the equilibrium we might expect to carry over from the environment with observable actions. Clearly, with short-term contracts funding should still stop at $\alpha_T = \alpha_S$. Moreover as $\alpha_0 \to 1$, the difference between observability and non-observability should become less important, as the changes in the posterior belief induced by the investment process converge to zero. In consequence, the difference between the private belief of entrepreneur and the investor are very small and become negligible as $\alpha_0 \to 1$. But the two period model discussed earlier indicated that some of the striking inefficiencies observed in the model with observable actions may be softened under asymmetric information. In particular the slow-down in the end phase of the project doesn't have to occur anymore. As the investor adopts a pessimistic attitude towards the project, the termination of the project becomes ever more likely as α_t decreases. The necessary strength in the incentives provided to the entrepreneur then depend on how valuable the option of diversion is. The value of the option is, as before, generally increasing in the discount factor δ. We find therefore discount factors low enough so that funding can be provided at the full level near the termination point of the project.

With full funding possible towards the end of the project, the question then arises whether the funding volume is still (weakly) decreasing over time. This in turn depends on the funding volume and hence the speed at which the project can be developed. While the funding volume is still monotone over time, the sign of the changes now depend on the size of the upper bound λ. For low levels of λ, the funding volume still decreases over time. However for large values the funding now increases over time. The contrast between the observable and unobservable environment is easiest understood by identifying λ as the parameter which describes the potential difference in the belief between investor and entrepreneur. Again, for small values of λ, these differences remain small and, in particular, they cannot increase too rapidly in any particular period. However as λ increases, the differences become more important. The reversal in the change of the financing volume is best understood through the recursive structure of the problem. For large λ, the value of the diversion becomes smaller as the ratio α_t/α_{t+1} increases, as in α_0/α_1 of (18) in the two period example. This eventually allows for more financing towards the end of the project. But as the financing towards the end of the project increases, the value of diverting the funds in earlier periods of project increases as well. The funding now has to slow down in the earlier phases, since its acceleration cannot be prevented in the final phase of the project, leading to a reversal in the monotonicity.

Theorem 4 (Short-Term Financing)

1. *The unique PBE in short-term contracts is a pooling equilibrium.*
2. *Funding stops at $\alpha_T = \alpha_S$.*
3. *Funding occurs with probability one at α_S if and only if*

$$\delta \leq 2\,(1 \Leftrightarrow \alpha_S)\,. \tag{20}$$

4. *The probability of financing is monotone. It is weakly increasing over time if and only if*

$$\lambda \leq 2 \Leftrightarrow \frac{1}{\alpha_S}. \tag{21}$$

Proof. See Appendix. ■

The joint results on the speed of financing can best be represented in the following matrix:

$$\begin{array}{lll} \lambda > 2 \Leftrightarrow \frac{1}{\alpha_S} & (c1)\,,(c2) & (d) \\ \lambda \leq 2 \Leftrightarrow \frac{1}{\alpha_S} & (a) & (b1)\,,\ (b2) \\ & \delta \leq 2\,(1 \Leftrightarrow \alpha_S) & \delta > 2\,(1 \Leftrightarrow \alpha_S) \end{array} \tag{22}$$

where $(b1)$ and $(c1)$ occur if

$$\delta \leq 2\frac{1 \Leftrightarrow \lambda}{2 \Leftrightarrow \lambda} \Leftrightarrow \lambda \leq 2\frac{1 \Leftrightarrow \delta}{2 \Leftrightarrow \delta}, \tag{23}$$

and $(b2)$ and $(c2)$ occur if the inequalities in (23) are reversed. The matrix (22) indicates that the size of the various regions depend on the size of the cost and benefits of the project. The funding probability in the various regimes is represented in the next matrix:

$$\begin{array}{lll}
\lambda > 2 \Leftrightarrow \frac{1}{\alpha_S} & p_T = 1,\ p_t \uparrow & p_t < 1,\ p_t \uparrow \\
 & & \\
\lambda \leq 2 \Leftrightarrow \frac{1}{\alpha_S} & p_t = 1, \forall t & p_T < 1,\ p_t \downarrow \\
 & & \\
 & \delta \leq 2\left(1 \Leftrightarrow \alpha_S\right) & \delta > 2\left(1 \Leftrightarrow \alpha_S\right)
\end{array}$$

The relationship between the size of the project and size of the various regimes is displayed in Fig.1. The large square dot indicates the intersection of the horizontal line $\lambda = 2 \Leftrightarrow \frac{1}{\alpha_S}$ and the vertical line $\delta = 2\left(1 \Leftrightarrow \alpha_S\right)$ in the (δ, λ) space for a given α_S. The bold curved line describes the evolution of the intersection of the horizontal and vertical line as a function of α_S. The arrows point in the direction of decreasing α_S and hence rich projects. The curve is described by the equality the relations described in (23):

$$\delta = 2\frac{1 \Leftrightarrow \lambda}{2 \Leftrightarrow \lambda} \Leftrightarrow \lambda = 2\frac{1 \Leftrightarrow \delta}{2 \Leftrightarrow \delta}, \tag{24}$$

and reaches $(\delta = 1, \lambda = 1)$ for all $\alpha_S \leq \frac{1}{2}$. The perfect symmetry between λ and δ is due to the fact that in the limit as $\alpha_0 \to 1$, the volume of financing and discounting are exchangeable as the true discount factor is $\delta\left(1 \Leftrightarrow \lambda\right)$. Thus for sufficiently rich projects, the only surviving regimes are $(c1)$ and $(c2)$. Similarly, very poor projects which are characterized by $\alpha_S \to 1$, only display the regions $(b1)$ and $(b2)$.

It can further be shown that the curve described by (24) jointly with the restrictions on the different regimes outlined in (22) is equivalent to the condition (11) which ascertained whether full funding is possible in the limit as $\alpha_0 \to 1$ in the observable environment. Thus suppose that the primitives (R, c, δ, λ) of the model satisfy the restrictions of regime (b), then

$$\delta \leq 2\frac{1 \Leftrightarrow \lambda}{2 \Leftrightarrow \lambda} \ \Leftrightarrow \delta \leq \frac{R \Leftrightarrow 2c}{R \Leftrightarrow 2c + \lambda c}, \tag{25}$$

and likewise for regime (c). The interaction between the speed of development, represented by λ, and the discount factor δ leads to a subtle structure about the intertemporal funding of the project. But considering δ or λ independently leads to the observation, that the option value of diverting increases in δ and λ and hence increases in the scale of the project or the discount factor lead to a uniform reduction in the funding volume.

Proposition 2

Consider the probability of financing $p(\alpha_t)$ in state α_t, then:

1. *for a given λ, $p(\alpha_t)$ is (weakly) decreasing in δ,*
2. *for a given δ, $p(\alpha_t)$ is (weakly) decreasing in λ.*

Proof. See Appendix. ■

However, independent of the dynamic structure in the provision of the incentives and the financing, the asymmetry reduces the ability of the entrepreneur to renegotiate at favorable terms and hence weakens the incentives for the entrepreneur to delay investment into the project.

Proposition 3

With short-term contracts, funding is provided faster under unobservable than under observable actions.

Proof. See Appendix. ■

4.4 Long-Term Financing

The funding volume in the different short-term financing regimes is rather suggestive as to how long-term contracts can improve upon short-term contracts. The equivalence in the conditions presented in (25) indicates that in the areas below the curve, namely in the regimes $(a), (b1), (c1)$, long-term contracts enhance the efficiency by extending the time horizon over which full funding can be extended to the entrepreneur. Long-term contracts modify short-term financing in these regimes in way similar to the observable environment. The most interesting modification occurs in the regime $(c2)$, where short-term contracts provide sufficient incentives towards the end of the project, but can't provide sufficient incentives at the beginning of the project to maintain full funding throughout. This situation occurs if the project is sufficiently rich and the rate of development λ is sufficiently large relative to the discount factor. Then long-term contracts can weaken the incentives of the entrepreneur at the end and instead increase the incentives towards the beginning of the project. The result is that full funding can be provided towards the beginning and towards the end of the project, but that the development will proceed slower than efficient in some intermediate stages.

Theorem 5 (Long-term financing)

1. *There is a unique PBE in long-term contracts.*
2. *Long-term financing strictly extends the region of full financing relative to short-term financing in $(a), (b1), (c1)$. It always extends the horizon beyond α_S in (a) and $(c1)$.*
3. *Long-term financing doesn't improve on short-term financing in $(b2), (d)$.*
4. *Long-term financing extends the region of full financing and leads to a U-shaped financing volume in $(c2)$.*

Proof. See Appendix. ■

We conclude with a remark on the absence of any notion of renegotiation-proofness in the environment with unobservable actions. Note first that the notion of renegotiation-proofness was introduced for observable actions only in the infinite horizon context. In the case of unobservable actions, we just argued that the investor continues to update his posterior belief independent of whether the entrepreneur invested the funds or not. Thus for any investment policy, which is bounded away from zero until funding ceases completely, the investor reaches the posterior belief α_S in some finite time T, or $\alpha_T = \alpha_S$. As the investor will reject any new funding proposal at any $\alpha_t < \alpha_S$, the game ends in finite time. The pooling property of the PBE thus naturally limits the horizon of the game and recursive arguments based on a terminal point are then sufficient to establish the equilibrium structure based on recursive efficiency.

5 Extensions

To Be Completed.

5.1 Competition and Project Size

5.2 Positive Wealth and Savings

5.3 Nonlinear Investment Costs

5.4 Multiple Projects

5.5 Multiple Signals

6 Conclusion

This paper considered the funding of a project with unknown returns, when the investment of the funds is subject to agency problems. The funding level was determined endogenously and depended on the scale of the project, the discount factor and the informational asymmetry between entrepreneur and investor. The impatience of the entrepreneur was an important determinant of the volume of funding as the severity of the incentive constraint increased with the discount factor. This is in contrast to the results in the theory of repeated moral hazard games, where discount factors close enough to one often allow the equilibrium set to reach the efficiency frontier. The funding problem without commitment displays properties similar to sequential contribution games as analyzed by **?**). This paper studied the funding of a single project. An interesting generalization of the analysis presented here might be the funding of a sequence of projects or other forms of repeated relationships, where reputation for truthful investment or for strict termination behavior may become valuable.

7 Appendix

This appendix contains the proofs to all propositions in the main body of the text.

Proof of Theorem 1. (1.) and (2.) We first construct an MPE in which the entrepreneur offers a break-even contract with full funding to the investor in every period and there are finitely many periods in which full funding is provided. In the following we count only the periods in which funding is provided as the state variable remains unchanged in all other periods. The equilibrium is in mixed strategies as the investor sometimes randomizes in his acceptance decision. After the construction, we then show that it is also the unique MPE. Without loss of generality we may assume that the entrepreneur has zero wealth when entering into final period. The incentive problem in the final period T is given by:

$$\alpha_T \lambda_T s_T R + \delta \left(1 \Leftrightarrow \alpha_T \lambda_T\right) E\left(\alpha_{T+1}\right) \geq \lambda_T c + \delta E\left(\alpha_T\right). \tag{26}$$

By definition, there is no financing after α_T and hence $E\left(\alpha_{T+1}\right) = 0$, so (26) is

$$\alpha_T \lambda_T s_T R \geq \lambda_T c + \delta E\left(\alpha_T\right). \tag{27}$$

Denote by p_T the probability by which a break-even contract receives financing in the state α_T. In the case of observable actions, the entrepreneur can consume and then ask for renewed financing. Thus if $p_T > 0$, $E\left(\alpha_T\right) > 0$, but then (27) cannot be satisfied at $\alpha_T \leq \alpha_S$, and hence $p_T = 0$. Notice that a reverse argument implies that $\alpha_T > \alpha_S \Rightarrow p_T > 0$. Consider next $\alpha_T > \alpha_S$. The equilibrium is constructed recursively. Start with any α_T such that $\alpha_{T+1}\left(\alpha_T, \lambda\right) \leq \alpha_S$. The expected payoff for the entrepreneur of such a contract is

$$E\left(\alpha_T\right) = \frac{p_T\left(\alpha_T \lambda R \Leftrightarrow c\lambda\right)}{1 \Leftrightarrow \delta\left(1 \Leftrightarrow p_T\right)} \tag{28}$$

and the incentive constraint conditional upon funding is

$$\left(\alpha_T \lambda R \Leftrightarrow c\lambda\right) \geq c\lambda + \delta E\left(\alpha_T\right). \tag{29}$$

If the incentive constraint (29) is not binding, then the entrepreneur can reduce her share by ε and offer the investor a proposal which has strictly positive expected profit, which is accepted by sequential rationality. Thus

$$p_T < 1 \Rightarrow (\alpha_T \lambda R \Leftrightarrow c\lambda) = c\lambda + \delta E(\alpha_T),$$

and using (28) and (29) we have

$$p_T = \frac{(1 \Leftrightarrow \delta)}{\delta} \frac{\alpha_T R \Leftrightarrow 2c}{c}.$$

Next we derive the evolution of the funding probabilities as a difference equation, and conclude by observing that if $p_t = 1 \Rightarrow p_s = 1$ for all $s < t$. Consider any two adjacent periods t and $t+1$ with $p_{t+1} < 1$. The incentive condition in period t is

$$(\alpha_t R \Leftrightarrow c)\lambda + (1 \Leftrightarrow \alpha_t \lambda)\delta E(\alpha_{t+1}) \geq \lambda c + \delta E(\alpha_t) \tag{30}$$

and with equality for $p_t < 1$. The value function of the entrepreneur is given by:

$$E(\alpha_t) = \frac{p_t}{1 \Leftrightarrow \delta(1 \Leftrightarrow p_t)} \left(\alpha_t \lambda R \Leftrightarrow c\lambda + (1 \Leftrightarrow \alpha_t \lambda_t)\delta E(\alpha_{t+1})\right) \tag{31}$$

and using the incentive constraint (30) it can be written as:

$$E(\alpha_t) = \frac{p_t \lambda c}{1 \Leftrightarrow \delta} \tag{32}$$

if $p_t < 1$. By using (30) and (32) the following difference equation is obtained:

$$p_t = (1 \Leftrightarrow \alpha_t \lambda) p_{t+1} + \frac{1 \Leftrightarrow \delta}{\delta c}(\alpha_t R \Leftrightarrow 2c). \tag{33}$$

The difference equation has the following unique solution:

$$p_t = \frac{1 \Leftrightarrow \delta}{\delta c} \sum_{\tau=t}^{T} (\alpha_\tau R \Leftrightarrow 2c) \prod_{k=t}^{\tau-1} (1 \Leftrightarrow \alpha_k \lambda), \tag{34}$$

under the initial condition

$$p_T = \frac{1 \Leftrightarrow \delta}{\delta c}(\alpha_T R \Leftrightarrow 2c). \tag{35}$$

and it is verified that p_t is strictly decreasing in t. The sequence of break even contracts with probabilistic funding leaves the entrepreneur with the expected payoff:

$$E(\alpha_t) = \sum_{s=t}^{T} \left(\prod_{u=t}^{s} \frac{p_u}{1 \Leftrightarrow \delta(1 \Leftrightarrow p_u)} \prod_{u=t}^{s-1} (1 \Leftrightarrow \alpha_u \lambda) \right) \delta^{s-t} (\alpha_s R \lambda \Leftrightarrow c\lambda). \tag{36}$$

As the incentive constraint with certain funding, or $p_t = 1$, requires that

$$E(\alpha_t) \geq \frac{\lambda c}{1 \Leftrightarrow \delta} \tag{37}$$

it follows that if (36) satisfies (37) with $p_t = 1$, a break even contract with $p_{t-1} = 1$ and associated value function $E(\alpha_{t-1})$ satisfies (37) as well.

The uniqueness of the MPE is shown next. The proof proceeds in two steps. First we show that there is final period of financing in which full funding is provided, i.e. there exists an α_T such that $p_T \leq 1$, $\lambda_T = \lambda$ and for $\alpha_{T+1}(\alpha_T, \lambda)$ we have $p_{T+1} = 0$. In the second step, we then show that less than full funding cannot arise in equilibrium for any $\alpha_t > \alpha_T$. In both steps, we proceed by contradiction. Consider then any such α_t with $\lambda_t < \lambda$. Suppose first that $p_{t+1} = 0$. The incentive problem is then

$$(\alpha_t R \Leftrightarrow c)\lambda_t \geq \lambda_t c + \delta E(\alpha_t) \tag{38}$$

and with

$$E(\alpha_t) = \frac{p_t(\alpha_t \lambda_t R \Leftrightarrow c\lambda_t)}{1 \Leftrightarrow \delta(1 \Leftrightarrow p_t)} \tag{39}$$

we can write (38) as

$$(\alpha_t R \Leftrightarrow c)\lambda_t = \lambda_t c + \delta \frac{p_t(\alpha_t \lambda_t R \Leftrightarrow c\lambda_t)}{1 \Leftrightarrow \delta(1 \Leftrightarrow p_t)}$$

and the equilibrium probability of acceptance is given by:

$$p_t = \frac{\alpha_t R \Leftrightarrow \alpha_t R\delta \Leftrightarrow 2c + 2c\delta}{c\delta},$$

which is independent of the funding volume. The resulting equilibrium value is

$$E(\alpha_t) = \frac{\lambda_t}{\delta}(\alpha_t R \Leftrightarrow 2c),$$

which is increasing in λ_t and hence the entrepreneur will always ask for the maximal funding conditional on no further funding. Suppose next that $p_{t+1} > 0$ for all t. This case has to be considered as there might be an equilibrium in which funding is extended with positive probability for infinitely many funding periods. This is possible without ever violating $\alpha_T \leq \alpha_S$ if p_t and λ_t decline sufficiently fast. Formally, there must then exist an $\varepsilon > 0$ and a corresponding τ such that $\lambda_t < \varepsilon$ for all $t > \tau$. In particular we can choose ε so that $2\varepsilon < \lambda$. Consider a strategy profile with two adjacent funding periods α_t and α_{t+1} and associated funding levels and probabilities, (λ_t, p_t) and (λ_{t+1}, p_{t+1}) respectively. We show that the entrepreneur has a profitable deviation strategy which suggests funding at the level $\hat{\lambda}$ in period t and which is accepted with probability one. Define $\hat{\lambda}$ by the equality

$$\frac{\alpha_t\left(1 \Leftrightarrow \hat{\lambda}\right)}{1 \Leftrightarrow \alpha_t \hat{\lambda}} = \frac{\alpha_t\left(1 \Leftrightarrow \lambda_t\right)\left(1 \Leftrightarrow \lambda_{t+1}\right)}{\alpha_t\left(1 \Leftrightarrow \lambda_t\right)\left(1 \Leftrightarrow \lambda_{t+1}\right) + 1 \Leftrightarrow \alpha_t}$$

so that the posterior belief is identical after a sequential investment of $(\lambda_t, \lambda_{t+1})$ or a single investment $\hat{\lambda}$, and it is verified that $\hat{\lambda} < \lambda_t + \lambda_{t+1}$. For any given λ_t, λ_{t+1} and $E\left(\alpha_{t+2}\right)$, we can then compute the corresponding funding probabilities by using

$$\left(\alpha_{t+1} R \Leftrightarrow c\right)\lambda_{t+1} + \left(1 \Leftrightarrow \alpha_{t+1}\lambda\right)\delta E\left(\alpha_{t+2}\right) = \lambda_{t+1} c + \delta E\left(\alpha_{t+1}\right)$$

and

$$E\left(\alpha_{t+1}\right) = \frac{p_{t+1}}{1 \Leftrightarrow \delta\left(1 \Leftrightarrow p_{t+1}\right)}\left(\alpha_{t+1}\lambda_{t+1} R \Leftrightarrow c\lambda_{t+1} + \left(1 \Leftrightarrow \alpha_{t+1}\lambda_{t+1}\right)\delta E\left(\alpha_{t+2}\right)\right).$$

and similar expressions for period t. We obtain

$$E\left(\alpha_t\right) = \left(1 \Leftrightarrow \alpha_t \lambda_t\right) E\left(\alpha_{t+1}\right) + \frac{\lambda_t}{\delta}\left(\alpha_t R \Leftrightarrow 2c\right) \tag{40}$$

and

$$E\left(\alpha_{t+1}\right) = \left(1 \Leftrightarrow \alpha_{t+1}\lambda_{t+1}\right) E\left(\alpha_{t+2}\right) + \frac{\lambda_{t+1}}{\delta}\left(\alpha_{t+1} R \Leftrightarrow 2c\right) \tag{41}$$

respectively. Suppose next that the entrepreneur would offer a break even contract with an associated funding level of $\hat{\lambda}$. The incentive constraint would be

$$\left(\alpha_t R \Leftrightarrow c\right)\hat{\lambda} + \left(1 \Leftrightarrow \alpha_t \hat{\lambda}\right)\delta E\left(\alpha_{t+2}\right) \geq \hat{\lambda} c + \delta E\left(\alpha_t\right). \tag{42}$$

Since $\hat{\lambda} > \lambda_t$, it follows that if the incentive constraint (42) is satisfied as a strict inequality, then we have shown that the deviation is accepted by the investor with probability one, and that the deviation is profitable for the entrepreneur. Rearranging (42) and using (40) and (41) yields

$$\alpha_{t+1} R \Leftrightarrow \frac{\alpha_{t+1}}{\alpha_t} 2c > \left(\alpha_{t+1} R \Leftrightarrow 2c\right),$$

which is indeed a strict inequality. It is then established that funding stops after a finite number of positive funding rounds. Moreover, the last round of financing provides full funding. It remains to be shown that full funding is also provided in all preceding funding periods. Consider then the first period, again looking from the end, at which less than full funding is observed. By an argument similar to the one just presented, we can then show that the investor would in fact be willing to accept full funding, and that the entrepreneur would benefit from full funding. Thus the candidate profile can't be a subgame perfect equilibrium, completing the argument on the uniqueness of the MPE.
(3.) As $T \to \infty$, or alternatively $\alpha_0 \to 1$, we have from (34):

$$\lim_{\alpha_0 \to 1} p_0 < 1 \Leftrightarrow \delta \leq \frac{R \Leftrightarrow 2c}{R \Leftrightarrow 2c + c\lambda}, \tag{43}$$

which completes the proof. ■

Proof of Theorem 2. To Be Completed. ■

Proof of Theorem 3. It is an immediate implication of the definition of the Markov perfect equilibrium that the equilibrium value functions of the players depend only on the payoff-relevant state of the game. The set of possible equilibrium values at any state α therefor is a singleton for every player, and it follows that any Markov perfect equilibrium is also a weakly renegotiation-proof equilibrium. To Be Completed. ■

Proof Theorem 4. (1) and (2) To Be Completed.
(3.) The problem is analyzed recursively. We first construct a pooling equilibrium and then show that it is unique. Suppose the project is funded fully in period T. Then $E(\alpha_T) = \lambda c$. The problem in period $T \Leftrightarrow 1$ is then given by:

$$\alpha_{T-1}\lambda R \Leftrightarrow cR + \delta (1 \Leftrightarrow \alpha_{T-1}\lambda) E(\alpha_T) \geq \lambda c + \delta \frac{\alpha_{T-1}}{\alpha_T} E(\alpha_T), \tag{44}$$

or after using the fact that $\alpha_T = \alpha_S$, (44) reduces to

$$\alpha_S \leq \frac{2 \Leftrightarrow \delta}{2}, \tag{45}$$

which states that if the project is sufficiently rich, full funding will always be provided at the end, independent of the volume λ of feasible financing.

(4.) Next consider the speed at which the financing is provided over time. Consider any two adjacent periods t and $t+1$ with $p_{t+1} < 1$. The incentive constraint is

$$(\alpha_t R - c)\,\lambda + (1 - \alpha_t \lambda)\,\delta E\,(\alpha_{t+1}) \geq \lambda c + \delta \frac{\alpha_t}{\alpha_{t+1}} E\,(\alpha_{t+1}) \tag{46}$$

where

$$E\,(\alpha_t) = \frac{p_t\,((\alpha_t R - c)\,\lambda + (1 - \alpha_t \lambda)\,\delta E\,(\alpha_{t+1}))}{1 - \delta\,(1 - p_t)}. \tag{47}$$

and with the time indices pushed forward by one unit for $E\,(\alpha_{t+1})$. We observe that the value of $E\,(\alpha_t)$ is increasing in p_{t+1}, but that the lhs is increasing slower than the rhs of (46). Thus the randomization in period $t+1$ eases the incentive constraint in period t. If $p_{t+1} < 1$, then (46) has to hold as an equality and we obtain $E\,(\alpha_{t+1})$ as

$$E\,(\alpha_{t+1}) = \frac{\alpha_{t+1}}{\alpha_t} \frac{\alpha_t R - 2c}{\delta}. \tag{48}$$

With $p_t < 1$ we can then solve for p_t by using (46)-(48) and obtain:

$$p_t = \frac{1 - \delta}{\delta} \frac{\alpha_t R - 2c + 2c\lambda\,(1 - \alpha_t)}{\lambda c\,(2\alpha_t - 1)} \tag{49}$$

and forwarding by one period and taking differences we obtain

$$p_t - p_{t+1} = (R + 2\lambda c - 4c)\,\frac{\alpha_t\,(1 - \alpha_t)\,(1 - \delta)}{(1 + \alpha_t \lambda - 2\alpha_t)\,c\delta\,(2\alpha_t - 1)}$$

Hence for $p_t, p_{t+1} \in (0,1)$, it follows that

$$p_t - p_{t+1} \geq 0 \Leftrightarrow \alpha_S \geq \frac{1}{2 - \lambda}, \tag{50}$$

which completes the proof. ■

Proof of Proposition 2. To Be Completed. ■

Proof of Proposition 3. To Be Completed. ■

Proof of Theorem 5. To Be Completed. ■