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Abstract. *Bidding in first-price auctions crucially depends on the beliefs of the bidders about their competitors' willingness to pay. We analyze bidding behavior in a first-price auction in which the knowledge of the bidders about the distribution of their competitors' valuations is restricted to the support and the mean. To model this situation, we assume that under such uncertainty a bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution. This introduces a novel way to endogenize beliefs in games of incomplete information. We find that for a bidder with a given valuation her worst-case belief just puts sufficient probability weight on lower valuations of her competitors to induce a high bid. At the same time the worst-case belief puts as much as possible probability weight on the same valuation in order to minimize the bidder's winning probability. This implies that even though the worst-case beliefs are type dependent in a non-monotonic way, an efficient equilibrium of the first-price auction exists.*

JEL classification: D44, D81, D82

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1. Introduction

Consider a company preparing a bid for a first-price procurement auction. The company's optimal bidding strategy will crucially depend on their belief about the costs of its competitors. Typically, this company would spend a considerable amount of resources to reverse-engineer the products of their competitors and learn about their cost structure. However, such learning has its limits. For example, reverse-engineering may inform the company about the used components and the general complexities in producing this part. But it cannot inform about the production processes and the used equipment of its competitors. Thus, it is reasonable to assume that learning about the distribution of the competitors' costs is not perfect and just specifies some summary statistic of the underlying distribution like the support and the mean. How to weigh the probabilities of certain costs within this support is subjective and hard to objectify. Thus, in order to submit a bid in the auction, the company has to form a subjective belief.

In this paper we consider the problem of a bidder in a first-price auction whose only information about the valuations of her competitors is the support and the mean of their distribution. Given such a large uncertainty, it seems natural for this bidder to prepare for the worst case.[1] Thus, we assume that for a given bidding strategy of her competitors the bidder will tailor her

[1]From our own experience in consulting bidders in high-stakes (procurement) auctions, it is a typical approach taken by bidders to generate several scenarios with respect to the valuations (costs) of their competitors and than to tailor their strategy to the worst-case.

bid to be optimal given that she expects to face the worst distribution of of her competitors' valuations among all distributions with the same support and mean. Worst distribution, in this context, means the bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution. In other words, the worst-case belief of a bidder minimizes her maximum possible expected utility. We assume that every bidder in the auction follows a similar logic when preparing her bid. In this case, a profile of bids is an equilibrium if each bidder chooses her optimal bid given her valuation (type), the bidding strategy of her competitors, and the worst-case belief as defined above. In particular, this implies that the worst-case belief of a bidder will crucially depend on her type (valuation) in a non-monotonic way.

Our contribution is threefold. Our first contribution is to introduce a novel way to model endogenous beliefs in a first-price auction. Endogenous, in this case, means that a bidder's beliefs about the valuations of the competitors are not assumed as a primitive of the environment but arise naturally as worst-case beliefs from the game induced by the rules of the first-price auction.[2] This can be viewed as a relaxation of the paradigm of symmetric independent private value (IPV) auctions that each bidder's valuation for the object is drawn from a distribution that is common knowledge among all bidders. Moreover, our solution concept constitutes a novel way to analyze games with asymmetric information and can be straightforwardly extended to any kind of such game.

Our second contribution is to show that even though the endogenous beliefs that arise from our solution concept are type dependent in a non-monotonic way, an ex-post efficient equilibrium exists. That is, even though the worst-case beliefs of bidders with a higher valuation do not imply that they believe to face a stronger competition in the auction than bidders with a lower valuation, in equilibrium the object is allocated with probability one to the bidder with the highest valuation.

Our third contribution is to introduce a novel proof method that we use in order to derive the worst-case strategies and beliefs in the efficient equilibrium. The method encompasses an elegant way to compare the solutions of an infinite set of minimization problems. To fix ideas and to gain some intuition for our results, consider the case that the valuation of a bidder can take one of three valuations 0, θ and 1. Suppose furthermore that it is common knowledge among the bidders that the mean of the distribution of valuations is μ with $\mu < \theta$. In this case, the efficient equilibrium takes the following form: all bidders with valuation 0 bid 0, all bidders with valuation θ mix between 0 and some $\overline{b}_\theta$, and all bidders with a valuation of 1 mix between $\overline{b}_\theta$ and some $\overline{b}_1$. The beliefs of a bidder with valuation 0 are arbitrary as she will always bid 0 and expect a utility of 0. A bidder with valuation θ believes that she is facing only bidders

[2] A different auction format would generate different worst-case beliefs.

with valuations 0 and θ with probabilities such that the mean of her belief is μ. A bidder with valuation 1 believes that she is facing bidders with valuations 0, θ, and 1 with probabilities such that she is indifferent between mixing in $\left[\bar{b}_\theta, \bar{b}_1\right]$ and bidding 0 and such that the mean of her belief is μ. Given their beliefs, all bidders best reply to the bidding strategies of their competitors. Given the bidding strategies, the beliefs make each bidder worst off given her type. It may appear counterintuitive that, given her bid, the worst-case scenario for a bidder with valuation θ is that she is the strongest bidder. However, given the bidding strategies in the efficient equilibrium, the utility of a bidder with a valuation of θ depends only on the probability that she is facing bidders with a valuation of 0. Given that the mean of the belief is fixed, this probability is minimized if the probability of facing bidders with a valuation of 1 is zero. In other words, for a bidder with a valuation of θ it is the worst-case that the probability that she will face only bidders with a valuation of 0, against whom she will win for sure, is minimized.

For bidders with a valuation of 1, the worst-case is determined by minimizing her winning probability while keeping the incentives intact to bid above $\bar{b}_\theta$. Thus, the belief of a bidder with a valuation of 1 puts just enough probability weight on 0 and θ such that she will bid above the highest bid of a bidder with a valuation of θ and then as much probability as possible on 1.

The intuitions from the case with three types carry over to the general model. In particular, the worst-case belief of a bidder with a given valuation just puts enough probability weight on lower valuations to induce that for this bidder it is optimal to outbid each bidder with a lower valuation. The remaining probability weight is put on the valuation of the bidder in question in order to minimize her winning probability. It follows directly that such beliefs induce bidding that leads to an efficient allocation.

In order to show that the proposed strategies indeed constitute an equilibrium with worst-case beliefs, it remains to show that there is no other belief that would induce a bid that would make a bidder worse off than in the proposed equilibrium. For this we introduce a novel proof method. The underlying idea of the proof is to show that we can switch from comparing different beliefs and their induced utilities to comparing different bids and their induced utilities. This is due to the fact that a given best reply b can be induced by a multitude of beliefs (given the bidder's valuation and the other bidders' strategies). It follows that every bid b can be identified by a minimization problem: among all distribution functions with mean μ which induce bid b as a best reply it suffices to consider the belief which leads to the minimum utility. Using this concept, we can map every bid to a belief and a corresponding utility. Therefore, checking whether the utility induced by b is lower than the utility induced by some other b' establishes a transitive total order on the set of bids.

We use three different tools with which we can compare different bids with respect to the introduced transitive order. The first tool is to show that for certain types there exists only one distribution which induces a particular bid. This allows to directly compute the minimum expected utility which can be induced by this bid for these types. The second tool constitutes a connection between binding constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. Third, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type. Using these three tools, we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bids are the lowest. Due to the transitivity of our relation, this excludes all other bids as possible deviations from the proposed equilibrium strategy.

Besides specifying an efficient worst-case belief equilibrium, we provide a comparison of expected revenues of a second-price auction and a first-price auction under endogenous worst-case beliefs for the case where bidders can have three discrete valuations 0, θ and 1. We show that for certain parameter constellations of θ and μ the first-price or the second-price auction perform better in terms of expected revenue independent of the true valuation distribution. There also exist parameters θ and μ such that the revenue-maximizing choice of the auction format depends on the true valuation distribution.

The remainder of the paper is organized as follows. We conclude the introduction with an overview over the related literature. The second section contains the formal model including the formal description of our solution concept, the worst-case belief equilibrium. In the third section we show the existence of an efficient worst-case belief-equilibrium and derive the corresponding beliefs and strategies for the special case of two bidders and three types. We consider this special case in order to focus on the intuition of the results and to illustrate the techniques of our proof. In the fourth section we conduct the revenue comparison between the first-price and the second-price auction under endogenous worst-case beliefs for the case of two bidders and three valuations. The fifth section contains the formal model and an outline of the proof for the general case with an arbitrary number of bidders and discrete valuations. We conclude in section six and section seven provides an overview over the most used notation and definitions. The appendix contains the proofs not provided in previous sections. We provide all proofs in the Appendix for the case of two bidders and three valuations and the general case separately. The proofs for the special case are provided in order to give an intuition for the general case. However, the model for the general case in the fifth section as well as the proofs for the general case can be also understood without reading the special case first.

Relation to the literature. Our paper complements two strands of literature: the literature on robust auction design and the literature on first-price auctions with non-standard priors. Both strands of literature relax the typically strict assumptions that are placed on the beliefs of the designer and the participants of an auction.

Contrary to the literature on robust auction design that focuses on the problem of the designer who does not have precise beliefs about the bidders, we focus on the problem of the bidder who does not have precise beliefs about her competitors. Departing from the ideas posed in this literature, we propose that not only the designer may be uninformed about the environment but also the bidders, if they do not interact frequently, may have some uncertainty. We then use modeling techniques developed in this literature and develop a novel solution concept to analyze this problem. For example, **?** consider optimal monopoly pricing under uncertainty about demand distribution with a seller who either maximizes worst-case expected utilities or minimizes the maximal regret. They find that the optimal pricing policy hedges against uncertainty by randomizing over a range of prices. Buyers with low valuations cannot generate substantial regret and are priced out of the market. **?** consider a robust version of the classic problem of optimal monopoly pricing with incomplete information. The seller faces model uncertainty and only knows that the true demand distribution is in the neighborhood of a given model distribution. They find that the equilibrium price under either criterion is lower then in the absence of uncertainty. The concern for robustness leads the seller to concede a larger information rent to all buyers with valuations below the optimal price without uncertainty. **?** analyze the optimal selling mechanism if the seller maximizes worst-case expected profits and is only informed about one moment of the distribution of the buyer's valuations. They show that the optimal mechanism entails distortions at the intensive margin, e.g., except for the highest valuation buyer, sales will take place with probability strictly smaller than one. The seller can implement such allocation by committing to post prices drawn from a non-degenerate distribution, so that randomizing over prices is an optimal robust selling mechanism. **?** considers the mechanism design problem of a seller who is uninformed about demand, while potential buyers are well-informed. The seller's goal is to maximize the minimum ratio between expected revenue and the expected efficient utility. He characterizes simple mechanisms that maximize the minimum extraction ratio. In these mechanisms, the seller runs a second-price auction and simultaneously surveys the beliefs of buyers about other's valuations. **?** considers a moral hazard problem where the principal is uncertain what the agent can and cannot do: She knows some actions available to the agent, but other, unknown actions may also exist. The principal demands robustness, evaluating possible contracts by their worst-case performance, over unknown actions the agent might potentially take. He finds that the optimal contract from the point of view of the principal is linear.

The literature on first-price auctions with non-standard priors relaxes the assumptions placed on the priors of the bidders by the standard IPV model. For example, **?** consider parametric examples of symmetric two-bidder private valuation auctions in which each bidder observes her own private valuation as well as noisy signals about her opponent's private valuation. They show that in such environments the revenue equivalence between the first and second-price auction (SPA) breaks down and there is no definite revenue ranking; while the SPA always allocates efficiently, the first price auction (FPA) may be inefficient; equilibria may fail to exist for the FPA. **?** study auctions in which bidders may know the types of some rival bidders but not others. They show that the first-price auction results in an inefficient allocation and that this inefficient allocation translates into a poor revenue performance. **?** characterize the set of all possible outcomes that may arise in a first-price auction under any given information structure among the bidders. They find that revenue is maximized when buyers know who has the highest valuation, but the highest valuation buyer has partial information about others' valuations. Revenue is minimized when buyers are uncertain about whether they will win or lose and incentive constraints are binding for all upward bid deviations. Contrary to this literature, we do not assume an exogenously given prior but rather introduce a novel way to model endogenous beliefs that will depend on the specific game structure. We find, in contrast to most findings in this literature, that the first-price auction allocates efficiently.

2. Model

2.1. **Setup.** There are n risk-neutral bidders competing in a first-price sealed-bid auction for one indivisible object. Before the auction starts, each bidder $i \in \{1, \ldots, n\}$ privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\}$. The valuation distributions are unknown to the bidders. However, it is common knowledge among the bidders that the mean of this distribution is μ. Hence, every bidder knows that the probability mass function of the other bidders' valuations is an element from

$$\mathscr{F}_\mu^{n-1} = \left\{ f_1 \times \cdots \times f_{n-1} : \Theta^{n-1} \to [0,1] \;\middle|\; \sum_{j=1}^{m} \theta^j f_i\left(\theta^j\right) = \mu \text{ for all } i \in \{1, \ldots, n-1\} \right\},$$

where for every $i \in \{1, \ldots, n-1\}$ and every $j \in \{1, \ldots, m\}$, $f_i(\theta^j)$ denotes the probability with which valuation θ^j occurs according to the probability mass function f_i. In other words, this is the set of all probability mass functions of independently drawn valuations from the set Θ for $n-1$ bidders with mean μ. For a shorter notation we will use the term probability function instead of probability mass function.

In the auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, we assume an efficient tie-breaking rule[3]. Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by [4]

[3]We assume an efficient tie-breaking rule since it simplifies notation. With a random tie-breaking rule one would need to assume a discrete bid grid (which may be arbitrarily fine) in order to ensure equilibrium existence. However, the equilibrium strategies under both tie breaking rules would differ by at most one bid step in the bid grid.

[4]For a vector $(v_1, \ldots, v_n)$ we denote by v_{-i} the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_n)$.

$$u_i(\theta_i, b_i, b_{-i}) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j\neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i > \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i < \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}(\theta_i - b_i) & \text{if } b_i = \max\limits_{j\neq i} b_j \text{ and } \theta_i = \max\limits_{j\neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j\neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k = \#\{\max\{\theta_j | b_j = b_i\}\}$.

A *(mixed) strategy* β_i of a bidder i maps the valuation (type) of a bidder to a distribution of bids:

$$\beta_i : \Theta \to \Delta\mathbb{R}^+$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\Delta\mathbb{R}^+$ is the set of all probability distributions on $\mathbb{R}^+$. For bidder i with valuation θ_i it is a cumulative distribution function of bids, denoted by $G^{\beta_i}_{\theta_i}$ with corresponding density $g^{\beta_i}_{\theta_i}$ and support $supp(\beta_i(\theta_i))$. A *pure strategy* of bidder i with valuation θ_i is a mapping

$$\beta_i : \Theta \to \mathbb{R}^+$$

$$\theta_i \mapsto \beta_i(\theta_i),$$

i.e. this is a mapping from the set of valuations to the set of bids.[5] The expected utility of a bidder i with valuation θ_i, belief $f_{-i} \in \mathscr{F}^{n-1}_\mu$ and bid b_i given that her competitors employ bidding strategies β_{-i} can be written as

$$U_i(\theta_i, f_{-i}, b_i, \beta_{-i}) = \int_{\theta_{-i}} \int_{b_{-i}} u_i(\theta_i, b_i, b_{-i}) \prod_{j\neq i} g^{\beta_j}_{\theta^j}(b_j) d\theta_{-j} f_{-i}(\theta_{-i}) \, d\theta_{-i}.$$

2.2. **Solution Concept.** We are interested in the bidding behavior of a bidder who apart from the support and mean has no information about the distribution of the valuations of her competitors. Thus, in order to derive a bid, the bidder has to form a subjective belief. We assume that this bidder will prepare for the worst case. Prepare means that the bidder will choose her optimal bid given she expects to face the worst-case distribution of valuations. That is, the bidder will expect to face the distribution of valuations that minimizes her expected utility, given her bid is an optimal reaction to the bids of her competitors induced by this distribution

[5] A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. We abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in $\mathbb{R}^+$ while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in $\Delta\mathbb{R}^+$. However, in the following it will be clear whether β_i is a pure or a mixed strategy. In addition, we will also use the notation $G^{\beta_i}_{\theta_i}$ instead of $\beta_i(\theta_i)$ in case of mixed strategies.

and their bidding strategy. We will introduce the concept in several steps. First, we define the best reply of bidder i to a given belief f_{-i} and a given bidding strategy of the competitors β_{-i}. Second, we introduce the worst-case belief for a given bidding strategy of the competitors β_{-i}. That is, we derive the belief that minimizes the expected utility of bidder i given her best reply to this belief and the bidding strategy of her competitors. Third, we will define the worst-case belief equilibrium in which each type of each bidder bids the optimal bid given her worst-case belief and the bidding strategy of her competitors.

Best reply to a belief and the competitors' strategies. For bidder i with valuation θ_i and for each belief f_{-i} about the other bidders' valuations and bidding strategies β_{-i}, the set of *best replies* of bidder i is given by

$$B_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right) = \arg\max_{b_i}\ U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right).$$

Bidder i's best reply induces an expected utility of

$$U\left(\theta_i, f_{-i}, b_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right)$$

for $b_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right) \in B_i^r\left(\theta_i, f_{-i}, \beta_{-i}\right)$.

Worst-case belief given a best reply and the competitors' strategies. As argued before, we will assume that a bidder prepares for the worst case, i.e. she will assume that the distribution of her competitors' valuations induces the worst utility given her best reply and the bidding strategy of her competitors. Since after forming a belief, a bidder will choose an optimal bid given this belief, a distribution induces the worst outcome for a bidder if it minimizes the expected utility of a bidder given her optimal bid. That is, the worst-case belief minimizes the maximum expected utility of the bidder. Formally, a worst-case belief $f_{-i}^{\theta_i}$ of bidder i with valuation θ_i is given by

$$\begin{aligned} f_{-i}^{\theta_i} &= \arg\min_{f_{-i}\in\mathscr{F}_\mu^{n-1}} U_i\left(\theta_i, f_{-i}, b^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right) \\ &= \arg\min_{f_{-i}\in\mathscr{F}_\mu^{n-1}} \max_{b_i}\ U_i\left(\theta_i, f_{-i}, b_i, \beta_{-i}\right). \end{aligned}$$

Given the other bidders' strategies β_{-i}, a bidder i with type θ_i calculates her best reply to each belief in $\mathscr{F}_\mu^{n-1}$ and the corresponding utility. The worst-case belief of bidder i is the one inducing the lowest utility. In other words, the worst-case belief minimizes the maximum possible expected utility of a bidder given her valuation and the other bidders' strategies. Note that a worst-case belief is not necessarily unique but every worst-case belief yields the same utility.

Worst-case belief equilibrium. In equilibrium, after forming a worst-case belief as described above, each bidder will choose an optimal bid given her valuation, her worst-case belief and the other bidders' strategies. That is, in equilibrium it has to hold for every valuation of every bidder that

(i) Given her valuation, her belief, and the other bidders' strategies the bid of a bidder maximizes her expected utility.

(ii) For every bidder there does not exists another belief such that a best reply to this belief induces a lower expected utility.

This leads to the following definition.

Definition 1 (Worst-case belief equilibrium)**.** *A profile of bidding strategies* $(\beta_1, \ldots, \beta_n)$ *together with a profile of beliefs* $\left([f_{-1}^{\theta^1}, \ldots, f_{-1}^{\theta_{m-1}}, f_{-1}^{\theta^m}], \ldots, [f_{-n}^{\theta^1}, \ldots, f_{-n}^{\theta_{m-1}} f_{-n}^{\theta^m}]\right) \in \left(\mathscr{F}_\mu^{n-1}\right)^m$ *form a* worst-case belief equilibrium *if for all* $i \in \{1, ..., n\}$*, all* $\theta_i \in \Theta$*, all* $f_{-i} \in \mathscr{F}_\mu^{n-1}$ *and all* $b_i \in supp\,(\beta_i(\theta_i))$ *it holds that*

$$b_i \in B_i^r\left(\theta_i, f_{-i}^{\theta_i}, \beta_{-i}\right) \tag{1}$$

and

$$U_i\left(\theta_i, f_{-i}^{\theta_i}, b_i, \beta_{-i}\right) \leq U_i\left(\theta_i, f_{-i}, b^r\left(\theta_i, f_{-i}, \beta_{-i}\right), \beta_{-i}\right). \tag{2}$$

In the following we will refer to the first condition as the best-reply condition *and to the second condition as the* worst-case belief condition*.*

3. Worst-case belief equilibrium: two bidders, three valuations

This section focuses on our main result which states that an efficient worst-case belief equilibrium exists. We characterize the beliefs and strategies in the worst-case belief equilibrium and illustrate the techniques of our proof. We start our analysis with the case of two bidders, A and B and and three possible valuations $0, \theta$ and 1. This allows us to focus on the main features of the concept without complex notation. The general case with n bidders and m types is analyzed in section **??**.

3.1. Efficient worst-case belief equilibrium.

Theorem 1. *In a first-price auction there exists an efficient worst-case belief equilibrium.*

In order to prove the existence of an efficient worst-case equilibrium, we specify a profile of increasing strategies and beliefs and show that they constitute a worst-case belief equilibrium. The underlying idea of the proof is to show that we can switch from comparing different beliefs and their induced utilities to comparing different bids and their induced utilities. This is due to the fact that a given best reply b can be induced by a multitude of beliefs (given the bidder's valuation and the other bidders' strategies). It follows that every bid b can be identified by a minimization problem: among all distribution functions with mean μ which induce bid b as a best reply it suffices to consider the belief which leads to the minimum utility. Using this concept, we can map every bid to a belief and a corresponding utility. Therefore, checking whether the utility induced by b is lower than the utility induced by some other b' establishes a transitive total order on the set of bids.

We use three different tools with which we can compare different bids with respect to the introduced transitive order. The first tool is to show that for certain types there exists only one distribution which induces a particular bid. This allows to directly compute the minimum expected utility which can be induced by this bid for these types. The second tool constitutes a connection between binding constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. Third, we show that for a given type there exist bids which can never be a best reply independent of the belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type. Using these three tools, we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bids are the lowest. Due to the transitivity of our relation, this excludes all other bids as possible deviations from the proposed worst-case strategy.

We start with the formal description of the strategies and beliefs we claim to constitute a worst-case belief equilibrium. We will consider two possible cases: $\theta \leq \mu$ and $\theta > \mu$.

3.2. **Characterization of the efficient worst-case belief equilibrium for $\boldsymbol{\theta \leq \mu}$.** We start with the simpler case $\theta \leq \mu$ and claim that the following strategies and beliefs constitute a worst-case belief equilibrium. The proof of this claim is provided in section **??**. Since both bidders will have symmetric beliefs and strategies, we omit the identity of the bidder in the notation of beliefs and strategies.

We denote the strategy which we claim to be played in a worst-case belief equilibrium by β^*. We define

$$\beta^*(0) = 0, \quad \beta^*(\theta) = \theta, \quad \beta^*(1) = G_1^*. \tag{3}$$

That is, a bidder with valuation zero bids zero, a bidder with valuation θ bids θ and a bidder with valuation 1 plays a mixed strategy on the interval $[\theta, \bar{b}_1]$ according to a continuous bid distribution G_1^*. We will calculate G_1^* and the exact valuation of $\bar{b}_1$ further below. One can immediately see that these strategies constitute an efficient equilibrium, that is, the bidder with the highest valuation wins the auction with probability 1.

We will denote the belief which we claim to constitute a worst-case belief equilibrium together with the strategies specified above, by $f^{\hat{\theta},*} = \left(f_0^{\hat{\theta},*}, f_\theta^{\hat{\theta},*}, f_1^{\hat{\theta},*}\right)$ for $\hat{\theta} \in \{0, \theta, 1\}$.[6] That is, $f_0^{\hat{\theta},*}$

[6]In the following we will refer to β^* and $f^{\hat{\theta}^*}$ as the worst-case strategy and the worst-case beliefs.

denotes the probability with which bidder A with valuation $\hat{\theta}$ believes that bidder B has valuation zero (and analogously for other valuations and bidder B).

The subjective worst-case beliefs are defined as follows. Type zero can have any belief from the set $\mathscr{F}_{\mu}^{n-1}$. A bidder with valuation θ has the subjective worst-case belief that the probability weight in the other bidder's probability function is solely distributed between valuations θ and 1. Since probabilities have to add up to one and the mean has to be preserved, it must hold that

$$f_0^{\theta,*} + f_\theta^{\theta,*} + f_1^{\theta,*} = 1$$

and

$$f_0^{\theta,*}0 + f_\theta^{\theta,*}\theta + f_1^{\theta,*}1 = \mu.$$

In the following we will refer to these two constraints as the first and second *probability constraint.* If it holds that $f_0^{\theta,*} = 0$, it follows from these constraints that

$$f_0^{\theta,*} = 0,\ f_\theta^{\theta,*} = \frac{1-\mu}{1-\theta},\ f_1^{\theta,*} = \frac{\mu-\theta}{1-\theta}. \tag{4}$$

We define the subjective worst-case belief of a bidder with valuation 1 to be the solution of the following minimization problem which we denote by $M^1_{<\theta}$:

$$\min_{(f_0,f_\theta,f_1)} \ (f_0 + f_\theta)$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0,\theta,1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = \mu$$

$$(f_0 + f_\theta)(1-\theta) \geq f_0.^{7}$$

The second and third constraints are the above described probability constraints. The last constraint ensures that bidding θ is weakly better for a bidder with valuation 1 than bidding any lower bid given the other bidder's strategy.[8] That is, there is just enough probability weight on lower types in order to induce a bid of at least θ for type 1. It is sufficient to consider only a possible deviation to bid 0 because all bids in the interval $(0,\theta)$ are placed with zero probability and therefore are never best replies. Note that the feasible set of this minimization problem is not empty since the worst-case belief of type θ is an element of the feasible set.

[7]We use the expression "the solution" instead of "a solution" since we will show that this minimization problem has a unique solution. Also in the remainder of the paper we will use the term "the solution" in order to indicate that we will show that the particular minimization problem has a unique solution.

[8]In the following we will use the notation with subscript "$<$" like in $M^1_{<\theta}$ in order to indicate that a minimization problem does not contain all possible constraints but only the constraints which ensure that bidding a given bid is weakly better than bidding any lower bid.

In the case with three types such that $\theta \leq \mu$ the solution of minimization problem $M^1_{<\theta}$ can be obtained directly. Consider the minimization problem

$$\min_{(f_0, f_\theta, f_1)} \ (f_0 + f_\theta)$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

The solution of this minimization problem puts zero probability weight on type θ. Such a solution does not fulfill the constraint

$$(f_0 + f_\theta)(1 - \theta) \geq f_0.$$

Since this is the only constraint besides the probability constraints, this constraint has to be binding in minimization problem $M^1_{<\theta}$. Therefore, the solution of minimization problem $M^1_{<\theta}$ is the unique solution of a system of three linear equations with three unknowns. It holds that

$$f_0^{1,*} = \frac{1-\mu}{1+\theta},\ f_\theta^{1,*} = \frac{\theta(1-\mu)}{1-\theta^2},\ f_1^{1,*} = \frac{\mu - \theta^2}{1-\theta^2}. \tag{5}$$

Given the subjective worst-case belief of a bidder with valuation 1, one can compute the upper endpoint of her bidding interval, denoted by $\bar{b}_1$, and the bid distribution, denoted by G_1.[9] The upper endpoint of this bidding interval is defined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = 1 - \bar{b}_1.$$

$$\Leftrightarrow \bar{b}_1 = f_1^{1,*} + \theta\left(f_0^{1,*} + f_\theta^{1,*}\right) = \frac{\mu - \theta^2 + \theta - \mu\theta}{1-\theta^2} = \frac{\theta+\mu}{1+\theta}.$$

The bid distribution is defined such that bidders A and B with valuation 1 make each other indifferent between any bid in their bidding interval. For every $s \in [\theta, \bar{b}_1]$ it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1(s)\right)(1-s) = \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta). \tag{6}$$

3.3. **Proving the best-reply and the worst-case belief condition for $\theta \leq \mu$.** After specifying the worst-case beliefs and strategy, we have to show that these indeed constitute a worst-case belief equilibrium. That is, we have to show that the best-reply and the worst-case belief condition are fulfilled.

[9] Since according to the worst-case strategy the support of the bid distribution for every type is an interval (which may consists only of one point), we use the term "bidding interval" for the support of the bid distribution prescribed by the worst-case strategy for a given type.

Proposition 1. *Given the worst-case strategy as defined in (**??**) and the worst-case beliefs as defined in (**??**) and (**??**), it holds for all $\hat{\theta} \in \{0, \theta, 1\}$ that*

(i) The best-reply condition given by

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidder's worst-case strategy.

(ii) The worst-case belief condition is fulfilled, i.e. for all $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ it holds that

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.^{10}$$

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: Due to the symmetry of beliefs and strategies, it is sufficient to show the best-reply condition for bidder A. The result is obvious for a bidder with valuation zero. Given the subjective worst-case belief of bidder A with valuation θ as defined in (**??**) and bidder B's strategy, bidder A with valuation θ considers θ to be the lowest bid placed by bidder B. Therefore, she expects a utility of zero and bidding θ is a best reply. It follows from the definition of the worst-case belief of bidder A with valuation 1 as defined in (**??**) that she does not earn a higher expected utility by bidding any bid lower than θ. Bids in the interval $(0, \theta)$ are never played according to β^* and therefore cannot be a best reply. The constraint

$$(f_0 + f_\theta)(1 - \theta) \geq f_0$$

in minimization problem $M^1_{<\theta}$ ensures that bidding zero does not induce a higher expected utility than bidding θ. Since bidder B does not place bids above $\bar{b}_1$, it cannot be a best reply for bidder A to bid above $\bar{b}_1$. The bid distribution G_1 is constructed in a way which makes bidder A with valuation 1 indifferent between any bid in $[\theta, \bar{b}_1]$ which completes the proof of part *(i)*. □

The remainder of this section is dedicated to proving the worst-case belief condition. That is, for every type we have to consider all probability functions over the valuations 0, θ and 1 with mean μ and have to show that none of these probability functions induces a lower expected utility than the worst-case beliefs of the given type. Before we can complete the proof of part (ii), we need to introduce several proof techniques.

As a first step, we will introduce the concept of *minimizing probability functions* which enables us to switch from comparing the induced utility of probability functions to comparing the induced

[10] Since utility functions are symmetric among bidders, we will omit the identitiy of the bidder in the notation of utility functions.

utility of bids. Afterwards, we will introduce different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy.

Minimizing probability functions. Consider the list of possible subjective beliefs from which bidder A chooses. Given the type of bidder A and bidder B's strategy, every probability function induces a best reply for bidder A. The best reply induces an expected utility:

probability function $f_B \rightarrow$	best reply $b^r(\theta_A, f_B, \beta_B) \rightarrow$	expected utility $U(\theta_A, f_B, b^r, \beta_B)$
f_B^a	$b^a(\theta_A, f_B^a, \beta_B)$	$U(\theta_A, f_B^a, b^a, \beta_B)$
f_B^b	$b^b(\theta_A, f_B^b, \beta_B)$	$U(\theta_A, f_B^b, b^b, \beta_B)$
$\vdots$	$\vdots$	$\vdots$

Here $\theta_A \in \{0, \theta, 1\}$ denotes a valuation of bidder A and $f_B^a, f_B^b, \ldots$ denotes a list of probability functions of bidder B's valuations among which bidder A chooses her subjective worst-case belief. Note that different probability functions can induce the same best reply. Therefore, the list can be rearranged by grouping those probability functions together which induce the same best reply:

probability function $f_B \rightarrow$	best reply $b^r(\theta_A, f_B, \beta_B) \rightarrow$	expected utility $U(\theta_A, f_B, b^r, \beta_B)$
f_B^a	$b^a(\theta_A, f_B^a, \beta_B)$	$U(\theta_A, f_B^a, b^a, \beta_B)$
$f_B^{a'}$	$b^a\left(\theta_A, f_B^{a'}, \beta_B\right)$	$U\left(\theta_A, f_B^{a'}, b^a, \beta_B\right)$
$\vdots$	$\vdots$	$\vdots$
f_B^b	$b^b(\theta_A, f_B^b, \beta_B)$	$U(\theta_A, f_B^b, b^b, \beta_B)$
$f_B^{b'}$	$b^b\left(\theta_A, f_B^{b'}, \beta_B\right)$	$U\left(\theta_A, f_B^{b'}, b^{b'}, \beta_B\right)$
$\vdots$	$\vdots$	$\vdots$

Among the probability functions which induce the same bid, it is sufficient to consider the probability functions which induce the minimum expected utility. That is, it is sufficient to select the probability functions inducing the minimum expected utility from each group and compare the induced utilities. Hence, we can switch from comparing probability functions to comparing bids. This is formalized in the following definition and observation which we provide for bidder A to simplify notation.

Definition 2. *For bidder A with valuation $\theta_A \in \{0, \theta, 1\}$, a bid b_A and the competitor's strategy β_B, the set of probability functions $\mathscr{F}^{min}(\theta_A, b_A, \beta_B)$ given by*

$$\mathscr{F}^{min}(\theta_A, b_A, \beta_B) = \underset{f_B \in \mathscr{F}_\mu}{\arg\min} \ \{U(\theta_A, f_B, b_A, \beta_B) \mid b_A \in B^r(\theta_A, f_B, \beta_B)\}$$

is called the set of minimizing probability functions *of bid b_A for a bidder with valuation θ_A given the other bidder's strategy β_B. Among all probability functions which induce bid b_A as a best reply, a minimizing probability function is a probability function which induces the minimum utility.*

Observation 1. *Let β_K be a strategy of bidder K for $K \in \{A, B\}$ and $\left(f_B^0, f_B^\theta, f_B^1\right)$ be a profile of beliefs bidder A has about bidder B's valuation. For a valuation $\theta_A \in \{0, \theta, 1\}$ of bidder A and a bid $b_A \in supp\left(\beta_A(\theta_A)\right)$ the worst-case belief condition for bid b_A, given by*

$$U\left(\theta_A, f_B^{\theta_A}, b_A, \beta_B\right) \leq U\left(\theta_A, f_B, b^r\left(\theta_A, f_B, \beta_B\right), \beta_B\right)$$

for all $f_B \in \mathscr{F}_\mu$, is equivalent to the following two conditions:

(i) The belief $f_B^{\theta_A}$ is an element in $\mathscr{F}^{min}\left(\theta_A, b_A, \beta_B\right)$, i.e. a minimizing probability function of bid b_A for a bidder with valuation θ_A given B's strategy β_B.

(ii) Let b'_A be a bid and f_B be an element in $\mathscr{F}^{min}\left(\theta_A, b'_A, \beta_B\right)$, i.e. a minimizing probability function of bid b'_A for a bidder with valuation θ_A given β_B. Then it holds

$$U\left(\theta_A, f_B^{\theta_A}, b_A, \beta_B\right) \leq U\left(\theta_A, f_B, b'_A, \beta_B\right).$$

Clearly, a belief cannot be a worst-case belief of a given type if this belief induces a bid as a best reply for this type but there exists another belief which induces the same bid but with a lower expected utility. Therefore, a worst-case belief has to be a minimizing probability function for all bids in the support of bidder A's bidding strategy, as stated in the first condition of the observation. Moreover, for every type and every bid in the support of the given type there cannot exist another bid which induces a lower expected utility together with a minimizing probability function for this type and this bid, as stated in the second condition. In other words, if we group together all probability functions which induce the same bid and consider the minimizing probability function in every group, we can compare the expected utility induced by bids instead the expected utility induced by beliefs.

That is, it is sufficient to compare bids if we compare them with respect to the expected utility they induce together with their minimizing probability function. In order to apply this technique, we need the following definitions.

Definition 3. *For a bidder with valuation 1 minimization problem M_b^1 of a bid $b \in [\theta, \bar{b}_1]$ is the minimization problem corresponding to its minimizing probability functions, i.e. all solutions of minimization problem M_b^1 are minimizing probability functions of b for a bidder with valuation 1 given the other bidder's worst-case strategy β^*. Formally, minimization problem M_b^1 is given* *by*

$$\min_{(f_0,f_\theta,f_1)} \left(f_0+f_\theta+f_1G_1\left(b\right)\right)\left(1-b\right)$$
$$s.t.\ f_{\hat{\theta}}\geq 0 \text{ for all } \hat{\theta}\in\{0,\theta,1\}$$
$$f_0+f_\theta+f_1=1$$
$$f_\theta\theta+f_1=\mu$$
$$\left(f_0+f_\theta+f_1G_1\left(b\right)\right)\left(1-b\right)\geq\left(f_0+f_\theta G_\theta\left(s\right)\right)\left(1-s\right) \text{ for all } s\in[0,\overline{b}_\theta]$$
$$\left(f_0+f_\theta+f_1G_1\left(b\right)\right)\left(1-b\right)\geq\left(f_0+f_\theta+f_1G_1\left(s\right)\right)\left(1-s\right) \text{ for all } s\in[\overline{b}_\theta,\overline{b}_1].$$

In other words, among all probability functions which induce bid b for type 1 as a best reply, the solutions of minimization problem M_b^1 induce the minimum expected utility. Note that since bids above $\overline{b}_1$ are never a best reply, it is not necessary to include constraints which ensure that bidding b induces at least the same expected utility as bids above $\overline{b}_1$.

Definition 4. *Apart from the constraints*

$$f_{\hat{\theta}}\geq 0 \text{ for all } \hat{\theta}\in\{0,\theta,1\}$$
$$f_0+f_\theta+f_1=1$$
$$f_\theta\theta+f_1=\mu,$$

every constraint in minimization problem M_b^1 compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U\left(1,f,b,\beta^*\right)\geq U\left(1,f,b',\beta^*\right).$$

We call such a constraint an incentive constraint corresponding to bid b'.

Definition 5. *For a type $\hat{\theta}\in\{0,\theta,1\}$ and bids b,b' we use the notation $b\leq^{\hat{\theta}} b'$ if for the $\hat{\theta}$-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidder's worst-case strategy β^*. Formally, let $f^{min}\left(\hat{\theta},b,\beta^*\right)\in\mathscr{F}^{min}\left(\hat{\theta},b,\beta^*\right)$ and $f^{min}\left(\hat{\theta},b',\beta^*\right)\in\mathscr{F}^{min}\left(\hat{\theta},b',\beta^*\right)$. Then it holds that*

$$U\left(\hat{\theta},f^{min}\left(\hat{\theta},b,\beta^*\right),b,\beta^*\right)\leq U\left(\hat{\theta},f^{min}\left(\hat{\theta},b',\beta^*\right),b',\beta^*\right)\ \Rightarrow b\leq^{\hat{\theta}} b',$$
$$U\left(\hat{\theta},f^{min}\left(\hat{\theta},b,\beta^*\right),b,\beta^*\right)< U\left(\hat{\theta},f^{min}\left(\hat{\theta},b',\beta^*\right),b',\beta^*\right)\ \Rightarrow b<^{\hat{\theta}} b'$$

and

$$U\left(\hat{\theta},f^{min}\left(\hat{\theta},b,\beta^*\right),b,\beta^*\right)= U\left(\hat{\theta},f^{min}\left(\hat{\theta},b',\beta^*\right),b',\beta^*\right)\ \Rightarrow b=^{\hat{\theta}} b'.$$

We also use the notation $b<^{\hat{\theta}} b'$ if b' does not have a minimizing probability function given $\hat{\theta}$ because it is never a best reply for a bidder with valuation $\hat{\theta}$, but b does have a minimizing *probability function. We use the notation* $b =^{\hat{\theta}} b'$ *if neither* b*, nor* b' *have a minimizing probability function.*

Given the notation provided in this definition and Observation **??**, we can state a condition which is equivalent to the worst-case belief condition but is more tractable:

Observation 2. *The worst-case belief condition for a bidder with valuation* $\hat{\theta} \in \{0, \theta, 1\}$*, bid* $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ *and the other bidder's strategy* β^* *given by*

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu$$

is equivalent to

(i) $$f^{\hat{\theta},*} \in \mathscr{F}^{min}\left(\hat{\theta}, b_{\hat{\theta}}, \beta^*\right)$$

(ii) $$b_{\hat{\theta}} \leq^{\hat{\theta}} b' \text{ for all } b' \in [0, \bar{b}_1].$$

In order to apply this observation, we will make use of the fact that the relation $\leq^{\hat{\theta}}$ constitutes a transitive order which allows us to build chains of the form

$$b_{\hat{\theta}} \leq^{\hat{\theta}} b_1 \cdots \leq^{\hat{\theta}} b_k$$

and exclude all bids $b_1, \ldots, b_k$ as bids which could induce a lower expected utility.

After reframing the worst-case belief condition, we prove two lemmas which correspond to two different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy.[11] The first tool is to show that for every bid in the interval $(\theta, \bar{b}_1)$ there exists only one probability function which induces this bid as a best reply for the 1-type. As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[\theta, \bar{b}_1]$ and show that the minimum utility is equal for all bids in the interval $[\theta, \bar{b}_1]$. This is formalized in the following Lemma and Corollary.

Lemma 1. *Let* $b \in (\theta, \bar{b}_1)$ *be such that* b *is an element in* $B^r\left(1, f^{1,b}, \beta^*\right)$ *for* $f^{1,b} \in \mathscr{F}_\mu$*. Then* $f^{1,b}$ *equals to* $f^{1,*} = \left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$*, the worst-case belief of a bidder with valuation 1.*

The intuition behind this is that the worst-case belief of a 1-type together with the strategy of the other 1-type makes her indifferent between any bid in the interval $[\theta, \bar{b}_1]$. Any change of the worst-case belief makes either a deviation to θ or to $\bar{b}_1$ more profitable. Hence, a bid $b \in \left(\theta, \bar{b}_1\right)$ cannot be induced by a belief different from the worst-case belief of the 1-type. The formal proof is relegated to Appendix **??**.

Corollary 1. *For every* $b \in [\theta, \bar{b}_1]$ *it holds that* $\theta =^1 b$*.*

[11] We will need a third tool in the case $\theta > \mu$.

That is, every bid in the interval $[\theta, \bar{b}_1]$ induces the same expected utility together with a minimizing probability function.

Proof. As defined in (**??**), the worst-case belief of a bidder with valuation 1, denoted by $f^{1,*}$, is the solution of minimization problem $M^1_{<\theta}$. Since we have shown that the best-reply condition is fulfilled for type 1, it holds that $f^{1,*}$ is an element of the feasible set of minimization problem M^1_θ. Since the constraints of minimization problem $M^1_{<\theta}$ are a subset of the constraints of minimization problem M^1_θ, it follows that $f^{1,*}$ is a solution of M^1_θ. It follows from Lemma **??** and the definition of the worst-case belief of the 1-type that every bid in $[\theta, \bar{b}_1)$ together with its unique minimizing probability function induces the same expected utility given by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta).$$

Independent of the probability function the expected utility of bidding $\bar{b}_1$ is equal to $1-\bar{b}_1$ which is equal to $\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta)$. Therefore, it holds for all $b \in [\theta, \bar{b}_1]$ that

$$\theta =^1 b.$$

□

The second tool constitutes a connection between binding incentive constraints in minimization problem M^1_b and bids which are lower than b with respect to the introduced transitive order $\leq^1$.

Lemma 2. *Let b be a bid and $f^{1,b}$ a solution of minimization problem M^1_b. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(1, f^{1,b}, b, \beta^*\right) = U\left(1, f^{1,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^1 b$.

Proof. Let L^1_b and $L^1_{\hat{b}}$ be the feasible sets, $f^{1,b} = \left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$ and $f^{1,\hat{b}} = \left(f_0^{1,\hat{b}}, f_\theta^{1,\hat{b}}, f_1^{1,\hat{b}}\right)$ solutions and $U\left(1, f^{1,b}, b, \beta^*\right)$ and $U\left(1, f^{1,\hat{b}}, \hat{b}, \beta^*\right)$ the values of the objective functions of minimization problem M^1_b and $M^1_{\hat{b}}$ respectively. In minimization problem $M^1_{\hat{b}}$ for every $s \in [\theta, \bar{b}_1]$ the incentive constraint corresponding to bid s given by

$$U\left(1, f, \hat{b}, \beta^*\right) \geq U\left(1, f, s, \beta^*\right)$$

is fulfilled for $f = f^{1,b}$ because it holds that

$$U\left(1, f^{1,b}, \hat{b}, \beta^*\right) = U\left(1, f^{1,b}, b, \beta^*\right) \geq U\left(1, f^{1,b}, s, \beta^*\right).$$

The equality follows from the fact that the incentive constraint corresponding to $\hat{b}$ is binding in minimization problem M^1_b. The inequality

$$U\left(1, f^{1,b}, b, \beta^*\right) \geq U\left(1, f^{1,b}, s, \beta^*\right)$$

holds because $f^{1,b}$ is a solution of minimization problem M^1_b. Since every constraint in $M^1_{\hat{b}}$ is fulfilled by $f^{1,b}$, it holds that $f^{1,b}$ is an element of $L^1_{\hat{b}}$. This also shows that the feasible set of minimization problem $M^1_{\hat{b}}$ is not empty. Therefore, in $M^1_{\hat{b}}$ the solution of minimization problem $M^1_{\hat{b}}$ has to induce a lower or equal utility than the solution of minimization problem M^1_b and it follows that

$$U\left(1, f^{1,\hat{b}}, \hat{b}, \beta^*\right) \leq U\left(1, f^{1,b}, \hat{b}, \beta^*\right) = U\left(1, f^{1,b}, b, \beta^*\right).$$

We conclude that bid b together with a minimizing probability function does not induce a lower expected utility than bid $\hat{b}$ together with a minimizing probability function and it therefore holds that $\hat{b} \preceq^1 b$. □

After introducing two tools with which we can compare bids with respect to the introduced transitive order, we can prove the second part of Proposition **??**.

Proof. Since by bidding zero a bidder with valuation zero expects a utility of zero and this is the lowest possible utility, the worst-case belief condition is fulfilled for type zero. The expected utility of a bidder with valuation θ induced by her worst-case belief and the other bidder's strategy is zero and therefore, the worst-case belief condition is fulfilled for type θ. It is left to show the worst-case belief condition for type 1. As stated in Observation **??**, the worst-case belief condition for type 1 is equivalent to

(i) $$f^{1,*} \in \mathscr{F}^{min}\left(1, b, \beta^*\right)$$

(ii) $$b \preceq^1 b' \text{ for all } b' \in [0, \overline{b}_1]$$

for all $b \in [\theta, \overline{b}_1]$. Analogously as in the proof of Corollary **??**, one can show that the worst-case belief of type 1 is a solution of minimization problem M^1_b for all $b \in [\overline{b}_\theta, \overline{b}_1]$. It follows from Lemma **??** that condition (i) is fulfilled for all bids in $[\overline{b}_\theta, \overline{b}_1)$. By definition of the worst-case belief of type 1, this belief induces $\overline{b}_1$ as best reply for a bidder with valuation 1. Since any probability function which induces $\overline{b}_1$ as a best reply for the 1-type yields an expected utility of $1 - \overline{b}_1$, any probability function with this property is a minimizing probability function. Therefore, condition (i) is also fulfilled for bid $\overline{b}_1$. Given the result in Corollary **??**, condition (ii) reduces to

(7) $$\theta \preceq^1 b \text{ for all } b \in [0, \theta).$$

The only candidate for a bid in the interval $[0, \theta)$ which could induce a lower expected utility than bid θ is 0 since all other bids cannot be a best reply independently of the belief. A minimizing probability function of zero is a solution of minimization problem M_0^1:

$$\min_{(f_0, f_\theta, f_1)} \quad f_0$$

$$s.t. \; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = 1 = \mu$$

$$f_0 \geq \left(f_0 + f_\theta + f_1 G_1(s)\right)(1 - s) \;\text{ for all } s \in [\theta, \bar{b}_1].$$

Note that it is not necessary to include incentive constraints with corresponding bid in the interval $(0, \theta)$ since such a bid is never a best reply. If only the constraints

$$s.t. \; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = 1 = \mu$$

would be considered, it would hold for the solution of M_0^1 that $f_1 = 0$. But then the constraint

$$f_0 \geq (f_0 + f_\theta)(1 - \theta)$$

would be violated. Therefore, one of the incentive constraints in M_0^1 has to be binding. Let $\hat{b}$ be the bid such that the corresponding incentive constraint is binding. It follows from Lemma **??** that $\hat{b} \leq^1 0$. Since bids in the interval $(0, \theta)$ are never a best reply, it must hold that $\hat{b} \in [\theta, \bar{b}_1]$. Using the transitivity of the relation $\leq^1$, we conclude that

$$0 \leq^1 \hat{b} =^1 \theta.$$

Thus, we have shown (**??**) which completes the proof. □

After proving the best-reply and the worst-case belief condition, we conclude that the strategies and beliefs specified in **??** indeed constitute a worst-case belief equilibrium. This completes the example with two bidders and three types such that $\theta \leq \mu$ and now we turn to the case where $\theta > \mu$. As before, we first specify the worst-case strategy and beliefs.

3.4. **Characterization of the efficient worst-case belief equilibrium for $\boldsymbol{\theta > \mu}$.** Again, we denote the worst-case strategy by β^* and define

$$\beta^*(0) = 0, \;\; \beta^*(\theta) = G_\theta, \;\; \beta^*(1) = G_1. \tag{8}$$

That is, type zero bids zero, type θ plays a mixed strategy on the interval $[0, \bar{b}_\theta]$ and type 1 plays a mixed strategy on the interval $[\bar{b}_\theta, \bar{b}_1]$. As before, one can immediately see that this constitutes an efficient equilibrium. We denote the worst-case belief of a bidder with valuation $\hat{\theta}$ by $\left(f_0^{\hat{\theta},*}, f_\theta^{\hat{\theta},*}, f_1^{\hat{\theta},*}\right)$ for $\hat{\theta} \in \{0, \theta, 1\}$. Type zero can have any belief. The worst-case belief of type θ is the solution of the following minimization problem, denoted by $M^\theta_{<0}$:

$$\min_{(f_0, f_\theta, f_1)} \; f_0$$

$$s.t. \; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

Recall that in the case $\theta \leq \mu$, by definition, the worst-case belief of a bidder with a given type contained all incentive constraints with corresponding bids which are lower than the lower endpoint of the type's bidding interval. This also holds for the case $\theta > \mu$. Since type θ plays a mixed strategy on an interval beginning with zero, there are no incentive constraints in this minimization problem. Any solution of minimization problem M^θ_0 has to fulfill the two probability constraints:

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu.$$

Rearranging the second probability constraint w.r.t. to f_θ and plugging in into the first probability constraint gives:

$$f_0 + \frac{\mu - f_1}{\theta} + f_1 = 1$$

$$\Leftrightarrow f_0 = \frac{\theta - \mu}{\theta} + \frac{f_1 (1 - \theta)}{\theta}.$$

Thus, the minimum value for f_0 is given by $\frac{\theta - \mu}{\theta}$ and the solution of minimization problem $M^\theta_{<0}$ is given by

$$f_0^{\theta,*} = \frac{\theta - \mu}{\theta}, \; f_\theta^{\theta,*} = \frac{\mu}{\theta}, \; f_1^{\theta,*} = 0. \tag{9}$$

The upper endpoint of the bidding interval of a bidder with valuation θ is obtained by the equation

$$f_0^{\theta,*} \theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right) \left(\theta - \bar{b}_\theta\right) \tag{10}$$

$$\Leftrightarrow \bar{b}_\theta = f_\theta^{\theta,*} \theta = \mu.$$

The bid distribution of bidders A and B with valuation θ makes them indifferent between any bid in their bidding interval. That is, it for every $s \in [0, \bar{b}_\theta]$ it holds that

$$f_0^{\theta,*}\theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s). \tag{11}$$

The subjective worst-case belief of a bidder with valuation 1 is the solution of the following minimization problem, denoted by $M^1_{<\bar{b}_\theta}$:

$$\min_{(f_0, f_\theta, f_1)} \quad f_0 + f_\theta$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

$$(f_0 + f_\theta)\left(1 - \bar{b}_\theta\right) \geq (f_0 + f_\theta G_\theta(s))(1 - s) \text{ for all } s \in [0, \bar{b}_\theta].$$

As before, the minimization problem contains all incentive constraints with corresponding bids which are lower than the lower endpoint of type 1's bidding interval. This implies that there is just enough probability weight on types zero and θ in order to incentivize the 1-type to play a mixed strategy on an interval beginning with $\bar{b}_\theta$. The upper endpoint of type 1's bidding interval is obtained by the equation

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}\right)\left(1 - \bar{b}_1\right).$$

The bid distribution of bidders A and B with valuation 1 makes them indifferent between any bid in their bidding interval. That is, for every $s \in [\bar{b}_\theta, \bar{b}_1]$ it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1(s)\right)(1 - s). \tag{12}$$

Note that in contrast to previous minimization problems we cannot derive the solution of minimization problem $M^1_{<\bar{b}_\theta}$ directly since we have to consider an uncountable number of incentive constraints . For now, we proceed with the given definition of the worst-case belief of a bidder with valuation 1 and provide the explicit solution of the minimization problem later on. However, it is easy to see that the feasible set of minimization problem $M^1_{<\bar{b}_\theta}$ is not empty since the worst-case belief of type θ is an element of this set.

3.5. **Proving the best-reply and the worst-case belief condition for $\theta > \mu$.** After specifying the worst-case strategy and beliefs, we have to show that these indeed constitute a worst-case belief equilibrium. That is, we have to show the optimality and the worst-case belief condition.

Proposition 2. *Given the worst-case strategy as defined in (??) and the worst-case beliefs as defined in (??) and (??), it holds for all $\hat{\theta} \in \{0, \theta, 1\}$ that*

(i) The best-reply condition given by

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidder's worst-case strategy.

(ii) The worst-case belief condition is fulfilled, i.e. for all $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ it holds that

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: The result is obvious for a bidder with valuation zero. The worst-case belief of bidder A with valuation θ is that bidder B has valuation 1 with probability zero. Hence, bidder A expects $\bar{b}_\theta = \mu$ to be the highest bid placed by bidder B. The bid distribution of type θ makes bidder A with valuation θ indifferent between any bid in the interval $[0, \bar{b}_\theta]$. Therefore, she has no incentive to deviate. It follows from the definition of the worst-case belief of bidder A with valuation 1 that she does not earn a higher expected utility by bidding any bid lower than $\bar{b}_\theta$. Since bidder B does not play a bid above $\bar{b}_1$, it cannot be a best reply for bidder A to bid above $\bar{b}_1$. The bid distribution G_1 is constructed in a way which makes bidder A with valuation 1 indifferent between any bid in $[\bar{b}_\theta, \bar{b}_1]$ which completes the proof. □

The remainder of the section is dedicated to proving the worst-case belief condition. Since type zero expects the lowest possible utility of zero by bidding zero, the worst-case belief condition is fulfilled for type zero. We will prove the worst-case belief condition for types θ and 1 separately, i.e. we divide part *(ii)* of Proposition **??** into two different parts

(ii.1) The worst-case belief condition is fulfilled for type θ, i.e. for all $b \in [0, \bar{b}_\theta]$ it holds that

$$U\left(\theta, f^{\theta,*}, b, \beta^*\right) \leq U\left(\theta, f, b^r\left(\theta, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

(ii.2) The worst-case belief condition is fulfilled for type 1, i.e. for all $b \in [\bar{b}_\theta, \bar{b}_1]$ it holds that

$$U\left(1, f^{\theta,*}, b, \beta^*\right) \leq U\left(1, f, b^r\left(1, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu.$$

We begin with part *(ii.1)*. Similarly, as in the case $\theta \leq \mu$, we prove three lemmas which correspond to three different tools with which we can compare the utility induced by different bids.[12]

[12]In contrast to the case $\theta \leq \mu$, in the case $\theta > \mu$ we will make use of three tools.

The first lemma provides a similar result as Lemma **??** and Corollary **??**. That is, we show that for every bid in the interval $(0, \bar{b}_\theta)$ there exists only one probability function which induces this bid as a best reply for the θ-type. As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[0, \bar{b}_\theta]$ and show that the minimum utility is equal for all bids in the interval $[0, \bar{b}_\theta]$.

Lemma 3. *Let $b \in (0, \bar{b}_\theta)$ be such that b is an element in $B^r\left(\theta, f^{\theta,b}, \beta^*\right)$ for $f^{\theta,b} \in \mathscr{F}_\mu$. Then $f^{\theta,b}$ equals to $f^{\theta,*} = \left(f_0^{\theta,*}, f_\theta^{\theta,*}, f_1^{\theta,*}\right)$, the worst-case belief of a bidder with valuation θ.*

We omit the formal proof since it works similarly to the proof of Lemma **??** and is also covered by the general case.

Corollary 2. *For every $b \in [0, \bar{b}_\theta]$ it holds that $0 =^\theta b$.*

That is, every bid in the interval $[0, \bar{b}_\theta]$ induces the same expected utility together with a minimizing probability function.

Proof. Analogously as in the proof of Corollary **??**, one can conclude that every bid in $[0, \bar{b}_\theta)$ together with its unique minimizing probability function induces the same expected utility given by $f_0^{\theta,*}\theta$.

It is left to show that $0 =^\theta \bar{b}_\theta$. Any probability function (f_0, f_θ, f_1) which induces bid $\bar{b}_\theta = \mu$ as a best reply for type θ has to fulfill

$$(f_0 + f_\theta)(\theta - \mu) \geq f_0\theta. \tag{13}$$

Since due to the probability constraints the smallest possible value for f_0 is given by $\frac{\theta-\mu}{\theta}$, it must hold that

$$(f_0 + f_\theta)(\theta - \mu) \geq \theta - \mu$$

from which follows that $f_0 + f_\theta = 1$. Hence, f_0 and f_θ are uniquely determined by the two probability constraints. Any probability function which fulfills the probability constraints and inequality (**??**) coincides with the worst-case belief of type θ.

Therefore, the worst-case belief of type θ is the only probability function which induces $\bar{b}_\theta = \mu$ as a best reply for type θ. Hence, the worst-case belief is the unique minimizing probability function for bid $\bar{b}_\theta$ and it follows from the definition of the worst-case belief that bids 0 and $\bar{b}_\theta$ induce the same expected utility together with a minimizing probability function given by

$$\left(f_0^{\theta,*} + f_0^{\theta,*}\right)(\theta - \mu) = f_0^{\theta,*}\theta.$$

Therefore, it holds for all $b \in [0, \bar{b}_\theta]$ that

$$0 =^1 b.$$

□

The second lemma corresponds to Lemma **??**. That is, it establishes a connection between binding incentive constraints in minimization problem M_b^θ and bids which are lower than b with respect to the order $\leq^\theta$

Lemma 4. *Let b be a bid and $f^{\theta,b}$ a solution of minimization problem M_b^θ. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) = U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^\theta b$.

The same proof as for Lemma **??** applies. For the third tool, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 5. *The feasible set of minimization problem M_b^θ for all $b \in (\bar{b}_\theta, \bar{b}_1]$ is empty.*

Assume there exists a bid b in the interval $(\bar{b}_\theta, \bar{b}_1]$ such that the feasible set of minimization problem M_b^θ is not empty. Then in a solution of minimization problem M_b^θ, denoted by $\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right)$, there must be strictly positive probability weight on the 1-type because otherwise there would be no incentive to bid higher than $\bar{b}_\theta$. In contrast, the worst-case equilibrium belief of the θ-type has no probability weight on the 1-type. Hence, in order to preserve the mean, the probability weight on the zero-type or the θ-type in the solution of minimization problem M_b^θ must be higher than in the worst-case belief. Given the worst-case belief, the θ-type is indifferent among all bids in the interval $[0, \bar{b}_\theta]$. If the probability weight of the zero-type is increased, it is optimal for the θ-type to bid zero. Therefore, the probability weight on the 0-type cannot be increased. Similarly, if the probability weight on the θ-type is increased, it is optimal for the 1-type to bid $\bar{b}_\theta$ or lower. Therefore, the belief $\left(f_0^{\theta,b}, f_\theta^{\theta,b}, f_1^{\theta,b}\right)$ cannot induce a bid above $\bar{b}_\theta$ for the θ-type. The formal proof is relegated to Appendix **??**.

After introducing the three tools, we can start with the proof of part *(ii.1)*.

Proof. As stated in Observation **??**, the worst-case belief condition for type θ is equivalent to

(i) $$f^{\theta,*} \in \mathscr{F}^{min}\left(\theta, b, \beta^*\right)$$

(ii) $$b \leq^\theta b' \text{ for all } b' \in [0, \bar{b}_1]$$

for all $b \in [0, \bar{b}_\theta]$.[13] Analogously as in the proof of Corollary **??**, one can show that the worst-case belief of type θ is a solution of minimization problem M_b^θ for all $b \in [0, \bar{b}_\theta]$. It follows from

[13]We use the notation provided in Definitions **??**-**??** also for the case $\theta > \mu$ but use β^* as defined in **??**.

Lemma **??** that condition (i) is fulfilled for all bids in $[0, \bar{b}_\theta)$. As shown in the proof of Corollary **??**, the worst-case belief of the θ-type is the only probability function which induces $\bar{b}_\theta$ as a best reply. Therefore, condition (i) is fulfilled. Given the result in Corollary **??**, condition (ii) reduces to

$$0 \leq^\theta b \text{ for all } b \in (\bar{b}_\theta, \bar{b}_1].$$

It follows from Lemma **??** that for all $b \in (\bar{b}_\theta, \bar{b}_1]$ it holds that $0 <^\theta b$ which completes the proof of part *(ii.1)*. □

It is left to show part *(ii.2)* of Proposition **??**, i.e. the worst-case belief condition for type 1. Again, we prove three lemmas which correspond to the three tools presented above.

The first lemma provides a similar result as Lemma **??** and Corollary **??** (and as Lemma **??** and Corollary **??** in the case $\theta \leq \mu$).

Lemma 6. *Let $b \in (\bar{b}_\theta, \bar{b}_1)$ be such that b is an element in $B^r\left(1, f^{1,b}, \beta^*\right)$ for $f^{1,b} \in \mathscr{F}_\mu$. Then $f^{1,b}$ equals to $f^{1,*} = \left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, the worst-case belief of a bidder with valuation 1.*

Corollary 3. *For every $b \in [\bar{b}_\theta, \bar{b}_1]$ it holds that $\bar{b}_\theta =^1 b$.*

We omit the proofs of the Lemma and the Claim since they work with the same arguments as before and are covered by the proof of the general case. The following lemma provides the second tool and corresponds to Lemma **??** and Lemma **??**.

Lemma 7. *Let b be a bid and $f^{1,b}$ a solution of minimization problem M_b^1. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

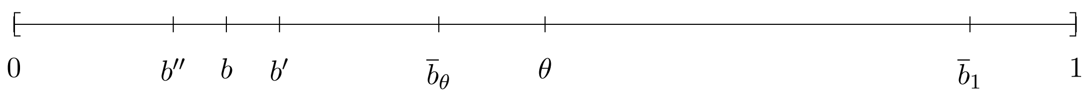
$$U\left(1, f^{1,b}, b, \beta^*\right) = U\left(1, f^{1,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \leq^1 b$.

The same proof as for Lemma **??** applies. The third tool in the proof of the worst-case belief condition for the 1-type is similar to the third tool (Lemma **??**) in the proof of the worst-case belief condition for type θ. That is, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 8. *The feasible set of minimization problem M_b^1 for all $b \in \left(0, \bar{b}_\theta\right)$ is empty.*

Assume there exists a bid $b \in \left(0, \bar{b}_\theta\right)$ such that the minimization problem M_b^1 has a solution which we denote by $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$. Consider two bids b', b'' with $0 \leq b'' < b < b' \leq \bar{b}_\theta$.



Given the belief $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$, the utility for the 1-type of bidding b must be at least as high as the utilities of bidding b'' or b'. The higher $f_0^{1,b}$, the lower is the optimal bid for type 1. Therefore, the incentive constraint corresponding to bid b'' sets a lower bound on the value of $f_0^{1,b}$ while the incentive constraint corresponding to bid b' sets an upper bound. We will show that the conditions resulting from these two bounds contradict each other. Intuitively, a bidder bidding in the interval $[0, \bar{b}_\theta]$ faces the bid distribution G_θ of the θ-type which is constructed in order to make the other θ-type indifferent. Thus, only for the θ-type the upper and the lower bound are compatible. The formal proof is relegated to Appendix **??** .

Given the three tools, we can show part *(ii.2)*.

Proof. As stated in Observation **??**, the worst-case belief condition for type 1 is equivalent to

(i) $$f^{1,*} \in \mathscr{F}^{min}(1, b, \beta^*)$$

(ii) $$b \leq^1 b' \text{ for all } b' \in [0, \bar{b}_1]$$

for all $b \in [\theta, \bar{b}_1]$. Condition (i) can be proven analogously as in the proof of Proposition **??** and due to Corollary **??**, the second condition reduces to

$$\bar{b}_\theta \leq^1 b' \text{ for all } b' \in [0, \bar{b}_\theta).$$

It follows from Lemma **??** that for all $b \in \left(0, \bar{b}_\theta\right)$ it holds that $\bar{b}_\theta <^1 b$. Therefore, in order to show the worst-case belief condition for type 1, it is left to show that

$$\bar{b}_\theta \leq^\theta 0. \tag{14}$$

As a next step, we use Lemma **??**, in order to calculate the worst-case belief of a bidder with valuation 1. This belief is the solution of minimization problem $M^1_{<\bar{b}_\theta}$:

$$\min_{(f_0, f_\theta, f_1)} \quad f_0 + f_\theta$$

$$s.t.\ f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

$$(f_0 + f_\theta)\left(1 - \bar{b}_\theta\right) \geq (f_0 + f_\theta G_\theta(s))(1 - s) \text{ for all } s \in [0, \bar{b}_\theta].$$

The solution of the reduced minimization problem which contains only the constraints

$$f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta \theta + f_1 = \mu$$

would distribute the probability weight solely between type zero and one which would violate the incentive constraint corresponding to bid zero. Hence, at least one of the incentive constraints has to be binding. Let $\hat{b}$ be a bid such that the corresponding incentive constraint is binding. Since we have shown that the best-reply condition is fulfilled for type 1, it holds that $f^{1,*}$ is an element of the feasible set of minimization problem $M^1_{\bar{b}_\theta}$. Since the constraints of minimization problem $M^1_{<\bar{b}_\theta}$ are a subset of the constraints of minimization problem $M^1_{\bar{b}_\theta}$, it follows that $f^{1,*}$ is a solution of $M^1_{\bar{b}_\theta}$. Therefore, it follows from Lemma **??** that $\hat{b} \leq^1 \bar{b}_\theta$. Due to Lemma **??**, it holds that $\bar{b}_\theta <^1 b$ for all $b \in \left(0, \bar{b}_\theta\right)$ from which follows that $\hat{b} = 0$. Therefore, the worst-case belief of type 1 is the unique solution of a system of three linear equation with three unknowns given by

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*}\theta + f_1^{1,*} = \mu$$

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \bar{b}_\theta\right) = f_0^{1,*}$$

The solution is given by

$$f_0^{1,*} = \frac{(1-\mu)^2}{1-\mu\theta},\; f_\theta^{1,*} = \frac{\mu(1-\mu)}{1-\mu\theta},\; f_1^{1,*} = \frac{\mu(1-\theta)}{1-\mu\theta}.$$

Consider minimization problem M^1_0:

$$\min_{(f_0,f_\theta,f_1)} f_0$$

$$s.t.\; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = 1 = \mu$$

$$f_0 \geq \left(f_0 + f_\theta G_\theta(s)\right)(1-s) \;\text{ for all } s \in [0, \bar{b}_\theta].$$

$$f_0 \geq \left(f_0 + f_\theta + f_1 G_1(s)\right)(1-s) \;\text{ for all } s \in [\theta, \bar{b}_1].$$

The solution of the reduced minimization problem which contains only the constraints

$$s.t.\; f_{\hat{\theta}} \geq 0 \text{ for all } \hat{\theta} \in \{0, \theta, 1\}$$

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = 1 = \mu$$

would distribute the probability weight solely between types zero and θ which would violate the incentive constraint corresponding to bid θ. Therefore, one of the incentive constraints with corresponding bid different from zero has to be binding. Let $\hat{b}'$ be the bid corresponding to the binding incentive constraint. It follows from Lemma **??** that $\hat{b}' \leq^1 0$.

As argued above, the worst-case belief of type 1 is an element of the feasible set of minimization problem $M^1_{\bar{b}_\theta}$. Since the incentive constraint corresponding to bid zero is binding in this minimization problem, it follows that the worst-case belief of type 1 is an element of the feasible set of minimization problem M^1_0. This implies that the feasible set of minimization problem M^1_0 is not empty. As stated in Lemma **??**, the feasible set of minimization problem M^1_b is empty for all $b \in (0, \bar{b}_\theta)$. Hence, it holds that $0 <^1 b$ for all $b \in (0, \bar{b}_\theta)$. Therefore, it holds that $\hat{b}' \in [\bar{b}_\theta, \bar{b}_1]$. It follows from Corollary **??** that $\bar{b}_\theta =^1 \hat{b}'$. Thus, we can construct the transitive chain

$$\bar{b}_\theta =^1 \hat{b}' \leq^1 0.$$

We have shown that (**??**) holds which we established as a sufficient condition for the worst-case belief condition for type 1. □

Since we have shown that the best-reply and the worst-case belief condition hold for all types, we conclude that the beliefs and strategies specified in **??** indeed constitute a worst-case belief equilibrium.

4. Revenue comparison of the first-price and second-price auction

We want to compare the revenue of a first-price and a second-price auction in a setting where bidders do not know the distribution of their competitors' valuations. As described in the model, we assume that the number of bidders, the set of possible valuations Θ and the exogenously given mean μ of valuations is common knowledge. In a second-price auction bidding the own valuation is a weakly dominant strategy and thus independent of the belief about the other bidders' valuations. Therefore, we assume that in a second-price auction bidders bid their valuation. For the first-price auction we assume that bidders play the efficient worst-case belief equilibrium.

Since the computation of revenue of the first-price auction involves the computation of the worst-case beliefs and strategy which is computationally complex, we provide the formal revenue comparison for the simplified case of two bidders with three possible types 0,θ and 1. As we will see, it highly depends on the valuation distribution which auction leads to the higher revenue. Hence, we cannot state any general theorems. The revenue comparison for a given valuation distribution and a given number of bidders requires a computational solution.

4.1. **Revenue of the second-price auction.** In order to compute the revenue of the first-price or the second-price auction, we need to know the true valuation distribution which we denote by (f_0, f_θ, f_1). Given that in a second-price auction all bidders bid their valuation, the revenue of the second-price auction is obtained as follows. The expected revenue from type zero is zero. The expected revenue from type θ is determined by the probability that the θ-type meets another

θ-type against whom she wins with probability $\frac{1}{2}$ and pays θ which gives an expected revenue of $\frac{1}{2}f_\theta\theta$. The expected revenue from a 1-type is determined by the probability that she meets a θ-type, in this case the 1-type wins with probability 1 and pays θ, and by the probability that she meets a 1-type, in this case the 1-type wins with probability $\frac{1}{2}$ and pays 1. This results in an expected revenue of $f_\theta\theta + \frac{1}{2}f_1$. The total expected revenue of a second-price auction from one bidder is given by

$$\frac{1}{2}f_\theta^2\theta + f_1\left(f_\theta\theta + \frac{1}{2}f_1\right). \tag{15}$$

Due to the probability constraints given by

$$f_0 + f_\theta + f_1 = 1$$

$$f_\theta\theta + f_1 = \mu,$$

there is only one degree of freedom in left in the choice of the probability function (f_0, f_θ, f_1). The probability constraints can be rewritten as

$$f_0 = 1 - f_\theta - f_1$$

$$f_1 = \mu - f_\theta\theta$$

which gives

$$f_0 = 1 - (1-\theta) f_\theta - \mu$$

$$f_1 = \mu - f_\theta\theta.$$

Substituting the expression for f_1 in (**??**) gives a revenue of

$$\frac{1}{2}f_\theta^2\theta + (\mu - f_\theta\theta)\left(f_\theta\theta + \frac{1}{2}(\mu - f_\theta\theta)\right)$$

$$= \frac{1}{2}\left(f_\theta^2\theta - f_\theta^2\theta^2 + \mu^2\right).$$

4.2. **Revenue of the first-price auction.** For the revenue calculation of the first-price auction with worst-case beliefs we have to differentiate between the case $\mu \geq \theta$ and $\mu < \theta$. We start with the case $\mu \geq \theta$. In this case the θ-type bids θ. The winning probability of the θ-type is $f_0 + \frac{1}{2}f_\theta$ which gives an expected revenue of

$$\theta\left(f_0 + \frac{1}{2}f_\theta\right) = \theta\left(1 - (1-\theta) f_\theta - \mu + \frac{1}{2}f_\theta\right) = \theta\left(1 - \mu + f_\theta\left(\theta - \frac{1}{2}\right)\right).$$

As shown in in section **??**, the worst-case belief of the 1-type, which we denote by $\left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, is the unique solution of the following system of linear equations:

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*}\theta + f_1^{1,*} = \mu$$
$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = f_0^{1,*}$$

which leads to

$$f_0^{1,*} = \frac{1-\mu}{1+\theta},\ f_\theta^{1,*} = \frac{\theta(1-\mu)}{1-\theta^2},\ f_1^{1,*}\frac{\mu-\theta^2}{1-\theta^2}.$$

The bid distribution of the 1-type, denoted by G_1, is determined by the equation

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*}G_1(s)\right)(1-s)$$

for $s \in [\theta, \bar{b}_1]$ where $\bar{b}_1$ is defined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta) = 1 - \bar{b}_1,$$

i.e. $G_1\left(\bar{b}_1\right) = 1$. This is equivalent to

$$\bar{b}_1 = 1 - \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\theta)$$
$$\Leftrightarrow \bar{b}_1 = 1 - \frac{1-\mu}{1-\theta^2}(1-\theta) = \frac{\theta+\mu}{1+\theta}.$$

After plugging in the values for $f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}$ into

$$G_1(s) = \frac{\left(f_0^{1,*} + f_\theta^{1,*}\right)(s-\theta)}{(1-s)\, f_1^{1,*}}$$

we get

$$G_1(s) = \frac{(1-\mu)(s-\theta)}{(1-s)(\mu-\theta^2)}$$

and

$$\frac{dG_1(s)}{ds} = \frac{(1-\mu)(1-s)\left(\mu-\theta^2\right) + (1-\mu)(s-\theta)\left(\mu-\theta^2\right)}{(1-s)^2(\mu-\theta^2)^2} = \frac{(1-\mu)(1-\theta)}{(1-s)^2(\mu-\theta^2)}.$$

The expected revenue from a 1-type is given by

$$\int_\theta^{\bar{b}_1} (f_0 + f_\theta + f_1 G_1(s))\, s dG_1(s)\, ds$$
$$= \int_\theta^{\frac{\theta+\mu}{1+\theta}} \left(1 - \mu + \theta f_\theta + (\mu - f_\theta\theta)\frac{(1-\mu)(s-\theta)}{(1-s)(\mu-\theta^2)}\right) s \frac{(1-\mu)(1-\theta)}{(1-s)^2(\mu-\theta^2)} ds.$$

The total expected revenue from one bidder in a first-price auction with $\theta \le \mu$ is given by

$$f_\theta\theta\left(f_0 + \frac{1}{2}f_\theta\right) + f_1\int_\theta^{\bar{b}_1} (f_0 + f_\theta + f_1 G_1(s))\, s dG_1(s)\, ds$$

$$= f_\theta \theta \left(1 - \mu + f_\theta \left(\theta - \frac{1}{2}\right)\right)$$

$$+ (\mu - f_\theta \theta) \int_\theta^{\frac{\theta+\mu}{1+\theta}} \left(1 - \mu + \theta f_\theta + (\mu - f_\theta \theta) \frac{(1-\mu)(s-\theta)}{(1-s)(\mu - \theta^2)}\right) s \frac{(1-\mu)(1-\theta)}{(1-s)^2 (\mu - \theta^2)} ds.$$

Now we will calculate the revenue of a first-price auction if $\theta > \mu$. Let $\left(f_0^{\theta,*}, f_\theta^{\theta,*}, f_1^{\theta,*}\right)$ denote the worst-case equilibrium belief of type θ. As shown in section **??**, the θ-type believes that there is no 1-type and therefore it follows from the probability constraints that

$$f_0^{\theta,*} = \frac{\theta - \mu}{\theta},\ f_\theta^{\theta,*} = \frac{\mu}{\theta},\ f_1^{1,*} = 0.$$

The θ-type plays a mixed strategy on the interval $[0, \bar{b}_\theta]$ where $\bar{b}_\theta$ is defined by

$$\theta - \bar{b}_\theta = f_0^{\theta,*} \theta$$

$$\Leftrightarrow \bar{b}_\theta = \theta - \frac{\theta - \mu}{\theta} \theta = \mu.$$

For all $s \in [0, \bar{b}_\theta]$ the bid distribution G_θ is defined by

$$f_0^{\theta,*} \theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s)$$

$$\Leftrightarrow G_\theta(s) = \frac{s f_0^{\theta,*}}{f_\theta^{\theta,*}(\theta - s)} = \frac{s(\theta - \mu)}{\mu(\theta - s)}.$$

It follows that

$$\frac{dG_\theta(s)}{ds} = \frac{(\theta-\mu)\mu(\theta - s) + s(\theta - \mu)\mu}{\mu^2 (\theta - s)^2} = \frac{(\theta - \mu)\theta}{\mu(\theta - s)^2}.$$

The expected revenue from a bidder with valuation θ is given by

$$\int_0^\mu (f_0 + f_\theta G_\theta(s)) s dG_\theta(s) ds$$

$$= \int_0^\mu \left(1 - \mu - (1-\theta) f_\theta + f_\theta \frac{s(\theta - \mu)}{\mu(\theta - s)}\right) s \frac{(\theta - \mu)\theta}{\mu(\theta - s)^2} ds.$$

The belief of the 1-type, denoted by $\left(f_0^{1,*}, f_\theta^{1,*}, f_1^{1,*}\right)$, is the unique solution of the following system of linear equations

$$f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} = 1$$

$$f_\theta^{1,*} \theta + f_1^{1,*} = \mu$$

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1 - \mu) = f_0^{1,*}$$

which leads to

$$f_0^{1,*} = \frac{(1-\mu)^2}{1 - \mu\theta},\quad f_\theta^{1,*} = \frac{\mu(1-\mu)}{1-\mu\theta},\quad f_1^{1,*} = \frac{\mu(1-\theta)}{1-\mu\theta}.$$

The 1-type plays a mixed strategy on the interval $[\bar{b}_\theta, \bar{b}_1]$ where $\bar{b}_1$ is determined by

$$1 - \bar{b}_1 = \left(f_0^{1,*} + f_\theta^{1,*}\right)(1 - \mu)$$

$$\Leftrightarrow \bar{b}_1 = 1 - \left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu) = 1 - \frac{(1-\mu)^2}{1-\mu\theta} = \frac{2\mu - \mu - \mu\theta}{1-\mu\theta}.$$

For all $s \in [\bar{b}_\theta, \bar{b}_1]$ the bid distribution G_1 is determined by

$$\left(f_0^{1,*} + f_\theta^{1,*}\right)(1-\mu) = \left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1(s)\right)(1-s)$$

$$\Leftrightarrow G_1(s) = \frac{\left(f_0^{1,*} + f_\theta^{1,*}\right)(s-\mu)}{f_1^{1,*}(1-s)} = \frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}$$

from which follows that

$$\frac{dG_1(s)}{ds} = \frac{(1-\mu)(1-b)\mu(1-\theta) + (1-\mu)(b-\mu)\mu(1-\theta)}{\mu^2(1-\theta)^2(1-s)^2} = \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2}.$$

The expected utility from a bidder with type 1 is given by

$$\int_\mu^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s)\, ds$$

$$= \int_\mu^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(1 - \mu + \theta f_\theta + (\mu - f_\theta\theta)\frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}\right) s \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2} ds.$$

The total expected revenue from a bidder is given by

$$f_\theta \int_0^\mu \left(f_0 + f_\theta G_\theta(s)\right) s dG_\theta(s)\, ds + f_1 \int_\mu^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(f_0 + f_\theta + f_1 G_1(s)\right) s dG_1(s)\, ds$$

$$= f_\theta \int_0^\mu \left(1 - \mu - (1-\theta) f_\theta + f_\theta \frac{s(\theta-\mu)}{\mu(\theta-s)}\right) s \frac{(\theta-\mu)\theta}{\mu(\theta-s)^2} ds$$
$$+ (\mu - f_\theta\theta) \int_\mu^{\frac{2\mu-\mu-\mu\theta}{1-\mu\theta}} \left(1 - \mu + \theta f_\theta + (\mu - f_\theta\theta)\frac{(1-\mu)(s-\mu)}{\mu(1-\theta)(1-s)}\right) s \frac{(1-\mu)^2}{\mu(1-\theta)(1-s)^2} ds.$$

4.3. **Revenue comparison.** After calculating the expected revenue of the first-price and the second-price auction we can compare the revenue for a given θ and μ in dependence of the valuation of f_θ. The minimum possible valuation for f_θ is zero. In case $\theta \leq \mu$ the maximum possible valuation of f_θ is obtained if $f_0 = 0$ and is equal to $\frac{1-\mu}{1-\theta}$. In case $\theta > \mu$, the maximum possible valuation of f_θ is obtained if $f_1 = 0$ and is equal to $\frac{\mu}{\theta}$.

The following graph illustrates the revenue comparison for the first-price auction (blue line) and the second-price auction (red line) for the parameters $\theta = 0.4$ and $\mu = 0.5$.

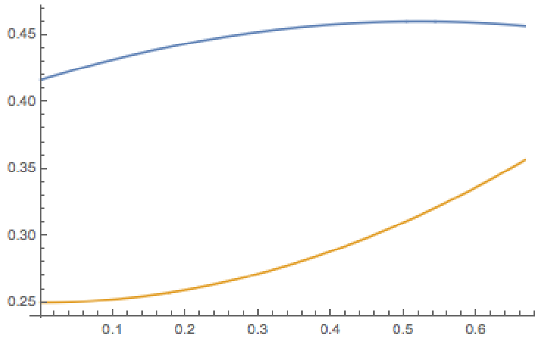


FIGURE 1. Revenue of the first-price auction (blue line) and second-price auction (red line) plotted against f_θ for $\theta = 0.4$ and $\mu = 0.5$

In this case the auctioneer would choose the first-price auction independent of the true valuation distributions. However, there exist valuations for θ and μ where the revenue functions cross, i.e. it depends on the true valuation distribution which auction leads to the higher revenue.

The following graph illustrates the revenue comparison for the first-price auction (blue line) and the second-price auction (red line) for the parameters $\theta = 0.6$ and $\mu = 0.5$.

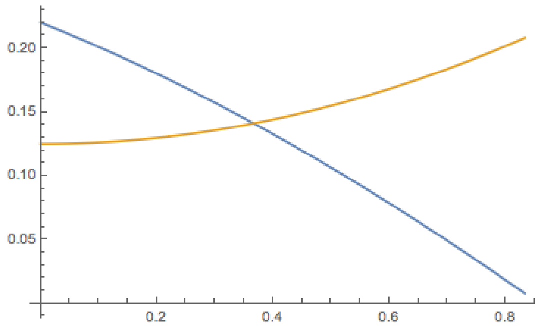


FIGURE 2. Revenue of the first-price auction (blue line) and second-price auction (red line) plotted against f_θ for $\theta = 0.6$ and $\mu = 0.5$

We conclude that the revenue comparison highly depends on the parameters θ and μ and depending on the valuation of θ and μ, it can depend on the true valuation distribution.

5. General Case: n bidders with m valuations

In this section we provide all definitions and results required for the general case with n bidders and m types. As before, the main result is that there exists an efficient worst-case belief equilibrium.

Theorem 2. *In a first-price auction there exists an efficient worst-case belief equilibrium.*

5.1. Characterization of the efficient worst-case belief equilibrium.

As in the case of two bidders and three types we begin with the characterization of the strategies and beliefs which we claim to constitute a worst-case belief equilibrium.[14] We denote the worst-case strategy by β^*.

[14]As before, we call the strategy and beliefs we claim to constitute a worst-case belief equilibrium worst-case strategy and worst-case beliefs.

The support of the bid distribution of a bidder with valuation θ^k is denoted by $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. As every bidder adopts the same worst-case belief-equilibrium, we omit the identity of the bidder in the notation. Every bidder has the same worst-case belief and moreover, in the worst-case belief of a bidder every other bidder has the same valuation distribution. Thus, we can denote the worst-case belief of a bidder with valuation $\theta^k \in \Theta$ by $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$, i.e. for $l \in \{1, \ldots, m\}$ let $f_{\theta^l}^{\theta^k,*}$ be the probability with which one of the other $n-1$ bidders has the θ^l-type in the belief of a bidder with valuation θ^k.

For a bidder with valuation $\theta^k \leq \mu$ we define the bidder's strategy to be a pure strategy with $\beta^*\left(\theta^k\right) = \theta^k$. Let θ^z be the lowest type which is strictly greater than μ. The belief of a bidder with valuation $\theta^k \leq \mu$ is the probability function which puts strictly positive weight only on $f_{\theta^k}^{\theta^k}$ and $f_{\theta^z}^{\theta^k}$. Therefore, the probability weight is determined by the equations

$$f_{\theta^k}^{\theta^k,*} + f_{\theta^z}^{\theta^k,*} = 1$$

$$f_{\theta^k}^{\theta^k,*}\theta^k + f_{\theta^z}^{\theta^k,*}\theta^z = \mu.$$

The unique solution of this system of linear equations is given by

$$f_{\theta^k}^{\theta^k,*} = \frac{\theta^z - \mu}{\theta^z - \theta^k}, \quad f_{\theta^k}^{\theta^z,*} = \frac{\mu - \theta^k}{\theta^z - \theta^k}.$$

Given this belief, it is a best reply for a bidder with valuation θ^k to bid θ^k since the lowest bid which such a bidder believes is played by another bidder is given by θ^k. This induces the lowest possible expected utility of zero and therefore, the strategies and beliefs specified for types $\theta^k \leq \mu$ fulfill the best-reply condition and the worst-case belief condition.

Now we define the bidding strategy and beliefs for a bidder with valuation θ^k with $\theta^k > \mu$. A bidder with type $\theta^k > \mu$ plays a mixed strategy on the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ where the upper limit of type θ^k's bidding interval is the lower limit of type θ^{k+1}'s bidding interval. We will derive the boundaries of this bidding interval inductively starting with the boundaries of the bidding interval of the θ^z-type, which we defined above as the lowest type strictly greater than μ. The θ^z-type plays a mixed strategy on the interval $[\bar{b}_{\theta^{z-1}}, \bar{b}_{\theta^z}]$ with $\bar{b}_{\theta^{z-1}} = \theta^{z-1}$. We define the worst-case belief of a bidder with valuation θ^z to be the solution of the following minimization problem which we denote by $M_{<\theta^{z-1}}^{\theta^z}$:

$$\min_{(f_{\theta^1}, \ldots, f_{\theta^m})} \left(f_{\theta^1} + \cdots + f_{\theta^{z-1}}\right)^{n-1} \left(\theta^z - \theta^{z-1}\right)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{l=1}^{m} f_{\theta^l} = 1$$

$$\sum_{l=1}^{m} f_{\theta^l}\theta^l = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{z-1}}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq \left(f_{\theta^1}\right)^{n-1}\theta^z$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{j-1}}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq \left(f_{\theta^1} + f_{\theta^2}\right)^{n-1}\left(\theta^z - \theta^2\right)$$

$$\vdots$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{z-1}}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right) \geq \left(f_{\theta^1} + \cdots + f_{\theta^{z-2}}\right)^{n-1}\left(\theta^z - \theta^{z-2}\right),$$

i.e. among all distributions with mean μ such that for a bidder with valuation θ^z it is weakly better to bid θ^{z-1} than any lower bid given the other bidders' strategies, it is the distribution inducing the minimum utility. We do not have to include the incentive constraints with corresponding bid b for $b \in (\theta^{h-1}, \theta^h)$ for $1 < h < z-1$ since these bids are never played according to the worst-case strategy and thus are never a best reply. Note that the feasible set of this minimization problem is non-empty since a distribution which puts strictly positive probability weight only on the θ^{z-1}- and the θ^z-type preserving the mean μ is an element of the feasible set. The upper endpoint of the bidding interval of the θ^z-type is obtained by the equation

$$\left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right) = (\theta^z - \bar{b}_{\theta^z}).$$

The bid distribution G_{θ^z} is defined such that every bidder with valuation θ^z is indifferent between every bid in her bidding interval given her belief and the other bidders' strategies, i.e. for every $s \in [\bar{b}_{\theta^{z-1}}, \bar{b}_{\theta^z}]$ where $\bar{b}_{\theta^{z-1}} = \theta^{z-1}$ it holds

$$\left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*} + f_{\theta^z}^{\theta^z,*}G_{\theta^z}(s)\right)^{n-1}(\theta^z - s) = \left(f_{\theta^1}^{\theta^z,*} + \cdots + f_{\theta^{z-1}}^{\theta^z,*}\right)^{n-1}\left(\theta^z - \theta^{z-1}\right).$$

After we have specified the strategies and beliefs for the θ^z-type, we can proceed inductively. Assume, that strategies and beliefs have been specified for types $1, \ldots, k-1$ with $z \leq k-1 < m$, then strategies and beliefs for type k are defined as follows. A bidder with valuation θ^k plays a mixed strategy on the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ where $\bar{b}_{\theta^{k-1}}$ is the upper bound of the bidding interval of the θ^{k-1}-type. We define the worst-case belief of type θ^k to be the solution of the following minimization problem which we denote by $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$:

$$\min_{(f_{\theta^1},\ldots,f_{\theta^m})} \left(f_{\theta^1} + \cdots + f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$\left(f_{\theta^1}+\cdots+f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k-\overline{b}_{\theta^{k-1}}\right) \geq \left(\left(\sum_{j=1}^{h-1} f_{\theta^j}\right)+f_{\theta^h}G_{\theta^h}(s)\right)^{n-1}\left(\theta^k-s\right)$$

for all $h \in \{1,\ldots,k-1\}$ and all $s \in [\overline{b}_{\theta^{h-1}}, \overline{b}_{\theta^h}]$,

i.e. among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any lower bid given the other bidders' strategies, it is the distribution inducing the minimum utility. The bid distribution G_{θ^k} and $\overline{b}_{\theta^k}$ are determined such that given this belief every bidder with valuation θ^k is indifferent between all bids in $[\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$. Formally, for every $s \in [\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$, $G_{\theta^k}(s)$ is defined by

$$\left(\sum_{j=1}^{k-1} f_{\theta^j}^{\theta^k,*}\right)^{n-1}\left(\theta-\overline{b}_{\theta^{k-1}}\right) = \left(\sum_{j=1}^{k-1} f_{\theta^j}^{\theta^k,*}+f_{\theta^k}^{\theta^k,*}G_{\theta^k}(s)\right)^{n-1}(\theta-s).$$

Obviously, the worst-case strategy is efficient. We will show in the next section that the feasible set of minimization problem $M^{\theta^k}_{<\overline{b}_{\theta^{k-1}}}$ is not empty. Moreover, in Lemma **??** in Appendix **??** we derive the unique solution of this minimization problem. We show that for the worst-case belief of a bidder with valuation θ^k it holds that $f_j^{\theta^k,*}=0$ for $j>k$ and that the vector $\left(f_{\theta^1}^{\theta^k,*},\ldots,f_{\theta^k}^{\theta^k,*}\right)$ is the unique solution of the system of k linear equations which includes the two probability constraints and the binding incentive constraints with corresponding bid $\overline{b}_{\theta^j}$ for $1 \leq j \leq k-2$.

5.2. **Proving the best-reply and the worst-case belief condition.** After specifying the worst-case strategy and beliefs, we have to show that they indeed constitute a worst-case belief equilibrium. That is, we have to show the best-reply and the worst-case belief condition.

Proposition 3. *Given the worst-case strategy and the worst-case beliefs as defined in* **??**, *it holds for all $\hat{\theta} \in \Theta$ that*

(i) The best-reply condition given by

$$b_{\hat{\theta}} \in B^r\left(\hat{\theta}, f^{\hat{\theta},*}, \beta^*\right) \text{ for all } b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$$

is fulfilled, i.e. every bidder plays a best reply given her valuation, her worst-case belief and the other bidders' worst-case strategy.

(ii) The worst-case belief condition is fulfilled, i.e. for all $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ it holds that

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}_\mu^{n-1}.$$ [15]

That is, there does not exist another belief such that a best reply to this belief induces a lower expected utility than the worst-case belief.

Proof. Part *(i)*: It follows directly from the definition of the worst-case beliefs, that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any lower bid. By construction, a bidder with valuation θ^k is indifferent between any bid in $[\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$. Hence, it is left to show that it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any bid higher than $\overline{b}_{\theta^k}$. In order to do so, we will compare the solutions of the following two minimization problems. Let $M^{\theta^k}_{<\overline{b}_{\theta_{k-1}}}$ be the minimization problem which corresponds to the worst-case belief of the bidder as defined above:

$$\min_{(f_{\theta^m},\dots,f_{\theta^m})} \left(f_{\theta^1} + \dots + f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^m f_{\theta^j} = 1$$

$$\sum_{j=1}^m f_{\theta^j}\theta^j = \mu$$

$$\left(f_{\theta^1} + \dots + f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right) \geq \left(\sum_{j=1}^{h-1} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1}\left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \dots, k-1\} \text{ and all } s \in [\overline{b}_{\theta^{h-1}}, \overline{b}_{\theta^h}],$$

i.e. among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any *lower* bid given the other bidders' strategies, it is the distribution inducing the minimum utility. Now we consider minimization problem $M^{\theta^k}_{\overline{b}_{\theta^{k-1}}}$ which determines the distribution inducing the minimum utility among all distributions with mean μ such that for a bidder with valuation θ^k it is weakly better to bid $\overline{b}_{\theta^{k-1}}$ than any *other* bid given the other bidders' strategies:

$$\min_{(f_{\theta^1},\dots,f_{\theta^m})} \left(f_{\theta^1} + \dots + f_{\theta^{k-1}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^m f_{\theta^j} = 1$$

[15] Since utility functions are symmetric among bidders, we will omit the identitiy of the bidder in utility function. Moreover, if there exists an asymmetric belief about the other bidders' valuations which violates the worst-case belief condition then due to the symmetry of the worst-case strategy, there exists also a symmetric belief. Therefore, it is sufficient to focus only on symmetric beliefs as possible deviations.

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(\left(\sum_{j=1}^{h} f_{\theta^j}\right) + f_{\theta^h}G_{\theta^h}(s)\right)^{n-1}\left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \ldots, m\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}].$$

Let $f^{\theta^k,*}$ and f^{θ^k} be solutions of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ and $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$ respectively. The constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ are a subset of the constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Therefore, it is sufficient to show that $f^{\theta^k,*}$ is an element of the feasible set of $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$.

In $f^{\theta^k,*}$ there is no probability weight on types above θ^k because this would require more probability weight on types below μ and hence increase the value of the objective function. If we plug in $f^{\theta^k,*}$ into $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$, then all constraints which correspond to a bid above $\bar{b}_{\theta^k}$ are fulfilled because there is no probability weight on types above θ^k. As argued above, all constraints with corresponding bid in the interval $[0, \bar{b}_{\theta^k}]$ are fulfilled. Therefore, $f^{\theta^k,*}$ is an element of the feasible set of $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$. □

Computing the worst-case belief of a bidder is equivalent to computing the distribution inducing the minimum utility of a bidder given the other bidders' strategies. Thus, one has to solve the trade-off between putting probability weight on lower types in order to induce a high bid and putting probability weight on higher types in order to reduce the winning probability.

This proof shows that this trade-off is solved such that the worst-case belief of a bidder with valuation θ^k puts just enough probability weight on lower types in order to induce the bid $\bar{b}_{\theta^{k-1}}$ and puts as much as possible probability weight on type θ^k in order to reduce the bidder's winning probability.

One can use this proof in order to show that the worst-case belief of the θ^{k-1}-type is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Hence, one can show by induction that for all $1 \leq k \leq m$ the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ is not empty.

By definition, the worst-case belief of the θ^{k-1}-type is a solution of minimization problem $M^{\theta^{k-1}}_{<\bar{b}_{\theta^{k-2}}}$. Moreover, it holds by definition of $\bar{b}_{\theta^{k-1}}$ that

$$\left(\sum_{j=1}^{k-2} f^{\theta^{k-1},*}_{\theta^j}\right)\left(\theta^{k-1} - \bar{b}_{\theta^{k-2}}\right) = \theta^{k-1} - \bar{b}_{\theta^{k-1}}.$$

Therefore, for all s with $s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}]$ with $h \leq k-1$ it holds that

$$\theta^{k-1} - \bar{b}_{\theta^{k-1}} = \left(\sum_{j=1}^{k-2} f^{\theta^{k-1},*}_{\theta^j}\right)^{n-1}\left(\theta^{k-1} - \bar{b}_{\theta^{k-2}}\right) \geq \left(\sum_{j=1}^{h-1} f^{\theta^{k-1},*}_{\theta^j} + G_{\theta^h}(s)\right)^{n-1}\left(\theta^{k-1} - s\right).$$

It follows that for all s with $s < \bar{b}_{\theta^{k-1}}$ the incentive constraint corresponding to s is fulfilled if plugging in $f^{\theta^{k-1},*}$ into $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$ because adding the inequalities

$$\theta^{k-1} - \bar{b}_{\theta^{k-1}} \geq \left(\sum_{j=1}^{h-1} f_{\theta^j}^{\theta^{k-1},*} + G_{\theta^h}(s) \right)^{n-1} \left(\theta^{k-1} - s \right)$$

and

$$\theta^{k} - \theta^{k-1} \geq \left(\sum_{j=1}^{h-1} f_{\theta^j}^{\theta^{k-1},*} + G_{\theta^h}(s) \right)^{n-1} \left(\theta^{k} - \theta^{k-1} \right)$$

yields

$$\theta^{k} - \bar{b}_{\theta^{k-1}} \geq \left(\sum_{j=1}^{h-1} f_{\theta^j}^{\theta^{k-1},*} + G_{\theta^h}(s) \right)^{n-1} \left(\theta^{k} - s \right).$$

Conclusively, $f^{\theta^{k-1},*}$ is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$.

We have already shown the worst-case belief condition for all types $\theta^k \leq \mu$. In order to show the worst-case belief condition for higher types, as in the case of three types and two bidders, we introduce the concept of minimizing probability functions and show that we can switch from comparing the induced utility of distributions to comparing the induced utility of bids. This is formalized in the following definition and observation.

Definition 6. *For a bidder with valuation θ_i, a bid b_i and a strategy β_{-i} of the other bidders, the set of probability functions $\mathscr{F}^{min}_{n-1}(\theta_i, b_i, \beta_{-i})$ given by*

$$\mathscr{F}^{min}_{n-1}(\theta_i, b_i, \beta_{-i}) = \underset{f_{-i} \in \mathscr{F}^{n-1}_{\mu}}{\arg\min} \left\{ U(\theta_i, f_{-i}, b_i, \beta_{-i}) \mid b_i \in B^r(\theta_i, f_{-i}, \beta_{-i}) \right\}$$

is called the set of minimizing probability functions *of bid b_i for a bidder with valuation θ_i given the other bidders' strategies β_{-i}. Among all probability functions which induce bid b_i as a best reply, a minimizing probability function is a probability function which induces the minimum utility.*

Observation 3. *Let $(\beta_1, \ldots, \beta_n)$ be a profile of strategies and $\left(f^{\theta^1}_{-i}, \ldots, f^{\theta^m}_{-i}\right)$ be a profile of beliefs bidder i has about the other bidders' valuations. For a valuation θ_i of bidder i and a bid $b_i \in supp(\beta_i(\theta_i))$ the worst-case belief condition for bid b_i, given by*

$$U\left(\theta_i, f^{\theta_i}_{-i}, b_i, \beta_{-i}\right) \leq U\left(\theta_i, f^{\theta_i}_{-i}, b^r(\theta_i, f_{-i}, \beta_i), \beta_{-i}\right)$$

for all $f_{-i} \in \mathscr{F}^{n-1}_{\mu}$, is equivalent to the following two conditions:

(i) The belief $f^{\theta_i}_{-i}$ is an element in $\mathscr{F}^{min}_{n-1}(\theta_i, b_i, \beta_{-i})$, i.e. a minimizing probability function of bid b_i for bidder i with valuation θ_i given the other bidders' strategies β_{-i}.

(ii) Let b_i' be a bid and f_{-i} be an element in $\mathscr{F}_{n-1}^{min}(\theta_i, b_i', \beta_{-i})$, i.e. a minimizing probability function of bid b_i' for bidder i with valuation θ_i. Then it holds

$$U\left(\theta_i, f_{-i}^{\theta_i}, b_i, \beta_{-i}\right) \leq U\left(\theta_i, f_{-i}, b_i', \beta_{-i}\right).$$

That is, it is sufficient to compare bids if we compare them with respect to the expected utility they induce together with a minimizing probability function. In order to apply this technique, we need the following definitions.

Definition 7. *For a bidder with valuation θ minimization problem M_b^θ of a bid $b \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ is the minimization problem corresponding to its minimizing probability functions, i.e. all solutions of minimization problem M_b^θ are minimizing probability function of b for a bidder with valuation θ given the other bidders' worst-case strategy β^*. Formally, minimization problem M_b^θ is given by*

$$\min_{(f_{\theta^1},\ldots,f_{\theta^m})} \left(\sum_{j=1}^{l-1} f_{\theta^j} + f_{\theta^l} G_l(b)\right)^{n-1} (\theta - b)$$

$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu$$

$$\left(f_{\theta^1} + \cdots + f_{\theta^{l-1}}\right)^{n-1} (\theta - b) \geq \left(\sum_{j=1}^{h} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} (\theta - s)$$

$$\text{for all } h \in \{1, \ldots, m\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}].$$

Definition 8. *Apart from the constraints*

$$f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j} \theta^j = \mu,$$

every constraint in minimization problem M_b^θ compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U(\theta, f, b, \beta^*) \geq U\left(\theta, f, b', \beta^*\right).$$

We call such a constraint an incentive constraint corresponding to bid b'.

Definition 9. *For a type θ and bids b, b' we use the notation $b \leq^{\theta} b'$ if for the θ-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidders' worst-case strategy β^*. Formally, let $f^{min}(\theta, b, \beta^*) \in \mathscr{F}^{min}_{n-1}(\theta, b, \beta^*)$ and $f^{min}(\theta, b', \beta^*) \in \mathscr{F}^{min}_{n-1}(\theta, b', \beta^*)$. Then it holds that*

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) \leq U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \Rightarrow b \leq^{\theta} b',$$

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) < U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \Rightarrow b <^{\theta} b'$$

and

$$U\left(\theta, f^{min}(\theta, b, \beta^*), b, \beta^*\right) = U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \Rightarrow b =^{\theta} b'.$$

We also use the notation $b <^{\theta} b'$ if b' does not have a minimizing probability function given θ because it is never a best reply for a bidder with valuation θ, but b does have a minimizing probability function. We use the notation $b =^{\theta} b'$ if neither b, nor b' have a minimizing probability function.

Given the notation provided in this Definition, we can state a condition which is equivalent to the worst-case belief condition but is more tractable:

Observation 4. *The worst-case belief condition for bidder i with valuation $\hat{\theta}$ and bid $b_{\hat{\theta}} \in supp\left(\beta^*\left(\hat{\theta}\right)\right)$ and bidder B's strategy β^* given by*

$$U\left(\hat{\theta}, f^{\hat{\theta},*}, b_{\hat{\theta}}, \beta^*\right) \leq U\left(\hat{\theta}, f, b^r\left(\hat{\theta}, f, \beta^*\right), \beta^*\right) \text{ for all } f \in \mathscr{F}^{n-1}_{\mu}$$

is equivalent to

(i) $$f^{\hat{\theta},*} \in \mathscr{F}^{min}_{n-1}\left(\hat{\theta}, b_{\hat{\theta}}, \beta^*\right)$$

(ii) $$b_{\hat{\theta}} \leq^{\hat{\theta}} b' \text{ for all } b' \in [0, \overline{b}_{\theta^m}].$$

As in the case with two bidders and three valuations, we prove three lemmas which correspond to three different tools with which we can compare the utility induced by different bids and therefore exclude bids as possible deviations from the proposed worst-case strategy. The first tool is to show that for every valuation $\theta^k \geq \theta^z$ for every bid in the interval $(\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ there exists only one probability function which induces this bid as a best reply for the θ^k-type.[16] As a consequence, one can directly compute the minimum utility which can be induced for a bid in the interval $[\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$ and show that the minimum utility is equal for all bids in the interval $[\overline{b}_{\theta^{k-1}}, \overline{b}_{\theta^k}]$. This is formalized in the following Lemma and Corollary.

[16] Recall that we defined θ^z to be the smallest valuation which is strictly greater than μ.

Lemma 9. *For a valuation θ^k with $\theta^z \le \theta^k \le \theta^m$ and $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ let $f^{min}\left(\theta^k, b, \beta^*\right)$ be an element in $\mathscr{F}^{min}_{n-1}\left(\theta^k, b, \beta^*\right)$. Then $f^{min}\left(\theta^k, b, \beta^*\right)$ equals to $f^{\theta^k,*}$, the worst-case belief of a bidder with valuation θ.*

The intuition behind this result works similarly as for the result for two bidders and three types in Lemma **??**. The formal proof is relegated to Appendix **??**.

Corollary 4. *For every valuation θ^k with $\theta^k \ge \theta^z$ and for every $b \in (\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ it holds that $b =^{\theta^k} \bar{b}_{\theta^{k-1}}$.*

That is, every bid in the interval $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ induces the same expected utility together with a minimizing probability function.

Proof. We have shown in the first part of Proposition **??** that the best-reply condition is fulfilled for all types. Hence, it holds that the worst-case belief of a bidder with valuation $\theta^k \ge \theta^z$, which is the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$, is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^k}}$. Since the constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ are a subset of $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$, it holds that $f^{\theta,*}$ is a solution of $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$. It follows from Lemma **??** and the definition of the worst-case belief of a bidder with valuation θ^k that every bid in $(\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ together with its unique minimizing probability function induces the same expected utility given by

$$\left(f^{\theta^k,*}_{\theta^1} + \cdots + f^{\theta^k,*}_{\theta^{k-1}}\right)\left(\theta^k - \bar{b}_{\theta^{k-1}}\right).$$

Thus, it holds for every $b \in (\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ that

$$b =^{\theta^k} \bar{b}_{\theta^{k-1}}.$$

□

The second tool constitutes a connection between binding incentive constraints in the minimization problem corresponding to a bid b and bids which are lower than b with respect to our order. It corresponds to Lemmas **??**,**??** and **??** in the case of two bidders and three types.

Lemma 10. *Let θ be a valuation, b a bid and $f^{\theta,b}$ a solution of minimization problem M^{θ}_{b}. If there exists a binding incentive constraint with corresponding bid $\hat{b}$, i.e.*

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) = U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right),$$

then it holds that $\hat{b} \le^{\theta} b$.

Proof. Let L^{θ}_{b} and $L^{\theta}_{\hat{b}}$ be the set of feasible solutions , $f^{\theta,b} = \left(f^{\theta,b}_{\theta^1}, \ldots, f^{\theta,b}_{\theta^m}\right)$ and $f^{\theta,\hat{b}} = \left(f^{\theta,\hat{b}}_{0}, \ldots, f^{\theta,\hat{b}}_{\theta^m}\right)$ solutions and $U\left(\theta, f^{\theta,b}, b, \beta^*\right)$ and $U\left(\theta, f^{\theta,\hat{b}}, \hat{b}, \beta^*\right)$ the values of the objective functions of minimization problem M_b^θ and $M_{\hat{b}}^\theta$ respectively. In minimization problem $M_{\hat{b}}^\theta$ for every bid s the incentive constraint corresponding to s given by

$$U\left(\theta, f, \hat{b}, \beta^*\right) \geq U\left(\theta, f, s, \beta^*\right)$$

is fulfilled for $f = f^{\theta,b}$ because it holds that

$$U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right) = U\left(\theta, f^{\theta,b}, b, \beta^*\right) \geq U\left(\theta, f^{\theta,b}, s, \beta^*\right).$$

The equality follows from the fact that the incentive constraint corresponding to $\hat{b}$ is binding in minimization problem M_b^θ. The inequality

$$U\left(\theta, f^{\theta,b}, b, \beta^*\right) \geq U\left(\theta, f^{\theta,b}, s, \beta^*\right)$$

holds because f_b^θ is a solution of minimization problem M_b^θ. Since every constraint in $M_{\hat{b}}^\theta$ is fulfilled by $f^{\theta,b}$, it holds that $f^{\theta,b}$ is an element of $L_{\hat{b}}^\theta$. Therefore in $M_{\hat{b}}^\theta$, the solution of minimization problem $M_{\hat{b}}^\theta$ has to induce a lower or equal utility than the solution of minimization problem M_b^θ and it follows that

$$U\left(\theta, f^{\theta,\hat{b}}, \hat{b}, \beta^*\right) \leq U\left(\theta, f^{\theta,b}, \hat{b}, \beta^*\right) = U\left(\theta, f^{\theta,b}, b, \beta^*\right).$$

We conclude that bid b together with a minimizing probability function does not induce a lower expected utility than bid $\hat{b}$ together with a minimizing probability function and therefore it holds that $\hat{b} \leq^\theta b$. □

For the third tool, we show that for a given type there exist bids which can never be a best reply independent of the subjective belief. Hence, these bids cannot be possible deviations from the proposed worst-case equilibrium for the given type.

Lemma 11. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-1}$ and every b with $\overline{b}_{\theta^{l-1}} < b < \overline{b}_{\theta^l}$ the feasible set of minimization problem $M_b^{\theta^k}$ is empty.*

Lemma 12. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^{k+1} \leq \theta^l$ and every b with $\overline{b}_{\theta^{l-1}} < b \leq \overline{b}_{\theta^l}$ the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$ is empty.*

The formal proof is relegated to Appendices **??** and **??**. The intuition for Lemma **??** is similar to Lemma **??** and the intuition for Lemma **??** is similar to Lemma **??**, i.e. as in the case of two bidders and three valuations. We provide a detailed intuition for both results at the end of this section.

After introducing these three tools, we can provide the proof of part *(ii)* of Proposition **??**. That is, we prove that the worst-case belief condition is fulfilled for all types. In this proof we construct a chain where all bids are arranged with respect to our order and the efficient equilibrium bid is the lowest. Due to the transitivity of our relation, this excludes all other distributions than the efficient worst-case beliefs as a potential deviation.

Proof. Analogously as in the proof of Corollary **??**, one can show that for all $\theta^k \geq \theta^z$ it holds that $f^{\theta,*}$ is a solution of $M_b^{\theta^k}$ for all $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$. It follows from Lemma **??** that

$$f^{\theta^k,*} \in \mathscr{F}^{min}\left(\theta^k, b, \beta^*\right)$$

for all $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ Thus, we can conclude from Observation **??** and Corollary **??** that in order to show the worst-case belief condition, it is left to show that for all $\theta^k \geq \theta^z$ it holds that

$$\bar{b}_{\theta^{k-1}} \leq^{\theta^k} b \text{ for all } b \in [0, \bar{b}_{\theta^m}] \backslash [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]. \tag{16}$$

Lemma **??** shows that if $b \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ was to induce a lower expected utility than $\bar{b}_{\theta^{k-1}}$ and $l < k$, then b needs to be either $\bar{b}_{\theta^{l-1}}$ or $\bar{b}_{\theta^l}$. Since every lower bound of a bidding interval is the upper bound of some interval, it is w.l.o.g. to assume that b is equal to $\bar{b}_{\theta^l}$ for an appropriate l. Lemma **??** shows that a lower expected utility can be achieved by inducing a bid only in the bidding interval of a lower type. Lemma **??** and **??** combined state that if $b \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ was to induce a lower expected utility than $\bar{b}_{\theta^{k-1}}$, then $b = \bar{b}_{\theta^l}$ for $0 \leq l \leq k-2$. In order to show that all bids $\bar{b}_{\theta^l}$ with $0 \leq l \leq k-2$ do not induce a lower expected utility than $\bar{b}_{\theta^{k-1}}$, we need to show the following two Lemmas.

Lemma 13. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-1}$ the unique solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ denoted by $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is obtained as follows. Choose the minimum $p \in \{1, \dots, m\}$ such that the probability vector $(f_{\theta^1}, \dots, f_{\theta^m})$ is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, i.e. it satisfies*

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \dots + f_{\theta^l})^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) \geq \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1, \dots, m\}.$$

$$f_{\theta^j} \geq 0 \quad \textit{for all } j \in \{1, \dots, m\}$$

where $(f_{\theta^1}, \dots, f_{\theta^{p+2}})$ is the unique solution of the system of linear equations with $p+2$ equations (or $p+1$ equations if $p \geq l$) given by

$$\sum_{j=1}^{p+2} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p+2} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^p})^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1, \ldots, p\}$$

and $f_{\theta^j} = 0$ *for all* $j > p+2$ *(or all* $j > p+1$ *if* $p \geq l$*).* [17] *Let* p^* *be the minimum* p*. Then* $\left(f_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, f_{\theta^{p+2}}^{\theta^k,\overline{b}_{\theta^l}}\right)$ *is the unique solution of the to system of equations*

$$\sum_{j=1}^{p^*+2} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+2} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1, \ldots, p^*\}$$

and it holds that $f_{\theta^j}^{\theta^k,\overline{b}_{\theta^l}} = 0$ *for all* $m \geq j > p^*+2$ *if* $p^* < l$*. If* $p^* \geq l$*, then there are* $p+1$ *equations and it holds that* $f_{\theta^j}^{\theta^k,\overline{b}_{\theta^l}} = 0$ *for all* $p^*+1 < j \leq m$*.*

The construction in this Lemma works as follows. We start with the equalities

$$\sum_{j=1}^{2} f_{\theta^j} = 1$$

$$\sum_{j=1}^{2} f_{\theta^j}\theta^j = \mu.$$

This is a linear system of two equations which gives a unique f_{θ^1} and f_{θ^2}. If with the probability vector $(f_{\theta^1}, f_{\theta^2}, 0 \ldots, 0)$ we obtain an element of the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$, then we stop. Otherwise we add the equation which is identical to the binding incentive constraint with corresponding bid 0, i.e.

$$(f_{\theta^1} + f_{\theta^2} + f_{\theta^3})^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = f_{\theta^1}^{n-1}\theta^k$$

and obtain a unique solution for $(f_{\theta^1}, f_{\theta^2}, f_{\theta^3})$ and check whether the vector $(f_{\theta^1}, f_{\theta^2}, f_{\theta^3}, 0 \ldots, 0)$ is an element of the feasible set and so forth until we find an element of the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$. Let $\overline{b}_{\theta^{p^*}}$ be the bid corresponding to this final binding incentive constraint. The solution of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$ is given by $\left(f_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, f_{\theta^{p^*+2}}^{\theta^k,\overline{b}_{\theta^l}}, 0, \ldots, 0\right)$ where $\left(f_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, f_{\theta^{p^*+2}}^{\theta^k,\overline{b}_{\theta^l}}\right)$ is the unique solution of the system of

[17] If $p \geq l$, then the number of equations equals to $p+1$ since the equation which is the binding incentive constraint corresponding to bid $\overline{b}_{\theta^l}$ is redundant.

equations given by the probability constraints and all added incentive constraints if $p^* < l-1$. In case $p^* \geq l-1$, the solution has p^*+1 variables which are greater than zero.

Lemma 14. *For every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-2}$ the minimum p for minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ is greater or equal then $l+1$.*

Finally, Lemma **??** states that the construction in Lemma **??** leads to a minimum p which is greater than l. This implies that the binding incentive constraint with corresponding bid $l+1$, i.e.

$$(f_{\theta^1}+\cdots+f_{\theta^l})^{n-1}\left(\theta^k-\bar{b}_{\theta^l}\right)=(f_{\theta^1}+\cdots+f_{\theta^{l+1}})^{n-1}\left(\theta^k-\bar{b}_{\theta^{l+1}}\right)$$

is added to the system of equations. Therefore, in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ the constraint corresponding to $\bar{b}_{\theta^{l+1}}$ is binding and it follows from Lemma **??** that $\bar{b}_{\theta^l} \geq^{\theta^k} \bar{b}_{\theta^{l+1}}$. With the same reasoning in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{l+1}}}$ the constraint corresponding to $\bar{b}_{\theta^{l+2}}$ is binding and it follows from Lemma **??** that $\bar{b}_{\theta^{l+1}} \geq^{\theta^k} \bar{b}_{\theta^{l+2}}$ so on. Therefore, we can construct the following transitive chain

$$\bar{b}_{\theta^l} \geq^{\theta^k} \bar{b}_{\theta^{l+1}} \geq^{\theta^k} \cdots \geq^{\theta^k} \bar{b}_{\theta^{k-1}}.$$

We conclude that there does not exist a bid which induces a lower expected utility than $\bar{b}_{\theta^{k-1}}$ which shows the statement in (**??**). This completes the proof of part *(ii)* of Proposition **??** which states that the worst-case belief condition is fulfilled for all types. □

We relegate the formal proofs of Lemma **??** and **??** to Appendices **??** and **??** and provide an intuition for Lemma **??**-**??**.

Intuition for Lemma ??-??. The intuition for Lemma **??** works similarly as for Lemma **??**: Assume there exists a solution of minimization problem $M^{\theta^k}_{b}$ such that $\bar{b}_{\theta^{l-1}} < b < \bar{b}_{\theta^l}$ and $\theta^z \leq \theta^l \leq \theta^{k-1}$, denoted by $\left(f^{\theta^k,b}_{\theta^1},\ldots,f^{\theta^k,b}_{\theta^m}\right)$. Consider two bids $b', b'' \in [\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ with $b'' < b < b'$. The utility for the θ^k-type of bidding b must be at least as high as the utilities of bidding b'' or b'. The higher $f^{\theta^k,b}_{\theta^{l-1}}$, the lower is the optimal bid for type θ^k (if we allow only for bids in the interval $[\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$). Therefore, the incentive constraint corresponding to bid b' sets an upper bound on the valuation of $f^{\theta^k,b}_{\theta^{l-1}}$ while the incentive constraint corresponding to bid b'' sets a lower bound. We will show that the conditions resulting from these two bounds contradict each other. Intuitively, a bidder bidding in the interval $[\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ faces the bid distribution G_{θ^l} of the θ^l-type which is constructed in order to make her indifferent between any bid in the interval $[\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$. Thus, only for the θ^l-type the upper and the lower bound imposed by the incentive constraints corresponding to bids b'' and b' are compatible.

In order to explain to intuition for Lemma **??** and Lemma **??**, we illustrate how to construct a solution of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$. Given some belief $(f_{\theta^1},\ldots,f_{\theta^m})$, the expected utility of bidder i with valuation θ^k and bid $\bar{b}_{\theta^l}$ is given by

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right).$$

Choosing a probability function which minimizes the expected utility is equivalent to choosing a distribution which minimizes the sum $f_{\theta^1} + \cdots + f_{\theta^l}$. If we would look for a probability function which minimizes the sum $f_{\theta^1} + \cdots + f_{\theta^l}$ considering only the first probability constraint, we would set $f_{\theta^1} + \cdots + f_{\theta^l}$ to zero and put all the probability weight on types above θ^l. If we add the constraint that the probability function must have mean μ, this is not longer possible because the mean would be too high. Therefore, one would select types on which to put a strictly positive probability weight in a way such that the mean of the probabilities of types equal or lower than θ^l is minimized. Then one would put as much as possible probability weight on types above θ^l without violating the constraint that the mean has to be μ. In other words, independently of the valuation of μ one would put strictly positive probability weight only on types 0 and θ^{l+1} because this choice minimizes the mean of the probabilities of types equal or lower than θ^l. Then we would choose f_{θ^1} and $f_{\theta^{l+1}}$ such that the mean is μ. If we add the incentive constraints, one would shift only so much probability weight on types above 0 as it is necessary to fulfill the incentive constraints. In particular, one would put probability weight on some type θ^j only if the probability weight on lower types cannot be increased without violating a constraint.

The statement in Lemma **??** reflects exactly this reasoning. Consider the system of equations given by the probability constraints and the equations which are identical to the binding incentive constraints with corresponding bids $\bar{b}_{\theta^1}, \ldots, \bar{b}_{\theta^{p-1}}$, i.e.

$$\sum_{j=1}^{p+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p-1\}.$$

Assume that for the solution $f_{\theta^1}, \ldots, f_{\theta^{p+1}}$ of this system of equations (or $f_{\theta^1}, \ldots, f_{\theta^p}$ if $p-1 \geq l$) it does not hold that $(f_{\theta^1}, \ldots, f_{\theta^{p+1}}, 0, \ldots 0)$ is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$. If we now add the equation with corresponding bid $\bar{b}_{\theta_p}$, i.e.

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) = (f_{\theta^1} + \cdots + f_p)^{n-1} \left(\theta^k - \bar{b}_{\theta^p}\right),$$

then in the solution of the extended system of equations it holds that $f_{p+2} > 0$ (or $f_{\theta^{p+1}} > 0$ if $p \geq l$). We have to check whether the vector $(f_{\theta^1}, \ldots, f_{\theta^{p+2}}, 0, \ldots 0)$ is an element of the feasible set of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$. Since the new vector has positive probability weight on $f_{\theta^{p+2}}$, it must hold that there is less probability weight on types below f_{p+2} than in the old vector $(f_{\theta^1}, \ldots, f_{\theta^{p+2}}, 0, \ldots 0)$ (and analogously for the case $p \geq l$). Therefore, the construction in Lemma **??** ensures that probability weight on a higher type is shifted only if a constraint in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ is not fulfilled and shifting weight on lower types is not possible because all constraints corresponding to lower types already hold with equality.

This reasoning also explains the intuition of Lemma **??**. It states that for every $k \in \{1, \ldots, m\}$ and $l > k$ the feasible set of minimization problem $M^{\theta^k}_{b}$ for $b \in (\bar{b}_{\theta^{l-1}} \bar{b}_{\theta^l}]$ is empty. The belief of type θ^k is constructed such that there is just enough probability weight on types below θ^k in order to induce a mixed strategy in the bidding interval of the θ^k-type. As argued above, the choice of types on which there is strictly positive probability weight minimizes the mean of the probabilities of types below θ^k. If one would try to induce a bid $\bar{b}_{\theta^l}$ above $\bar{b}_{\theta^k}$, the probability weight on the θ^l-type has to be increased. In order to preserve the mean, this would imply a decrease of the probability weight on lower types. This is not possible without violating a constraint since the belief of type θ^k had already the lowest possible mean of the probabilities of types below θ^k.

In order to understand Lemma **??**, consider minimization problem $M^{\theta^l}_{\bar{b}_{\theta^{l-1}}}$ for $1 \leq l \leq m$. As shown in the proof of part *(i)* of Proposition **??**, the solution of this minimization problem is the worst-case belief of type θ^l denoted by $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^m}\right)$. Since in this proof we have also shown that in the worst-case belief of the θ^l-type there is no probability weight on types above θ^l, one can also write $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}, 0, \ldots 0\right)$. In Appendix **??** we prove Lemma **??** which states that the solution of minimization problem $M^{\theta^l}_{\bar{b}_{\theta^{l-1}}}$ is the solution of the system of l equations given by the two probability constraints and the $l-2$ incentive constraints given by the bids $\bar{b}_{\theta^j}$ for $1 \leq j \leq l-2$. Hence, for this minimization problem the minimum p equals to $l-2$. Now consider minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{l-1}}}$ for $1 \leq l \leq k-2$. Let $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^m}\right)$ be the solution of the system of l equations given by the two probability constraints and the binding incentive constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq l-2$. How does the vector $\left(\tilde{f}^{\theta^k}_{0}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$ differ from $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}\right)$? In minimization problem $M^{\theta^l}_{\bar{b}_{\theta^{l-1}}}$ a constraint with corresponding bid $\bar{b}_{\theta^j}$ given by

$$\left(f_{\theta^1} + \ldots + f_{\theta^{l-1}}\right)\left(\theta^l - \bar{b}_{\theta^{l-1}}\right) \geq \left(f_{\theta^1} + \ldots + f_{\theta^j}\right)\left(\theta^l - \bar{b}_{\theta^j}\right)$$

is equivalent to

$$\left(f_{j+1} + \ldots + f_{\theta^{l-1}}\right)\left(\theta^l - \bar{b}_{\theta^{l-1}}\right) - \left(f_{\theta^1} + \ldots + f_{\theta^j}\right)\left(\bar{b}_{\theta^l} - \bar{b}_{\theta^j}\right) \geq 0.$$

In minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{l-1}}}$ the same incentive constraint is equivalent to

$$(f_{j+1} + \ldots + f_{\theta^{l-1}}) \left(\theta^k - \bar{b}_{\theta^l}\right) - (f_{\theta^1} + \ldots + f_{\theta^j}) \left(\bar{b}_{\theta^l} - \bar{b}_{\theta^j}\right) \geq 0.$$

This shows that in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{l-1}}}$ it is possible to put more probability weight on lower types. Thus, the value of the objective function is lower under $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$ than under $\left(f^{\theta^l,*}_{\theta^1}, \ldots, f^{\theta^l,*}_{\theta^l}\right)$. But then the constraint corresponding to bid $\bar{b}_{\theta^l}$ is not fulfilled under $\left(\tilde{f}^{\theta^k}_{\theta^1}, \ldots, \tilde{f}^{\theta^k}_{\theta^l}\right)$. Hence, one has to add an additional constraint. Since the constraint corresponding to bid $\bar{b}_{\theta^{l-1}}$ is redundant, one has to add the constraint corresponding to bid $\bar{b}_{\theta^l}$. Thus, the minimum p in minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{l-1}}}$ is greater than $l-1$.

After proving the best-reply and the worst-case belief condition, we conclude that the strategies and belief specified in **??** indeed constitute a worst-case belief equilibrium.

6. Conclusion

We provide a novel approach to endogenize beliefs in games of incomplete information and apply this approach to bidding in first-price auctions. Our model is based on the assumption that bidders in a first-price auction who, apart from the mean of the distribution, have little information about the valuations of their competitors prepare for the worst case. Preparing for the worst-case means that the bidders assume that given the bidding strategies of their competitors they will face ex-ante the worst distribution of valuations. Given that all bidders prepare in the same way a worst-case belief equilibrium arises whenever all bidders best-reply to the bidding strategies of their competitors and their corresponding worst-case beliefs. In particular, this implies there is no other belief such that the best reply to this belief will yield a higher pay-off than in equilibrium. The resulting beliefs are type-dependent and due to the assumption of a constant mean of the distribution the beliefs cannot be strictly ordered by first-order stochastic dominance. In particular this implies that bidders with higher valuations not necessarily face higher competition. Nevertheless, we show that a worst-case equilibrium exists that allocates the object to the bidder with the highest valuation with probability one.

Our concept of the worst-case belief equilibrium can be easily extended to any game of incomplete information and provides a very intuitive way to endogenize beliefs. This is in particular helpful when modeling situations in which players only interact infrequently and thus may not be able to form reasonable objective beliefs.

7. Notation

- $U_i(\theta_i, f_{-i}, b_i, \beta_{-i})$ denotes the expected utility of a bidder i with valuation θ_i, belief about the other bidders' valuations f_{-i}, bid b_i given the other bidders' strategies β_{-i}.
- For bidder i with valuation θ_i and for each belief f_{-i} about the other bidders' valuations and bidding strategies β_{-i}, the set of *best replies* of bidder i is given by

$$B_i^r(\theta_i, f_{-i}, \beta_{-i}) = \arg\max_{b_i} \; U_i(\theta_i, f_{-i}, b_i, \beta_{-i}).$$

- For $\theta^k \in \Theta$, $\beta^*(\theta^k)$ denotes the worst-case strategy of a bidder with valuation θ^k and

$$\left(f_{\theta^1}^{\theta^k,*}, \dots, f_{\theta^m}^{\theta^k,*}\right)$$

  denotes the worst-case belief of a bidder with valuation θ^k. The bid distribution of a bidder with valuation θ^k which is prescribed by the worst-case strategy is denoted by G_{θ^k}, i.e. $\beta^*(\theta^k) = G_{\theta^k}$. The support of this bid distribution is given by $[\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$.
- The worst-case belief of a bidder with valuation θ^k is the solution of minimization problem $M_{<\bar{b}_{\theta^{k-1}}}$ which is defined by

$$\min_{(f_{\theta^1},\dots,f_{\theta^m})} \left(\sum_{j=1}^{k-1} f_{\theta^j}\right)^{n-1} (\theta^k - \bar{b}_{\theta^{k-1}})$$

$$s.t.\; f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \dots + f_{\theta^{l-1}})^{n-1} (\theta^k - \bar{b}_{\theta^{k-1}}) \geq \left(\sum_{j=1}^{h-1} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} \left(\theta^k - s\right)$$

$$\text{for all } h \in \{1, \dots, k-1\} \text{ and all } s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}].$$

  That is, minimization problem $M_{<\bar{b}_{\theta^{k-1}}}$ ensures that bidding $\bar{b}_{\theta^{k-1}}$ induces at least the expected utility than bidding any *lower* bid given the other bidders' worst-case strategy β^*.
- For a bidder with valuation θ_i, a bid b_i and a strategy β_{-i} of the other bidders, the set of probability functions $\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i})$ given by

$$\mathscr{F}_{n-1}^{min}(\theta_i, b_i, \beta_{-i}) = \arg\min_{f_{-i} \in \mathscr{F}_{\mu}^{n-1}} \{U(\theta_i, f_{-i}, b_A, \beta_{-i}) \mid b_A \in B^r(\theta_i, f_{-i}, \beta_{-i})\}$$

is called the *set of minimizing probability functions* of bid b_i for a bidder with valuation θ_i given the other bidders' strategies β_{-i}.

- For a bidder with valuation θ minimization problem M_b^θ of a bid $b \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ is the minimization problem corresponding to its minimizing probability function, i.e. all solutions of minimization problem M_b^θ are minimizing probability function of b for bidder with valuation θ given the other bidders' worst-case strategy β^*. Formally, minimization problem M_b^θ is given by

$$\min_{(f_{\theta^1},\dots,f_{\theta^m})} \left(\sum_{j=1}^{l-1} f_{\theta^j} + f_{\theta^l} G_l(b)\right)^{n-1} (\theta - b)$$
$$s.t.\ f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$
$$\sum_{j=1}^{m} f_{\theta^j} = 1$$
$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$
$$(f_{\theta^1} + \cdots + f_{\theta^{l-1}})^{n-1} (\theta - b) \geq \left(\sum_{j=1}^{h} f_{\theta^j} + f_{\theta^h} G_{\theta^h}(s)\right)^{n-1} (\theta - s)$$
$$\text{for all } h \in \{1,\dots,m\} \text{ and all } s \in [\overline{b}_{\theta^{h-1}}, \overline{b}_{\theta^h}],$$

- We denote the solution of minimization problem M_b^θ by $f^{\theta,b} = \left(f_{\theta^1}^{\theta,b},\dots,f_{\theta^m}^{\theta,b}\right)$.
- Apart from the constraints

$$f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$
$$\sum_{j=1}^{m} f_{\theta^j} = 1$$
$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu,$$

every constraint in minimization problem M_b^θ compares the utility of bidding b to the utility of bidding some other bid b', which is formalized by

$$U(\theta, f, b, \beta^*) \geq U\left(\theta, f, b', \beta^*\right).$$

We call such a constraint an *incentive constraint corresponding to bid b'.*

- For a type θ and bids b, b' we use the notation $b \leq^\theta b'$ if for the θ-type bid b' does not induce a strictly lower expected utility than bid b together with their minimizing probability functions given the other bidders' worst-case strategy β^*. Formally, let $f^{min}(\theta, b, \beta^*) \in \mathscr{F}^{min}(\theta, b, \beta^*)$ and $f^{min}(\theta, b', \beta^*) \in \mathscr{F}^{min}(\theta, b', \beta^*)$. Then it holds that

$$U\left(\theta, f^{min}\left(\theta, b, \beta^*\right), b, \beta^*\right) \leq U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \; \Rightarrow b \leq^{\theta} b',$$

$$U\left(\theta, f^{min}\left(\theta, b, \beta^*\right), b, \beta^*\right) < U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \; \Rightarrow b <^{\theta} b'$$

and

$$U\left(\theta, f^{min}\left(\theta, b, \beta^*\right), b, \beta^*\right) = U\left(\theta, f^{min}\left(\theta, b', \beta^*\right), b', \beta^*\right) \; \Rightarrow b =^{\theta} b'.$$

We also use the notation $b <^{\theta} b'$ if b' does not have a minimizing probability function given θ because it is never a best reply for a bidder with valuation θ, but b does have a minimizing probability function. We use the notation $b =^{\theta} b'$ if neither b, nor b' have a minimizing probability function.

Appendices

Appendix A. Proof of Lemma ??

Assume there exists a belief $f^{1,b} = \left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right) \in \mathscr{F}_\mu$ and a bid $b \in \left(\theta, \bar{b}_1\right)$ such that b is a best reply to $f^{1,b}$ for a bidder with valuation 1 but $f^{1,b}$ differs from the worst-case belief of the 1-type given by $f^{1,*}$. Let $\delta_0, \delta_\theta, \delta_1$ be real numbers such that

$$\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right) = \left(f_0^{1,*} + \delta_0, f_\theta^{1,*} + \delta_\theta, f_1^{1,*} + \delta_1\right).$$

Since $f^{1,b}$ has to fulfill the two probability constraints, it must hold that

$$\delta_0 + \delta_\theta + \delta_1 = 0 \tag{17}$$

$$\delta_\theta \theta + \delta_1 = 0. \tag{18}$$

Due to (??), (??) and $f^{1,b} \neq f^{1,*}$, it must hold that either $\delta_0 < 0$ or $\delta_0 > 0$. First, we consider the case $\delta_0 < 0$. Subtracting (??) from (??) gives

$$\delta_0 + \delta_\theta \left(1 - \theta\right) = 0$$

$$\Leftrightarrow \delta_\theta = -\frac{\delta_0}{1 - \theta} \tag{19}$$

from which follows that $\delta_\theta > 0$. Due to (??), it follows that $\delta_1 < 0$. By definition of the bid distribution of the 1-type in (??), it holds that

$$\left(f_0^{1,*} + f_\theta^{1,*} + f_1^{1,*} G_1\left(b\right)\right)\left(1 - b\right) = \left(f_0^{1,*} + f_\theta^{1,*}\right)\left(1 - \theta\right).$$

Since b is a best reply given $f^{1,b}$, it holds that

$$\left(f_0^{1,b}+f_\theta^{1,b}+f_1^{1,b}G_1(b)\right)(1-b)\geq\left(f_0^{1,b}+f_\theta^{1,b}\right)(1-\theta)$$

$$\Leftrightarrow\left(f_0^{1,*}+\delta_0+f_\theta^{1,*}+\delta_\theta+(f_1^{1,*}+\delta_1)G_1(b)\right)(1-b)\geq\left(f_0^{1,*}+\delta_0+f_\theta^{1,*}+\delta_\theta\right)(1-\theta)$$

from which follows that

$$\left(\delta_0-\frac{\delta_0}{1-\theta}+\delta_1\right)(1-b)\geq\left(\delta_0-\frac{\delta_0}{1-\theta}\right)(1-\theta)$$

$$\Leftrightarrow -\theta\delta_0(b-\theta)-\delta_1(1-\theta)(1-b)\leq 0.$$

Since $b>\theta$ and δ_0 and δ_1 are smaller than zero, this leads to a contradiction.

Now we consider the case $\delta_0>0$. It follows from (**??**) that $\delta_\theta<0$. Due to (**??**), it follows that $\delta_1>0$. By definition of the bid distribution of the 1-type in (**??**), it holds that

$$\left(f_0^{1,*}+f_\theta^{1,*}+f_1^{1,*}G_1(b)\right)(1-b)=\left(f_0^{1,*}+f_\theta^{1,*}+f_1^{1,*}\right)\left(1-\bar{b}_1\right).$$

Since b is a best reply given $f^{1,b}$, it holds that

$$\left(f_0^{1,b}+f_\theta^{1,b}+f_1^{1,b}G_1(b)\right)(1-b)\geq\left(f_0^{1,b}+f_\theta^{1,b}+f_1^{1,b}\right)\left(1-\bar{b}_1\right)$$

$$\begin{aligned}\Leftrightarrow&\left(f_0^{1,*}+\delta_0+f_\theta^{1,*}+\delta_\theta+\left(f_1^{1,*}+\delta_1\right)G_1(b)\right)(1-b)\\&\geq\left(f_0^{1,*}+\delta_0+f_\theta^{1,*}+\delta_\theta+f_1^{1,*}+\delta_1\right)\left(1-\bar{b}_1\right)\end{aligned}$$

from which follows that

$$\left(\delta_0-\frac{\delta_0}{1-\theta}+\delta_1G_1(b)\right)(1-b)\geq\left(\delta_0-\frac{\delta_0}{1-\theta}+\delta_1\right)\left(1-\bar{b}_1\right)$$

$$\Leftrightarrow -\theta\delta_0\left(\bar{b}_1-b\right)-\delta_1(1-\theta)\left(1-\bar{b}_1-G_1(b)(1-b)\right)\geq 0.$$

Since $1-\bar{b}_1>1-b>G_1(b)(1-b)$, $\bar{b}_1>b$ and δ_0 and δ_1 are greater than zero, this leads to a contradiction. We conclude that if a bid $b\in\left(\theta,\bar{b}_1\right)$ is a best reply to a belief for a bidder with valuation 1, then this belief coincides with the worst-case belief of the 1-type.

Appendix B. Proof of Lemma ??

Proof. Assume that the feasible set of minimization problem M_b^θ with $b\in(\bar{b}_\theta,\bar{b}_1]$ is not empty. Since $b>\theta$ is never a best reply, we can assume that $b\leq\theta$. Let $\left(f_0^{\theta,b},f_\theta^{\theta,b},f_1^{\theta,b}\right)$ denote a solution of this minimization problem. Let $\delta_0,\delta_\theta,\delta_1$ be real numbers such that

$$\left(f_0^{\theta,b},f_\theta^{\theta,b},f_1^{\theta,b}\right)=\left(f_0^{\theta,*}+\delta_0,f_\theta^{\theta,*}+\delta_\theta,f_1^{\theta,*}+\delta_1\right).$$

It holds that $f_1^{\theta,b} > 0$ because otherwise bidding above $\overline{b}_\theta$ is not a best reply. Since $f_1^{\theta,*} = 0$, it follows that $\delta_1 > 0$. Due to the probability constraints, it must hold

$$\delta_0 + \delta_\theta + \delta_1 = 0 \tag{20}$$

$$\delta_\theta \theta + \delta_1 = 0. \tag{21}$$

Hence, it must hold that $\delta_\theta < 0$ because otherwise (**??**) cannot be fulfilled. Subtracting (**??**) from (**??**) gives

$$\delta_0 + \delta_\theta - \delta_\theta \theta = 0.$$

Since $\delta_\theta - \delta_\theta\theta < 0$, it follows that $\delta_0 > 0$. Because the expected utility from bidding b must be as least as high as the expected utility from bidding any other bid, it holds that

$$\left(f_0^{\theta,b} + f_\theta^{\theta,b} + f_1^{\theta,b} G_1(b)\right)(\theta - b) \geq \left(f_0^{\theta,b}\right)\theta.$$

$$\Leftrightarrow \left(f_0^{\theta,*} + \delta_0 + f_\theta^{\theta,*} + \delta_\theta + \left(f_1^{\theta,*} + \delta_1\right) G_1(b)\right)(\theta - b) \geq \left(f_0^{\theta,*} + \delta_0\right)\theta.$$

Since $f_1^{\theta,*} = 0$ it holds that

$$\left(f_0^{\theta,*} + f_\theta^{\theta,*} + f_1^{\theta,*} G_1(b)\right)(\theta - b) = \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right)(\theta - b) < \left(f_0^{\theta,*} + f_\theta^{\theta,*}\right)\left(\theta - \overline{b}_\theta\right) = f_0^{\theta,*}\theta$$

where the last equality follows from the definition of $\overline{b}_\theta$ in (**??**). It follows that

$$\left(\delta_0 + \delta_\theta + \delta_1 G_1(b)\right)(\theta - b) > \delta_0 \theta > 0.$$

Because $b \leq \theta$, it must hold that

$$\delta_0 + \delta_\theta + \delta_1 G_1(b) > 0.$$

Since $\delta_1 > 0$ and $G_1(b) \leq 1$, it holds that

$$0 < \delta_0 + \delta_\theta + \delta_1 G_1(b) \leq \delta_0 + \delta_\theta + \delta_1 G_1(b) + \delta_1 (1 - G_1(b)) = \delta_0 + \delta_\theta + \delta_1 = 0.$$

We conclude that the assumption that the feasible set of minimization problem M_b^θ with $b \in (\overline{b}_\theta, \overline{b}_1]$ is not empty, leads to a contradiction. □

Appendix C. Proof of Lemma ??

Proof. The formal proof works by contradiction. Assume that there exists a $b \in \left(0, \overline{b}_\theta\right)$ such that there exists an element of the feasible set of minimization problem M_b^1 denoted by $\left(f_0^{1,b}, f_\theta^{1,b}, f_1^{1,b}\right)$. Then for every $s', s'' \in [0, \overline{b}_\theta]$ it holds

$$\left(f_0^{1,b} + f_\theta^{1,b} G_\theta(b)\right)(1 - b) \geq \left(f_0^{1,b} + f_\theta^{1,b} G_\theta(s')\right)(1 - s') \tag{22}$$

$$\left(f_0^{b1,b} + f_\theta^{1,b} G_\theta(b)\right)(1-b) \geq \left(f_0^{1,b} + f_\theta^{1,b} G_\theta\left(s''\right)\right)\left(1-s''\right). \tag{23}$$

Let $s'' < b < s'$ be such that

$$s' - b = b - s'' = \alpha \tag{24}$$

for some appropriate $\alpha > 0$. Rearranging of (**??**) gives

$$\Leftrightarrow f_0^{1,b} \geq \frac{f_\theta^{1,b} G_\theta(s')(1-s') - f_\theta^{1,b} G_\theta(b)(1-b)}{s'-b}. \tag{25}$$

Rearranging of (**??**) gives

$$\Leftrightarrow f_0^{1,b} \leq \frac{f_\theta^{1,b} G_\theta(b)(1-b) - f_\theta^{1,b} G_\theta(s'')(1-s'')}{b-s''}. \tag{26}$$

If we show that

$$\frac{f_\theta^{1,b} G_\theta(b)(1-b) - f_\theta^{1,b} G_\theta(s'')(1-s'')}{b-s''} < \frac{f_\theta^{1,b} G_\theta(s')(1-s') - f_\theta^{1,b} G_\theta(b)(1-b)}{s'-b}, \tag{27}$$

we find a contradiction between inequalities (**??**) and (**??**). Due to (**??**), inequality (**??**) is equivalent to

$$f_\theta^{1,b} G_\theta(b)(1-b) - f_\theta^{1,b} G_\theta\left(s''\right)\left(1-s''\right) < f_\theta^{1,b} G_\theta\left(s'\right)\left(1-s'\right) - f_\theta^{1,b} G_\theta(b)(1-b).$$

If b is a best reply to $f^{1,b}$, it must hold that $f_\theta^{1,b} > 0$ because otherwise bidding zero or above $\bar{b}_\theta$ would be strictly better. Therefore, the inequality is equivalent to

$$-2G_\theta(b)(1-b) + G_\theta\left(s''\right)\left(1-s''\right) + G_\theta\left(s'\right)\left(1-s'\right) > 0.$$

Due to (**??**), this is equivalent to

$$-2G_\theta(b)\left(1-s'+\alpha\right) + G_\theta\left(s''\right)\left(1-s'+2\alpha\right) + G_\theta\left(s'\right)\left(1-s'\right) > 0$$

$$\Leftrightarrow \left(1-s'\right)\left[-2G_\theta(b) + G_\theta\left(s''\right) + G_\theta\left(s'\right)\right] + \alpha\left[-2G_\theta(b) + 2G_\theta\left(s''\right)\right] > 0. \tag{28}$$

As defined in (**??**), for all $s \in [0, \bar{b}_\theta]$ the distribution G_θ is given by the equation

$$f_0^{\theta,*}\theta = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta(s)\right)(\theta - s)$$

$$\Leftrightarrow G_\theta(s) = \frac{f_0^{\theta,*} s}{f_\theta^{\theta,*}(\theta - s)}.$$

If $b \leq \frac{\mu}{2}$, we choose $s'' = 0$ and it holds that $s' = 2b \leq \mu = \bar{b}_\theta$.

Then inequality (**??**) is equivalent to

$$(1 - s') \left(\frac{-2 f_0^{\theta,*} b}{f_\theta^{\theta,*} (\theta - b)} + \frac{f_0^{\theta,*} s'}{f_\theta^{\theta,*} (\theta - s')} \right) - \frac{2\alpha f_0^{\theta,*} b}{f_\theta^{\theta,*} (\theta - b)} > 0. \tag{29}$$

It holds that

$$\theta - b - (\theta - 2b) > 0$$

$$\Leftrightarrow -2b (\theta - 2b) + 2b (\theta - b) > 0.$$

Due to (**??**), this is equivalent to

$$-2b \left(\theta - s'\right) + s' (\theta - b) > 0$$

$$\Leftrightarrow \frac{-2 f_0^{\theta,*} b}{f_\theta^{\theta,*} (\theta - b)} + \frac{f_0^{\theta,*} s'}{f_\theta^{\theta,*} (\theta - s')} > 0.$$

It follows that in order to show (**??**), it is sufficient to show that

$$(\theta - s') \left(\frac{-2 f_0^{\theta,*} b}{f_\theta^{\theta,*} (\theta - b)} + \frac{f_0^{\theta,*} s'}{f_\theta^{\theta,*} (\theta - s')} \right) - \frac{2\alpha f_0^{\theta,*} b}{f_\theta^{\theta,*} (\theta - b)} \geq 0.$$

Multiplying the inequality with $(\theta - b)$ and plugging in $\alpha = (s' - b)$ reduces the problem to

$$-2b \left(\theta - s'\right) + s' (\theta - b) - 2b \left(s' - b\right) \geq 0. \tag{30}$$

It holds that

$$s' \geq 2b$$

$$\Leftrightarrow -2b (\theta - b) + s' (\theta - b) \geq 0$$

$$\Leftrightarrow -2b\theta + s'\theta - s'b + 2b^2 \geq 0$$

$$-2b\theta + 2bs' + s'\theta - s'b - 2bs' + 2b^2 \geq 0$$

$$-2b \left(\theta - s'\right) + s' (\theta - b) - 2b \left(s' - b\right) \geq 0.$$

Thus, we have shown inequality (**??**) from which follows that inequality (**??**) holds. This shows that inequalities (**??**) and (**??**) lead to a contradiction in case $b \leq \frac{\mu}{2}$.

By definition of $\overline{b}_\theta = \mu$, it holds for all $s \in [0, \mu]$ that

$$\left(f_0^{\theta,*} + f_\theta^{\theta,*}\right) (\theta - \mu) = \left(f_0^{\theta,*} + f_\theta^{\theta,*} G_\theta (s)\right) (\theta - s)$$

$$\Leftrightarrow G_\theta (s) = \frac{-f_0^{\theta,*} (\mu - s) + f_\theta^{\theta,*} (\theta - \mu)}{f_\theta^{\theta,*} (\theta - s)}.$$

If $b > \frac{\mu}{2}$, then we set $s' = \mu$ and it holds that $s'' = 2b - \mu > 0$. Then inequality (**??**) is equivalent to

$$(1-\mu)\left(\frac{2f_0^{\theta,*}(\mu-b)-2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-b)}+\frac{-f_0^{\theta,*}(\mu-(2b-\mu))+f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-(2b-\mu))}+1\right)$$
$$+\alpha\left(\frac{2f_0^{\theta,*}(\mu-b)-2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-b)}+\frac{-2f_0^{\theta,*}(\mu-(2b-\mu))+2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-(2b-\mu))}\right)>0$$

$$\Leftrightarrow (1-\mu)\left(\frac{2f_0^{\theta,*}(\mu-b)-2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-b)}+\frac{-2f_0^{\theta,*}(\mu-b)+f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-2b+\mu)}+1\right)$$
$$+\alpha\left(\frac{2f_0^{\theta,*}(\mu-b)-2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-b)}+\frac{-4f_0^{\theta,*}(\mu-b)+2f_\theta^{\theta,*}(\theta-\mu)}{f_\theta^{\theta,*}(\theta-2b+\mu)}\right)>0$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)(\theta-2b+\mu)(1-\mu+\alpha)-2f_0^{\theta,*}(\mu-b)(\theta-b)(1-\mu+2\alpha)$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[\theta-\theta\mu+\theta\alpha-2b+2b\mu-2b\alpha+\mu-\mu+\mu\alpha-(\theta-\theta\mu+2\theta\alpha-b+b\mu+2b\alpha)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[-\alpha\theta-b+\mu b+\mu-\mu+\mu\alpha]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

By definition of α in (**??**), this is equivalent to

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[-(\mu-b)\theta-b+\mu b+\mu-\mu+\mu(\mu-b)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

$$\Leftrightarrow 2f_0^{\theta,*}(\mu-b)[\mu-b-\theta(\mu-b)]$$
$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)\geq 0.$$

Since $2f_0^{\theta,*}(\mu-b)[\mu-b-\theta(\mu-b)]>0$, it is sufficient to show that

(31) $$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)$$
$$+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

It holds that

$$\mu>b$$
$$\Leftrightarrow(\mu-b)(1-\theta)(-\theta+\mu+\theta-2b+\mu)>0$$
$$\Leftrightarrow-(\theta-\mu)(\mu-b)(1-\theta)+(\theta-2b+\mu)(\mu-b)(1-\theta)>0$$
$$\Leftrightarrow(\theta-\mu)[-\theta b+b-\mu+\mu\theta]+(\theta-2b+\mu)[\theta b+\mu-b-\mu\theta]>0$$

$$\Leftrightarrow(\theta-\mu)[-\theta+\theta b+2b-2b^2-\mu+\mu b+\theta-2b\theta+\theta\mu-b+2b^2-b\mu]$$
$$+(\theta-2b+\mu)[-\theta+\theta b+\mu-\mu b+\theta-b-\mu\theta+\mu b]>0$$

$$\Leftrightarrow(\theta-\mu)[-(\theta-2b+\mu)(1-b)+(\theta-b)(1-2b+\mu)]$$
$$+(\theta-2b+\mu)[-(\theta-\mu)(1-b)+(1-\mu)(\theta-b)]>0$$

Since $f_\theta^{\theta,*}>0$, this is equivalent to

$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-b)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-2b+\mu)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

Since $-b=-\mu+\alpha$ and $-2b+\mu=-\mu+2\alpha$, this is equivalent to

$$-2f_\theta^{\theta,*}(\theta-\mu)(\theta-2b+\mu)(1-\mu+\alpha)+f_\theta^{\theta,*}(\theta-\mu)(\theta-b)(1-\mu+2\alpha)$$
$$+f_\theta^{\theta,*}(1-\mu)(\theta-2b+\mu)(\theta-b)>0.$$

Thus, we have shown inequality (**??**) from which follows that inequality (**??**) holds. This shows that inequalities (**??**) and (**??**) lead to a contradiction in case $b>\frac{\mu}{2}$. We conclude that in any possible case the assumption that the feasible set of minimization problem M_b^1 with $b\in(0,\bar{b}_\theta)$ is not empty, leads to a contradiction. □

Appendix D. Proof of Lemma ??

Proof. We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-1}$ and every b with $\overline{b}_{\theta^{l-1}} < b < \overline{b}_{\theta^l}$ the feasible set of minimization problem $M_b^{\theta^k}$ is empty. Assume that there exist l and b with $\overline{b}_{\theta^{l-1}} < b < \overline{b}_{\theta^l}$ such that there exists an element of the feasible set of minimization problem $M_b^{\theta^k}$ denoted by $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$. Then for every $s', s'' \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ it holds

$$\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s')\right)^{n-1} \left(\theta^k - s'\right) \tag{32}$$

$$\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'')\right)^{n-1} \left(\theta^k - s''\right). \tag{33}$$

Let $s'' < b < s'$ be such that

$$\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b} = \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'} = \alpha \tag{34}$$

$$\Leftrightarrow \sqrt[n-1]{\theta^k - s'} + \alpha = \sqrt[n-1]{\theta^k - b}\,, \quad \sqrt[n-1]{\theta^k - s'} + 2\alpha = \sqrt[n-1]{\theta^k - s''}$$

for some appropriate $\alpha > 0$. Rearranging of (**??**) gives

$$\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b} \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s')\right) \sqrt[n-1]{\theta^k - s'}$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b}\right) \left(\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}\right) \geq f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}$$

$$\Leftrightarrow f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} \geq \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}}. \tag{35}$$

Rearranging of (**??**) gives

$$\left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b} \geq \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'')\right) \sqrt[n-1]{\theta^k - s''}$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b}\right)\left(\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}\right)$$
$$\leq f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}\left(s''\right) \sqrt[n-1]{\theta^k - s''}$$

$$\Leftrightarrow f_{\theta^1}^{\theta^k,b} + \ldots + f_{\theta^{l-1}}^{\theta^k,b} \leq \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}}. \tag{36}$$

If we show that

$$\frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b}} < \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'}}, \tag{37}$$

we find a contradiction between inequalities (??) and (??). Inequality (??) is equivalent to

$$\frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s'') \sqrt[n-1]{\theta^k - s''}}{\alpha} < \frac{f_{\theta^l}^{\theta^k,b} G_{\theta^l}(s') \sqrt[n-1]{\theta^k - s'} - f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b}}{\alpha}$$

$$\Leftrightarrow -2 f_{\theta^l}^{\theta^k,b} G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}\left(s''\right) \sqrt[n-1]{\theta^k - s''} + f_{\theta^l}^{\theta^k,b} G_{\theta^l}\left(s'\right) \sqrt[n-1]{\theta^k - s'} > 0.$$

If bid b is a best reply, it must hold that $f_{\theta^l}^{\theta^k,b} > 0$ and therefore the inequality is equivalent to

$$-2 G_{\theta^l}(b) \sqrt[n-1]{\theta^k - b} + G_{\theta^l}\left(s''\right) \sqrt[n-1]{\theta^k - s''} + G_{\theta^l}\left(s'\right) \sqrt[n-1]{\theta^k - s'} > 0.$$

$$\Leftrightarrow -2 G_{\theta^l}(b)\left(\sqrt[n-1]{\theta^k - s'} + \alpha\right) + G_{\theta^l}\left(s''\right)\left(\sqrt[n-1]{\theta^k - s'} + 2\alpha\right) + G_{\theta^l}\left(s'\right) \sqrt[n-1]{\theta^k - s'} > 0$$

$$\Leftrightarrow \sqrt[n-1]{\theta^k - s'}\left(-2 G_{\theta^l}(b) + G_{\theta^l}\left(s''\right) + G_{\theta^l}\left(s'\right)\right) + \alpha\left(-2 G_{\theta^l}(b) + 2 G_{\theta^l}\left(s''\right)\right) > 0$$

$$\Leftrightarrow \sqrt[n-1]{\theta^k - s'}\left(-2 G_{\theta^l}(b) + G_{\theta^l}\left(s''\right) + G_{\theta^l}\left(s'\right)\right) > \alpha\left(2 G_{\theta^l}(b) - 2 G_{\theta^l}\left(s''\right)\right). \tag{38}$$

For all $s \in [\overline{b}_{\theta^{l-1}}, \overline{b}_{\theta^l}]$ the distribution G_{θ^l} is defined by the equation

$$\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}\left(\theta^l - \overline{b}_{\theta^{l-1}}\right) = \left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*} + f_{\theta^l}^{\theta^l,*} G_{\theta^l}(s)\right)^{n-1}\left(\theta^l - s\right)$$

$$\Leftrightarrow G_{\theta^l}(s) = \frac{\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\sqrt[n-1]{\theta^l - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^l - s}\right)}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s}}$$

where $\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_m^{\theta^l,*}\right)$ denotes the worst-case belief of the θ^l-type.

Let b^* be defined by

$$\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*} = \sqrt[n-1]{\theta^k - b^*} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}}.$$

If $b \leq b^*$, we choose $s'' = \bar{b}_{\theta^{l-1}}$ and it holds that

$$\sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - s'} = \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b}$$
$$\leq \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*} = \sqrt[n-1]{\theta^k - b^*} - \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}}$$
$$\Rightarrow \sqrt[n-1]{\theta^k - s'} \geq \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - b^*} + \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}} \geq \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}}$$
$$\Rightarrow s' \leq \bar{b}_{\theta^l}.$$

Moreover, we define

$$\alpha_1^{s'} := \sqrt[n-1]{\theta^l - \bar{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^l - b} \quad \text{and } \alpha_2^{s'} := \sqrt[n-1]{\theta^l - b} - \sqrt[n-1]{\theta^l - s'}.$$

Then inequality (**??**) is equivalent to

$$\sqrt[n-1]{\theta^k - s'} \frac{\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}}{f_{\theta^l}^{\theta^l,*}} \left(\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}} + \frac{\alpha_1^{s'} + \alpha_2^{s'}}{\sqrt[n-1]{\theta^l - s'}} \right)$$
$$> \frac{\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)^{n-1}}{f_{\theta^l}^{\theta^l,*}} \alpha \frac{2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}}.$$

Since $1 > f_{\theta^l}^{\theta^l,*} > 0$ and $\sum_{j=1}^{l} f_{\theta^j}^{\theta^l,*} = 1$, it holds that $\sum_{j=1}^{l-1} f_{\theta^j}^{\theta^l,*} > 0$ and therefore the inequality is equivalent to

$$\sqrt[n-1]{\theta^k - s'} \left(\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}} + \frac{\alpha_1^{s'} + \alpha_2^{s'}}{\sqrt[n-1]{\theta^l - s'}} \right) > \alpha \frac{2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}}.$$

Since $\sqrt[n-1]{\ }$ is concave, it holds that $\alpha_1^{s'} \leq \alpha_2^{s'}$ from which follows that

$$\frac{-2\alpha_1^{s'}}{\sqrt[n-1]{\theta^l - b}} + \frac{\alpha_1^{s'} + \alpha_2^{s'}}{\sqrt[n-1]{\theta^l - s'}} > 0.$$

Hence, if $\theta^k > \theta^l$, it is sufficient to show that

$$-2\alpha_1^{s'} \sqrt[n-1]{\theta^l - s'} + \left(\alpha_1^{s'} + \alpha_2^{s'}\right) \sqrt[n-1]{\theta^l - b} \geq 2\alpha\alpha_1^{s'}.$$

Since $\alpha_1^{s'} \leq \alpha_2^{s'}$, it is sufficient to show that

$$-2\alpha_1^{s'} \sqrt[n-1]{\theta^l - s'} + 2\alpha_1^{s'} \sqrt[n-1]{\theta^l - b} \geq 2\alpha\alpha_1^{s'}$$
$$\Leftrightarrow -\sqrt[n-1]{\theta^l - s'} + \sqrt[n-1]{\theta^l - b} \geq \alpha$$

which is true since $\sqrt[n-1]{\ }$ is concave. Thus, we have shown inequality (**??**) and conclude that in the case $b \leq b^*$ the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ is not empty, leads to a contradiction.

If $b > b^*$, then we choose $s' = \overline{b}_{\theta^l}$ and it holds that

$$\sqrt[n-1]{\theta^k - s''} - \sqrt[n-1]{\theta^k - b} = \sqrt[n-1]{\theta^k - b} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}}$$

$$\leq \sqrt[n-1]{\theta^k - b^*} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} = \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*}$$

$$\Rightarrow \sqrt[n-1]{\theta^k - s''} \leq \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}} - \sqrt[n-1]{\theta^k - b^*} + \sqrt[n-1]{\theta^k - b} \leq \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{l-1}}}$$

$$\Rightarrow s'' \geq \overline{b}_{\theta^{l-1}}.$$

Moreover, we define

$$\alpha_1^{s''} := \sqrt[n-1]{\theta^l - b} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \quad \alpha_2^{s''} := \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^l - b}. \tag{39}$$

By definition of $\overline{b}_{\theta^l}$, it holds that

$$\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^l}^{\theta^l,*}\right)^{n-1} \left(\theta^l - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*} + f_{\theta^l}^{\theta^l,*} G_{\theta^l}(s)\right)^{n-1} \left(\theta^l - s\right)$$

$$\Leftrightarrow G_{\theta^l}(s) = \frac{-\left(f_{\theta^1}^{\theta^l,*} + \ldots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\sqrt[n-1]{\theta^l - s} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}\right) + f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s}}.$$

Then inequality (**??**) is equivalent to

$$\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} \left(\frac{2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''} - 2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - b}} + 1 \right.$$
$$\left. + \frac{-\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right) + f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}} \right)$$
$$> \alpha \left(\frac{-2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - b}} \right.$$
$$\left. + \frac{2\left(f_{\theta^1}^{\theta^l,*} + \cdots + f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''} + \alpha_2^{s''}\right) - 2 f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}}}{f_{\theta^l}^{\theta^l,*} \sqrt[n-1]{\theta^l - s''}} \right)$$

$$\Leftrightarrow 2\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\alpha_1^{s''}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}$$
$$-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}$$
$$-\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}>0.$$

Since $\sqrt[n-1]{\ }$ is concave, it holds that $\alpha_1^{s''}\geq\alpha_2^{s''}$ and therefore, it is sufficient to show that

$$\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\alpha_2^{s''}$$
$$-\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\alpha\sqrt[n-1]{\theta^l-b}$$
$$-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}>0.$$

If $\theta^k>\theta^l$, it holds that $\alpha\sqrt[n-1]{\theta^l-b}<\alpha\sqrt[n-1]{\theta^k-b}=\alpha\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)$ and therefore it is sufficient to show that

$$\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\alpha_2^{s''}$$
$$-\left(f_{\theta^1}^{\theta^l,*}+\cdots+f_{\theta^{l-1}}^{\theta^l,*}\right)\left(\alpha_1^{s''}+\alpha_2^{s''}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\alpha$$
$$-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}\geq 0.$$

Since $\sqrt[n-1]{\ }$ is concave, it holds that $\alpha_2^{s''}\geq\alpha$ and it is sufficient to show that

$$-2f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+\alpha\right)\sqrt[n-1]{\theta^l-s''}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-s''}\sqrt[n-1]{\theta^l-b}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}$$
$$+f_{\theta^l}^{\theta^l,*}\sqrt[n-1]{\theta^l-\overline{b}_{\theta^l}}\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^l}}+2\alpha\right)\sqrt[n-1]{\theta^l-b}\geq 0.$$

By definition of α in (**??**), it holds that $\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \alpha = \sqrt[n-1]{\theta^k - b}$ and hence, we have to show that

$$\Leftrightarrow -2 \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - b} \sqrt[n-1]{\theta^l - s''} + \sqrt[n-1]{\theta^l - s''} \sqrt[n-1]{\theta^l - b} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} \\ + \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - s''} \sqrt[n-1]{\theta^l - b} \geq 0$$

$$\Leftrightarrow \sqrt[n-1]{\theta^l - s''} \left(\sqrt[n-1]{\theta^l - b} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - b} \right) \\ - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \left(\sqrt[n-1]{\theta^k - b} \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^k - s''} \sqrt[n-1]{\theta^l - b} \right) \geq 0.$$

Since

$$\sqrt[n-1]{\theta^l - b} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - b} > 0, \\ \sqrt[n-1]{\theta^k - b} \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^k - s''} \sqrt[n-1]{\theta^l - b} > 0$$

and

$$\sqrt[n-1]{\theta^l - s''} \geq \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}},$$

it is sufficient to show that

$$\Leftrightarrow \left(\sqrt[n-1]{\theta^l - b} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - b} \right) \\ - \left(\sqrt[n-1]{\theta^k - b} \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^k - s''} \sqrt[n-1]{\theta^l - b} \right) \geq 0.$$

It holds that

$$\left(\sqrt[n-1]{\theta^l - b} \sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} - \sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} \sqrt[n-1]{\theta^k - b} \right) \\ - \left(\sqrt[n-1]{\theta^k - b} \sqrt[n-1]{\theta^l - s''} - \sqrt[n-1]{\theta^k - s''} \sqrt[n-1]{\theta^l - b} \right) \\ = \sqrt[n-1]{\theta^l - b} \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^l}} + \sqrt[n-1]{\theta^k - s''} \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} + \sqrt[n-1]{\theta^l - s''} \right).$$

By definition of α in (**??**), this is equal to

$$\sqrt[n-1]{\theta^l - b} \left(\sqrt[n-1]{\theta^k - b} - \alpha + \sqrt[n-1]{\theta^k - b} + \alpha \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - \overline{b}_{\theta^l}} + \sqrt[n-1]{\theta^l - s''} \right).$$

By definition of α_1 and α_2 in (**??**), this is equal to

$$= \sqrt[n-1]{\theta^l - b} \left(2 \sqrt[n-1]{\theta^k - b} \right) - \sqrt[n-1]{\theta^k - b} \left(\sqrt[n-1]{\theta^l - b} - \alpha_1 + \sqrt[n-1]{\theta^l - b} + \alpha_2 \right) \\ \geq \sqrt[n-1]{\theta^l - b} \left(2 \sqrt[n-1]{\theta^k - b} \right) - \sqrt[n-1]{\theta^k - b} \left(2 \sqrt[n-1]{\theta^l - b} \right) = 0.$$

Thus, we have shown inequality (**??**) and conclude that also in the case $b > b^*$ the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ for $\theta^k > \theta^l$ and $\bar{b}_{\theta^{l-1}} < b < \bar{b}_{\theta^l}$ is not empty, leads to a contradiction.

□

For the proofs to follow we will need the following Lemma.

Appendix E. Lemma ??

Lemma 15. *For every valuation $\theta^k \geq \theta^z$ the worst-case belief $f^{\theta^k,*}$ is the solution of the following system of equations:*

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^h}\right) \quad \textit{for all } h \in \{1, \ldots, k-2\}.$$

Proof. As defined in **??**, the worst-case belief of type θ^k is the solution of the minimization problem with objective function

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$

which consists of the two probability constraints and all incentive constraints with corresponding bid lower than $\bar{b}_{\theta^{k-1}}$. We denote this minimization problem by $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. The constraints of this minimization problem can be summarized as

$$\sum_{j=1}^{m} f_{\theta^j} = 1$$

$$\sum_{j=1}^{m} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^{k-1}})^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(\left(\sum_{j=1}^{h-1} f_{\theta^j}\right) + f_{\theta^h} G_{\theta^h}(s)\right)\left(\theta^k - s\right)$$

for all $h \in \{1, \ldots, k-1\}$ and all $s \in [\bar{b}_{\theta^{h-1}}, \bar{b}_{\theta^h}]$.

According to part *(i)* of Proposition **??**, the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ is also a solution of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^{k-1}}}$. Hence, an incentive constraint corresponding to a bid b with $\bar{b}_{\theta^{j-1}} < b < \bar{b}_{\theta^j}$ with $1 < j < k-1$ cannot be binding because otherwise it would follow from Lemma **??** that the solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ would be an element of the feasible set of minimization problem $M_b^{\theta^k}$. But this would be a contradiction to Lemma **??**.

Hence, the set of possible binding incentive constraints is a subset of the incentive constraints with corresponding bids $\bar{b}_{\theta^j}$ with $j \in \{1, \ldots, k-2\}$. It is left to show that every incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ with $j \in \{1, \ldots, k-2\}$ is binding. As shown in the proof of part *(i)* of Proposition **??**, in the worst-case belief of type θ^k there is no probability weight on types above θ^k and therefore we can write the worst-case belief of type θ^k as $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*}, 0, \ldots, 0\right)$. Assume that under $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*}, 0, \ldots, 0\right)$ an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ for some $j \in \{1, \ldots, k-2\}$ is not binding in minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Then we will construct a feasible solution of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ which leads to a lower value of the objective function than $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*}, 0, \ldots, 0\right)$ given by

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right).$$

Given the intuition provided above for Lemma **??**, this should not come as a surprise. We stated that in the solution of a minimization problem $M^{\theta^k}_{\bar{b}_{\theta^k-1}}$ the probability weight on lower types should be as high as possible without violating a constraint because this allows to put probability weight on high types without violating the second probability constraint. More precisely, if a constraint with corresponding bid $\bar{b}_{\theta^j}$ is not binding, this implies that one can reduce the probability weight on $f_{\theta^{j-1}}$ and increase probability weight on f_{θ^j} without violating an incentive constraint. This reduces the mean and therefore one can increase the probability weight on f_{θ^k}. This results in a lower value of the objective function. The rest of the proof formalizes this idea.

Case 1: $j = 1$. Let

$$l = \min_{\hat{l} > 1} \left\{ \hat{l} \mid f_{\theta^{\hat{l}}}^{\theta^k,*} > 0 \right\},$$

i.e. let θ^l be the smallest valuation such that $f_{\theta^l}^{\theta^k,*}$ is strictly greater than zero. We claim that the vector

$$f_\epsilon^{\theta^k} = \left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1}, f_{\theta^2}^{\theta^k,*}, \ldots, f_{\theta^l}^{\theta^k,*} - \epsilon_{\theta^l}, f_{\theta^{l+1}}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*} - \epsilon_{\theta^k}, f_{\theta^{k+1}}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$$

fulfills all constraints of $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$ but leads to a lower value of the objective function. Here $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ are strictly positive real numbers such that it holds

$$\epsilon_{\theta^1} - \epsilon_{\theta^l} + \epsilon_{\theta^k} = 0 \tag{40}$$

$$-\epsilon_{\theta^l}\theta^l + \epsilon_{\theta^k}\theta^k = 0. \tag{41}$$

First, we will show that such $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ exist, then we will show that the proposed vector is an element of the feasible set of minimization problem $M^{\theta^k}_{<\bar{b}_{\theta^{k-1}}}$. Since it follows from (**??**) that

$\epsilon_{\theta^1} - \epsilon_{\theta^l} < 0$, it follows directly that the constructed vector leads to a lower value of the objective function than $f^{\theta^k,*}$.

Equations (**??**) and (**??**) are solved by any choice of $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ that fulfill

$$\epsilon_{\theta^1} - \epsilon_{\theta^l} + \epsilon_{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^1} - \epsilon_{\theta^l} + \frac{\epsilon_{\theta^l}\theta^l}{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}$$

$$\Leftrightarrow \epsilon_{\theta^1} = \frac{\epsilon_{\theta^l}\left(\theta^k - \theta^l\right)}{\theta^k}, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^l}\theta^l}{\theta^k}.$$

This shows that $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ can be chosen as strictly positive real numbers. Moreover, it holds that the smaller the valuation of ϵ_{θ^1}, the smaller the valuation of ϵ_{θ^l}. Therefore, $\epsilon_{\theta^1}, \epsilon_{\theta^l}, \epsilon_{\theta^k}$ can be chosen such that the incentive constraint corresponding to bid zero given by

$$\left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} - \epsilon_{\theta^l} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1}\right)^{n-1}\theta^k$$

is fulfilled. The probability constraints are fulfilled by construction. Since all incentive constraints with corresponding bid b with $\bar{b}_{\theta^{h-1}} < b < \bar{b}_{\theta^h}$ and $h < k-1$ are not binding under $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*}, 0, \ldots, 0\right)$, they will be fulfilled under $f_\epsilon^{\theta^k}$ if ϵ_{θ^1} and ϵ_{θ^l} are sufficiently small. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $1 < j < l$, all incentive constraints with corresponding bid $\bar{b}_{\theta^h}$ with $1 < h < l$ are fulfilled if ϵ_{θ^l} is sufficiently small. Every other incentive constraint given by

$$\left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1} + f_{\theta^2}^{\theta^k,*} - \epsilon_{\theta^2} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \epsilon_{\theta^1} + f_{\theta^2}^{\theta^k} - \epsilon_{\theta^2} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^h}\right)$$

for $l \leq h \leq k-2$ is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} - + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $\epsilon_{\theta^1} - \epsilon_{\theta^2} < 0$. Hence, we have found a vector of probabilities which fulfills all probability and all incentive constraints while inducing a lower value of the objective function. Since $f_{\theta^l}^{\theta^k,*} > 0$, the constraint that all probabilities have to be non-negative is also fulfilled if ϵ_{θ^l} is sufficiently small. We conclude that the assumption that the incentive constraint with corresponding bid 0 is not binding in the worst-case belief of type θ^k, leads to a contradiction.

Case 2: $j > 1$. If the non-binding incentive constraint is an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ with $j > 1$, we proceed similarly, by constructing a vector which is an element of the feasible set of minimization problem $M_{<\bar{b}_{\theta^{k-1}}}^{\theta^k}$ but leads to a lower value of the objective function. Let

$$l' = \min_{\hat{l} > j}\left\{\hat{l} \mid f_{\theta^{\hat{l}}}^{\theta^k,*} > 0\right\},$$

then it must hold that $l' \leq k-1$ because otherwise bidding $\overline{b}_{\theta^{k-1}}$ would never be a best reply. We claim that the desired vector is given by

$$f_\epsilon^{\theta^k} = \left(f_{\theta^1}^{\theta^k,*} - \epsilon_{\theta^1}, f_{\theta^2}^{\theta^k,*}, \ldots, f_{\theta^j}^{\theta^k,*} + \epsilon_{\theta^j}, f_{\theta^{j+1}}^{\theta^k,*}, \ldots, f_{\theta^{l'}}^{\theta^k,*} - \epsilon_{\theta^{l'}}, f_{\theta^{l'}+1}^{\theta^k,*}, \ldots, f_{\theta^k}^{\theta^k,*} + \epsilon_{\theta^k}, f_{\theta^{k+1}}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$$

where $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ are strictly positive real numbers such that it holds

(42) $$-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \epsilon_{\theta^k} = 0$$

(43) $$\epsilon_{\theta^j}\theta^j - \epsilon_{\theta^{l'}}\theta^{l'} + \epsilon_{\theta^k}\theta^k = 0.$$

Since it follows from (**??**) that $-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^l} < 0$, it follows directly that the constructed vector leads to a lower value of the objective function than $f^{\theta^k,*}$. In addition, we choose $\epsilon_{\theta^{l'}}$ sufficiently small such that the non-binding incentive constraint

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right) > \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^j}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^j}\right)$$

is still fulfilled, i.e. it holds that

(44) $$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^j}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^j}\right).$$

Again, we will first show that such $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^j}, \epsilon_{\theta^k}$ exist, then we will show that the proposed vector is an element of the feasible set of minimization problem $M^{\theta^k}_{<\overline{b}_{\theta^{k-1}}}$.

Equations (**??**) and (**??**) are solved by any choice of $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^j}, \epsilon_{\theta^k}$ which fulfills

$$
\begin{aligned}
& -\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \epsilon_{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & -\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} + \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & -\epsilon_{\theta^1}\theta^k + \epsilon_{\theta^j}\theta^k - \epsilon_{\theta^{l'}}\theta^k - \epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'} = 0, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & \epsilon_{\theta^j}\left(\theta^k - \theta^j\right) = \epsilon_{\theta^{l'}}\left(\theta^k - \theta^j\right) + \epsilon_{\theta^1}\theta^k, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^j}\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\left(\epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}\right)\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k} \\
\Leftrightarrow\ & \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{-\epsilon_{\theta^{l'}}\left(\theta^k - \theta^j\right) + \epsilon_{\theta^1}\theta^k\theta^j + \epsilon_{\theta^{l'}}\theta^{l'}}{\theta^k\left(\theta^k - \theta^j\right)} \\
\Leftrightarrow\ & \epsilon_{\theta^j} = \epsilon_{\theta^{l'}} + \frac{\epsilon_{\theta^1}\theta^k}{\theta^k - \theta^j}, \quad \epsilon_{\theta^k} = \frac{\epsilon_{\theta^{l'}}\left(\theta^{l'} - \left(\theta^k - \theta^j\right)\theta^j\right) + \epsilon_{\theta^1}\theta^k\theta^j}{\theta^k\left(\theta^k - \theta^j\right)}.
\end{aligned}
$$

This shows that $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ can be chosen as strictly positive real numbers. Moreover, it holds that the smaller the valuation of ϵ_{θ^1} and $\epsilon_{\theta^{l'}}$, the smaller the valuation of ϵ_{θ^j} and ϵ_{θ^k}. Therefore, $\epsilon_{\theta^1}, \epsilon_{\theta^j}$ and $\epsilon_{\theta^{l'}}$ can be both chosen sufficiently small such that the incentive constraint (**??**) is fulfilled.

The probability constraints are fulfilled by construction. Since all incentive constraints with corresponding bid b with $\bar{b}_{\theta^{h-1}} < b < \bar{b}_{\theta^h}$ for $h < k-1$ are not binding under $f^{\theta^k,*}$, they will be fulfilled under $f_\epsilon^{\theta^k}$ if $\epsilon_{\theta^1}, \epsilon_{\theta^j}, \epsilon_{\theta^{l'}}, \epsilon_{\theta^k}$ are sufficiently small. Any incentive constraint with corresponding bid $\bar{b}_{\theta^h}$ for $1 \le h \le j-1$ given by

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*} - \epsilon_{\theta^1}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $0 > -\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} > -\epsilon_{\theta^1}$. The incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ is fulfilled by construction. Since $f_{\theta^h}^{\theta^k,*} = 0$ for $j < h < l'$, it holds that all incentive constraints with corresponding bid $\bar{b}_{\theta^h}$ with $j < h < l'$ are fulfilled if $\epsilon_{\theta^{l'}}$ is sufficiently small.

An incentive constraint with corresponding bid $\bar{b}_{\theta^h}$ for $l' \le h \le k-2$ given by

$$\left(f_{\theta^1}^{\theta^k} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*} - \epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

is fulfilled since it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,*} + f_{\theta^2}^{\theta^k,*} + \ldots + f_{\theta^h}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right)$$

and $-\epsilon_{\theta^1} + \epsilon_{\theta^j} - \epsilon_{\theta^{l'}} < 0$. Hence, we have found a vector of probabilities, $f_\epsilon^{\theta^k}$, which fulfills all probability and all incentive constraints while inducing a lower value of the objective function. We can assume that $f_{\theta^1}^{\theta^k} > 0$ because otherwise, the incentive constraint corresponding to bid $0 = \bar{b}_{\theta^1}$ is not binding and the first case applies. Since $f_{\theta^{l'}}^{\theta^k} > 0$, the constraint that probabilities are non-negative is also fulfilled if $\epsilon_{\theta^{l'}}$ is sufficiently small. We conclude that the assumption that an incentive constraint with corresponding bid $\bar{b}_{\theta^j}$ for $1 \le j \le k-2$ is not binding, leads to a contradiction.

□

Appendix F. Proof of Lemma ??

Proof. We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z \le \theta^{k+1} \le \theta^l$ and every b with $\bar{b}_{\theta^{l-1}} < b \le \bar{b}_{\theta^l}$ the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ is empty.

Assume that the feasible set of minimization problem $M_b^{\theta^k}$ for some $b \in (\bar{b}_{\theta^{l-1}}, \bar{b}_{\theta^l}]$ with $l > k$ is not empty. Let $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ denote a solution. We can write

$$\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right)$$

for some appropriate be real numbers $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$. We will prove the claim in four steps:

(1) For every j with $k+1 \le j \le m$ it holds $\delta_{\theta^j} \ge 0$.

(2) There exist strictly positive real numbers α and β such that

$$\sum_{j=1}^{k} \delta_{\theta^j} = -\alpha \quad \text{and} \quad \sum_{j=k+1}^{m} \delta_{\theta^j} = \beta$$

(3) Let $\left(\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^m}\right) =$

$$\arg\min \left\{ \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j \;\middle|\; \left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right) \text{ is element of feasible set of } M_b^{\theta^k}, \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha \right\},$$

then it holds that $\hat{\delta}_{\theta^j} \leq 0$ for all $1 \leq j \leq k-1$.

(4) We use steps (1)-(3) in order to show that the assumption that $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ is a solution of minimization problem $M_b^{\theta^k}$, leads to a contradiction.

Proof of step (1)

As shown in the proof of part *(i)* of Proposition **??**, it holds that $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$. Since probabilities cannot be negative, it follows that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$.

Proof of step (2)

Since $k < l$ and $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right) > \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^l,*} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right).$$

As $\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it must hold that

$$\left(f_{\theta^1}^{\theta^k} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right).$$

It follows that either $\sum_{j=1}^{k} \delta_{\theta^j} < 0$ or $\sum_{j=k+1}^{l} \delta_{\theta^j} > 0$. Due to the first probability constraint, it holds $\sum_{j=1}^{m} \delta_{\theta^j} = 0$. Assume that $\sum_{j=1}^{k} \delta_{\theta^j} \geq 0$. Then it must hold $\sum_{j=k+1}^{l} \delta_{\theta^j} > 0$. Since due to step (1) it holds that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$, it follows that $\sum_{j=1}^{m} \delta_{\theta^j} > 0$ which leads to a contradiction. Hence, it must hold that $\sum_{j=1}^{k} \delta_{\theta^j} < 0$. Therefore, there exist strictly positive be real numbers α and β such that $\sum_{j=1}^{k} \delta_{\theta^j} = -\alpha$ and $\sum_{j=k+1}^{m} \delta_{\theta^j} = \beta$.

Proof of step (3)

We start the proof of step (3) by showing the following claim: Let

$$\left(\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^m}\right) =$$

$$\arg\min \left\{ \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j \;\middle|\; \left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right) \text{ is element of feasible set of } M_b^{\theta^k}, \sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha \right\}.$$

Then it holds that under $\left(f_{\theta^1}^{\theta^k} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k} + \hat{\delta}_{\theta^m}\right)$ in minimization problem $M_b^{\theta^k}$ all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t \leq k-1$ are binding i.e.

$$(45) \quad \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \hat{\delta}_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^l,*} + \hat{\delta}_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) = \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right)$$

for all $t \leq k-1$.

In order to show this claim, consider all real numbers $\tilde{\delta}_{\theta^1}, \ldots, \tilde{\delta}_{\theta^m}$ such that $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha$ and $\left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right)$ is an element of feasible set of minimization problem $M_b^{\theta^k}$. If we would consider only the constraint $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} = -\alpha$, then one could achieve arbitrarily small values of the term $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j$ by choosing high values of $\delta_{\theta^1}, \ldots, \delta_{k-1}$ which results in a low value of δ_{θ^k}. Adding the constraint that $\left(f_{\theta^1}^{\theta^k,*} + \tilde{\delta}_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \tilde{\delta}_{\theta^m}\right)$ is an element of feasible set of minimization problem $M_b^{\theta^k}$, the value of the term $\sum_{j=1}^{k} \tilde{\delta}_{\theta^j} \theta^j$ is minimized if the values of all $\tilde{\delta}_{\theta^j}$ with $1 \leq j \leq k-1$ are as high as possible and the value of δ_{θ^k} is as low as possible without violating any incentive constraint.

An incentive constraint with corresponding bid $\overline{b}_{\theta^{t-1}} < b' < \overline{b}_{\theta^t}$ with $t < k$ cannot be binding because then due to Lemma **??**, $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ would be an element of the feasible set of minimization problem $M_{b'}^{\theta^k}$ which would be a contradiction to Lemma **??**. It follows that if all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t < k$ are binding, $\tilde{\delta}_{\theta^1}, \ldots, \tilde{\delta}_{\theta^{k-1}}$ cannot be increased without violating an incentive constraint in minimization problem $M_b^{\theta^k}$. A decrease of $\tilde{\delta}_{\theta^t}$ with $t \leq k-1$ would imply a higher $\tilde{\delta}_{\theta^k}$ which would lead to a higher value of the term $\sum_{j=1}^{k} \delta_{\theta^j} \theta^j$. We conclude that the values of all δ_{θ^j} with $1 \leq j \leq k-1$ are as high as possible if all constraints with corresponding bid $\overline{b}_{\theta^t}$ with $t \leq k-1$ are binding i.e.

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \hat{\delta}_{\theta^k} + \cdots + \left(f_{\theta^l}^{\theta^l,*} + \hat{\delta}_{\theta^l,}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) = \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right)$$

for $t \leq k-1$.

We will use this claim in order to show inductively that all $\hat{\delta}_{\theta^1}, \ldots, \hat{\delta}_{\theta^{k-1}}$ are non-positive. We start the induction by showing that $\hat{\delta}_{\theta^1} \leq 0$. Due to the first probability constraint, it holds that

$$\sum_{j=1}^{k} \hat{\delta}_{\theta^j} + \sum_{j=k+1}^{l} \hat{\delta}_{\theta^j} + \sum_{j=l+1}^{m} \hat{\delta}_{\theta^j} = 0$$

and since $l > k$, we can conclude with the same reasoning as in step (1) that

$$\sum_{j=l+1}^{m} \hat{\delta}_{\theta^j} \geq 0.$$

It follows that

$$\sum_{j=1}^{k} \hat{\delta}_{\theta^j} + \sum_{j=k+1}^{l} \hat{\delta}_{\theta^j} \leq 0.$$

Since $\hat{\delta}_{\theta^l} \geq 0$, it holds that

$$\hat{\delta}_{\theta^1} + \ldots + \hat{\delta}_{\theta^l} G_{\theta^l}(b) \leq 0.$$

Moreover, it follows from $f_{\theta^j}^{\theta^k,*} = 0$ for $j > k$, that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*} G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) < \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*}\right)^{n-1} \theta^k$$

where the equality follows from Lemma **??**. It also holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + \left(f_{\theta^l}^{\theta^k,*} + \hat{\delta}_{\theta^l}\right) G_{\theta^l}(b)\right)^{n-1} \left(\theta^k - b\right) \geq \left(f_{\theta^1}^{\theta^k} + \hat{\delta}_{\theta^1}\right)^{n-1} \theta^k$$

from which it follows that

$$\hat{\delta}_{\theta^1} \leq \frac{\left(\hat{\delta}_{\theta^1} + \cdots + \hat{\delta}_{\theta^l} G_{\theta^l}(b)\right) \sqrt[n-1]{\theta^k - b}}{\sqrt[n-1]{\theta^k}} \leq 0.$$

We now turn our attention to the inductive step. Assume it is already shown that $\hat{\delta}_t \leq 0$ for all $1 \leq t < k-2$. It follows from Lemma **??** that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{t+1}}^{\theta^k,*}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{t+1}}\right)$$

and due to the construction of $\hat{\delta}_{\theta^1}, \ldots \hat{\delta}_{\theta^m}$ it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^t}^{\theta^k,*} + \hat{\delta}_{\theta^t}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^t}\right) = \left(f_{\theta^1}^{\theta^k,*} + \hat{\delta}_{\theta^1} + \cdots + f_{\theta^{t+1}}^{\theta^k,*} + \hat{\delta}_{\theta^{t+1}}\right)^{n-1} \left(\theta^k - \overline{b}_{\theta^{t+1}}\right)$$

from which follows that

$$\hat{\delta}_{\theta^{t+1}} = \frac{\left(\hat{\delta}_{\theta^1} + \cdots + \hat{\delta}_{\theta^t}\right) \left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^t}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{t+1}}}\right)}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{t+1}}}} \leq 0.$$

We conclude that for all $1 \leq j \leq k-1$ it holds $\hat{\delta}_{\theta^j} \leq 0$.

Proof of step (4)

Recall that we defined $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ by

$$\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

According to step (3) it holds $\sum_{j=1}^{k} \hat{\delta}_{\theta^j}\theta^j \geq \sum_{j=1}^{k} \hat{\delta}_{\theta^j}\theta^k = -\alpha\theta^k$. Hence, the maximal possible valuation for the term $-\sum_{j=1}^{k} \delta_{\theta^j}\theta^j$ equals to $\alpha\theta^k$. Since due to step (1), $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$, it follows that $\sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j \geq \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^{k+1} = \beta\theta^{k+1}$. Hence, the maximal possible valuation for the term $-\sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j$ equals to $-\beta\theta^{k+1}$. It follows from the probability constraints that

$$-\alpha + \beta = 0$$
$$\sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j = 0.$$

Subtracting the second equation from the first gives

$$-\alpha - \sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \beta - \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j = 0.$$

It holds

$$-\alpha - \sum_{j=1}^{k} \delta_{\theta^j}\theta^j + \beta - \sum_{j=k+1}^{m} \delta_{\theta^j}\theta^j \leq -\alpha + \alpha\theta^k + \beta - \beta\theta^{k+1}.$$

Since $\alpha = \beta$ it holds

$$-\alpha + \alpha\theta^k + \beta - \beta\theta^{k+1} < 0.$$

Hence, the assumption that the feasible set of minimization problem $M_b^{\theta^k}$ is not empty, leads to a contradiction. □

Appendix G. Proof of Lemma ??

Let θ^l and θ^k be a pair of valuations such that $\theta^z \leq \theta^l \leq \theta^{k-2}$ and let p^* be the minimum p in the construction in Lemma ??. Such a minimum p exists since the worst-case belief of the θ^k-type is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ and due to Lemma ??, in $M_{\bar{b}_{\theta^l}}^{\theta^k}$ only the incentive constraints with corresponding bid $\bar{b}_{\theta^j}$ for $1 \leq j \leq k-1$ are binding if plugging in the worst-case belief. That is, the construction in Lemma ?? stops at the latest after adding the binding incentive constraint with corresponding bid $\bar{b}_{\theta^{k-1}}$. Let $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denote the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ as constructed in Lemma ??, i.e. if $p^* \geq l$ for all $j > p^*+1$ (and for all $j > p^*+2$ if $p^* < l$) it holds that $\tilde{f}_0^{\theta^k,\bar{b}_{\theta^l}} = 0$ and the vector

$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\ldots,\tilde{f}_{\theta^{p^*+1}}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is the unique solution of the system of equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j}\theta^j = \mu$$

$$(f_{\theta^1}+\cdots+f_{\theta^l})^{n-1}\left(\theta^k-\bar{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1}\left(\theta^k-\bar{b}_{\theta^h}\right) \quad \text{for all } h\in\{1,\ldots,p^*\}.$$

We will prove this Lemma using the following steps:

(1) Let $f^{\theta^k,\bar{b}_{\theta^l}} = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\ldots,f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ be a solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$. Then it holds that $f_{\theta^j}^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all $j > p^*+1$ if $p^* \geq l$ and for all $j > p^*+2$ if $p^* < l$.

(2) It holds for $f^{\theta^k,\bar{b}_{\theta^l}}$ that all constraints in $M_{\bar{b}_{\theta^l}}^{\theta^k}$ with corresponding bid $\bar{b}_{\theta^j}$ with $j \leq p^*$ have to be binding.

(3) It holds that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\ldots,\tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\ldots,f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

Proof of step (1)

If $p^* > k$, then the equation which is the binding incentive constraint corresponding to bid $\bar{b}_{\theta^l}$ is obviously redundant and therefore, the system of equations in Lemma **??** consist of two probability constraints and p^*-1 binding incentive constraints. This gives a system of p^*+1 equations for p^*+1 variables. We will provide the proof for the case $p^* \geq l$ since the case $p^* < l$ works analogously and we will show in Lemma **??** that it indeed holds that $p^* \geq l$.

Assume that there exists at least one h with $p^*+1 < h \leq m$ such that $f_{\theta^h}^{\bar{b}_{\theta^l}} > 0$. Let $\delta_{\theta^1},\ldots,\delta_{\theta^m}$ be such that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}+\delta_{\theta^1},\ldots,\tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}+\delta_{\theta^m}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}},\ldots,f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

It holds that $\tilde{f}_j^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all j with $j > p^*+1$. Therefore, it holds that $\delta_{\theta^j} \geq 0$ for all $p^*+1 < j \leq m$ and there exists at least one j with $p^*+1 < j \leq m$ such that $\delta_{\theta^j} > 0$.

Before we proceed with the proof, we introduce the concept of *δ-sequences.* We define a *δ-sequence* as a vector $\left(\delta_{\theta^{l^{min}}},\ldots,\delta_{\theta^{l^{max}}},\delta_{\theta^{k^{min}}},\ldots,\delta_{\theta^{k^{max}}}\right)$ with $\delta_{k^{min}} = \delta_{l^{max}+1}$ such that for all j with $l^{min} \leq j \leq l^{max}$ it holds $\delta_{\theta^j} < 0$ and for all $k^{min} \leq j \leq k^{max}$ it holds $\delta_{\theta^j} \geq 0$. If at least one δ_{θ^j} is not equal to zero, it holds

$$(46)\qquad \sum_{j=l^{min}}^{l^{max}} \delta_{\theta^j}\theta^j + \sum_{j=k^{min}}^{k^{max}} \delta_{\theta^j}\theta^j > \sum_{j=l^{min}}^{l^{max}} \delta_{\theta^j}\theta^{l^{max}} + \sum_{j=k^{min}}^{k^{max}} \delta_{\theta^j}\theta^{l^{max}} = \sum_{j=l^{min}}^{k^{max}} \delta_{\theta^j}\theta^{l^{max}}.$$

Every given vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ can be decomposed into δ-sequences. Let m' be the number of δ-sequences and $\left(\delta_{\theta^{j,l^{min}}}, \ldots, \delta_{\theta^{j,l^{max}}}, \delta_{\theta^{j,k^{min}}}, \ldots, \delta_{\theta^{j,k^{max}}}\right)$ be the $j-th$ δ-sequence. Let $\delta'_{\theta^j} := \sum_{s=j,l^{min}}^{j,k^{max}} \delta_{\theta^s}$ and $\theta^{j'} := \theta^{j,l^{max}}$. Then it holds

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j,l^{max}} = \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j'}.$$

The vector $\left(\delta'_{\theta^1}, \ldots, \delta'_{\theta^m}\right)$ can again be decomposed into δ-sequences. Let m'' be the number of δ-sequences in the vector $(\delta'_{\theta^m}, \ldots, \delta'_{\theta^m})$ and $\left(\delta'_{\theta^{j,l^{min}}}, \ldots, \delta'_{\theta^{j,l^{max}}}, \delta'_{\theta^{j,k^{min}}}, \ldots, \delta'_{\theta^{j,k^{max}}}\right)$ be the $j-th$ δ-sequence. Let $\delta''_{\theta^j} := \sum_{s=j,l^{min}}^{j,k^{max}} \delta'_{\theta^s}$ and $\theta^{j''} := \theta^{j,l^{max}}$. As in (**??**), we conclude that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m'} \delta'_{\theta^j}\theta^{j,l^{max}} \geq \sum_{j=1}^{m''} \delta''_{\theta^j}\theta^{j',l^{max}}.$$

If there does not exist a t with $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences ends with a δ-sequence of length 2, i.e. with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$. Since $\sum_{j=1}^{m} \delta_{\theta^j} = 0$, it holds that $\delta_1 = -\delta_2$ and there exists some θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \left(\delta_1^{final} + \delta_2^{final}\right)\theta^{final} = 0.$$

We illustrate the concept of δ-sequences with the following example.

Example 1. *Let*

$$(\delta_{\theta^1}, \ldots, \delta_{\theta^m}) = \left(-\frac{1}{12}, -\frac{1}{6}, \frac{1}{12}, -\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right).$$

The vector has two relevant properties. It holds that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$ and there does not exist a t with $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} = 0$. This vector can be decomposed into two δ-sequences given by $\left(-\frac{1}{12}, -\frac{1}{6}, \frac{1}{12}\right)$ and $\left(-\frac{1}{8}, \frac{1}{4}, \frac{1}{8}\right)$. It holds that

$$-\frac{1}{12}\theta^1 - \frac{1}{6}\theta^2 + \frac{1}{12}\theta^3 > -\frac{1}{12}\theta^2 - \frac{1}{6}\theta^2 + \frac{1}{12}\theta^2 = \sum_{1}^{3} \delta_{\theta^j}\theta^2$$

and

$$-\frac{1}{8}\theta^4 + \frac{1}{4}\theta^5 + \frac{1}{8}\theta^6 > -\frac{1}{8}\theta^4 + \frac{1}{4}\theta^4 + \frac{1}{8}\theta^4 = \sum_{4}^{6} \delta_{\theta^j}\theta^4.$$

We define $\delta'_1 = \sum_1^3 \delta_{\theta^j} = -\frac{1}{4}$ and $\delta'_2 = \sum_4^6 \delta_{\theta^j} = \frac{1}{4}$. It holds

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{3} \delta_{\theta^j}\theta^2 + \sum_{j=4}^{6} \delta_{\theta^j}\theta^4 = \delta'_1\theta^2 + \delta'_2\theta^4.$$

The new vector $(\delta_1', \delta_2') = \left(-\frac{1}{4}, \frac{1}{4}\right)$ *is a* δ*-sequence and it holds*

$$\delta_1'\theta^3 + \delta_2'\theta^4 = -\frac{1}{4}\theta^2 + \frac{1}{4}\theta^4 > -\frac{1}{4}\theta^4 + \frac{1}{4}\theta^4 = 0.$$

Hence, it holds that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \delta_1'\theta^2 + \delta_2'\theta^4 > 0.$$

Now we proceed with the proof of step (1) of Lemma ??. Recall that $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denotes a solution of minimization problem $M_{\bar{b}_{\theta^k}}^{\theta^l}$ and $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ denotes the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ as constructed in Lemma ??. Let the vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ be defined by

$$\left(\tilde{f}_0^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1}, \dots, \tilde{f}_1^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^m}\right) = \left(f_0^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_1^{\theta^k,\bar{b}_{\theta^l}}\right).$$

We can decompose the vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ into δ-sequences. Due to the two probability constraints it must hold that

$$\sum_{j=1}^{m} \delta_{\theta^j} = 0$$

and

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j = 0.$$

Assume that the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} > 0$ and $\delta_2^{final} < 0$. Then there exists some $1 \le t \le m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$.

First, we consider the case that $t > p^*$. It holds that $\tilde{f}_j^{\theta^k,\bar{b}_{\theta^l}} = 0$ for all $j > p^* + 1$. Thus, it holds that $\delta_{\theta^j} \ge 0$ for all $j > p^*+1$ from which follows that $\sum_{j=t+1}^{m} \delta_{\theta^j} \ge 0$. Since $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, it holds that $\sum_{j=1}^{m} \delta_{\theta^j} > 0$ which leads to a contradiction to the first probability constraint.

Second, we consider the case that $t \le p^*$. Since the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ cannot induce a lower value of the objective function than the solution of the minimization problem, it must hold that

$$\sum_{j=1}^{l} \delta_{\theta^j} \le 0. \tag{47}$$

Since the solution of the minimization problem $M_{\bar{b}_l}^{\theta^k}$, $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$, is an element of the feasible set of minimization problem $M_{\bar{b}_l}^{\theta^k}$ and we defined the real numbers δ_{θ^j} for $1 \le j \le m$ by

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1}, \dots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^m}\right) = \left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right).$$

it holds that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1} + \cdots + \tilde{f}_{\theta^l}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right)$$
$$\geq \left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^1} + \cdots + \tilde{f}_{\theta^t}^{\theta^k,\bar{b}_{\theta^l}} + \delta_{\theta^t}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^t}\right).$$

By construction of the vector $\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$, it holds that

$$\left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + \tilde{f}_{\theta^l}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) = \left(\tilde{f}_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}} + \cdots + \tilde{f}_{\theta^t}^{\theta^k,\bar{b}_{\theta^l}}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^t}\right)$$

from which follows that

$$\left(\delta_{\theta^1} + \ldots + \delta_{\theta^l}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}} \geq \left(\delta_{\theta^1} + \ldots + \delta_{\theta^t}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}}$$
$$\Leftrightarrow \left(\delta_{\theta^1} + \ldots + \delta_{\theta^l}\right) \geq \frac{\left(\delta_{\theta^1} + \ldots + \delta_{\theta^t}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}}}{\sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}}} > 0$$

which leads to a contradiction to (**??**). Therefore, the existence of $\delta_1^{final} > 0$ and $\delta_1^{final} < 0$ leads to a contradiction. Hence, there exists some θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j \geq \sum_{j=1}^{m} \delta_{\theta^j}\theta^{final}.$$

Since there exists a δ_{θ^h} for $p^* + 1 < h \leq m$ with $\delta_{\theta^h} > 0$, this inequality is strict and it holds

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m} \delta_{\theta^j}\theta^{final} = 0.$$

Since this is a contradiction to the second probability constraint, it follows that the assumption that there exists some h with $p^* + 1 < h \leq m$ such that $f_{\theta^h}^{\theta^k,\bar{b}_{\theta^l}} > 0$ leads to contradiction.

Proof of step (2)

It follows from Lemma **??** that

$$\left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right).$$

The worst-case belief of type θ^k is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^{k-1}}}$. Thus, for every $j \in \{1, \ldots, m\}$ and every $b \in [\bar{b}_{\theta^{j-1}}, \bar{b}_{\theta^j}]$ it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \bar{b}_{\theta^{k-1}}\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \ldots + f_{\theta^j}^{\theta^k,*}G_{\theta^j}(b)\right)^{n-1}\left(\theta^k - b\right).$$

Hence, the worst-case belief equilibrium of type θ^k is an element of the feasible set of minimization problem $M^{\theta^k}_{\overline{b}_{\theta^l}}$. Assume that the construction in Lemma **??** has reached the step where the constraint with corresponding bid $\overline{b}_{\theta^{k-1}}$ was added, i.e. all constraints with corresponding bid $\overline{b}_{\theta^j}$ for $1 \leq j \leq k-1$ were added and are binding. Consider the solution vector in this step i.e. the solution of the system of linear equations consisting of the two probability constraints and the binding incentive constraints with corresponding bid $\overline{b}_{\theta^j}$ for $1 \leq j \leq k-1$. According to Lemma **??**, this solution vector coincides with the worst-case belief equilibrium of type θ^k. As argued above, this is an element of the feasible set of minimization problem $M^{\theta^k}_{\overline{b}_{\theta^l}}$ and therefore the construction in Lemma **??** would stop. We conclude that it holds $p^* \leq k-1$.

It follows from step (1) that $f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^j} = 0$ for all $j > k$. Assume that there exists an incentive constraint with corresponding bid $\overline{b}_{\theta^h}$ with $1 \leq h \leq p^*$ which is not binding. Let $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ be defined such that

$$\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1}, \dots, f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^m}\right) = \left(f^{\theta^k,*}_{\theta^1} + \delta_{\theta^1}, \dots, f^{\theta^k,*}_{\theta^m} + \delta_{\theta^m}\right).$$

Then there exists j with $1 \leq j \leq m$ such that $\delta_j \neq 0$.

We consider the following two cases:

- Case 1: It holds that
$$\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1} + \dots + f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1} + \dots + f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right).$$
- Case 2: It holds that
$$\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1} + \dots + f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) > \left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1} + \dots + f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right).$$

Since by definition of $\left(f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^1}, \dots, f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^m}\right)$, this vector is an element of the feasible set of minimization problem $M^{\theta^k}_{\overline{b}_{\theta^l}}$, these two cases constitute all possible cases.

Case 1:

As before, we decompose the vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ into δ-sequences. If we can show that there does not exist a t with $1 \leq t \leq m$ such that

$$\sum_{j=1}^{t} \delta_{\theta^j} > 0, \tag{48}$$

the process of decomposing ends with some δ-sequence $\left(\delta^{final}_1, \delta^{final}_2\right)$ with $\delta^{final}_1 < 0$ and $\delta^{final}_2 > 0$. Assume there exists a t with $1 \leq t \leq m$ such that $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. Since $p^* \leq k-1$, it follows from step (1) that $f^{\theta^k,\overline{b}_{\theta^l}}_{\theta^j} = 0$ for all $j > k$. Because $f^{\theta^k,*}_{\theta^j} = 0$ for all $j > k$, it follows that $\delta_{\theta^j} = 0$ for all $j > k$. Due to the first probability constraint, it holds that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$

and therefore it must hold that $\sum_{j=1}^{k} \delta_{\theta^j} = 0$. Hence, t must be smaller than k. It follows from Lemma **??** that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^t}\right).$$

Since $\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right)$ is an element of the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$ and the incentive constraint with corresponding bid $\overline{b}_{\theta^k}$ is binding, it holds that

$$\begin{aligned}\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right) &= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) \\ &\geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \ldots, f_{\theta^t}^{\theta^k,*} + \delta_{\theta^t}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^t}\right).\end{aligned}$$

It follows that $\sum_{j=1}^{t} \delta_{\theta^j} \leq 0$ which is a contradiction to (**??**). Thus, the process of decomposing into δ-sequences ends with some δ-sequence $\left(\delta_{\theta^m}^{final}, \delta_2^{final}\right)$ with $\delta_{\theta^m}^{final} < 0$ and $\delta_2^{final} > 0$. Hence, it holds that

$$\sum_{j=1}^{m} \delta_{\theta^j}\theta^j > \sum_{j=1}^{m} \delta_{\theta^j}\theta^{final} = 0.$$

But then the solution of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$, given by

$$\left(f_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, f_{\theta^m}^{\theta^k,\overline{b}_{\theta^l}}\right) = \left(\tilde{f}_0^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^1}, \ldots, \tilde{f}_1^{\theta^k,\overline{b}_{\theta^l}} + \delta_{\theta^m}\right).$$

violates the second probability constraint.

Case 2:

As in the first case, if follows from the first probability constraint that $\sum_{j=1}^{k} \delta_{\theta^j} = 0$. Since $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$ is an element of the feasible set of minimization problem $M_{\overline{b}_{\theta^l}}^{\theta^k}$, it must hold that the value of the objective function if plugging in $\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right)$ is not greater than if plugging in $\left(\tilde{f}_{\theta^1}^{\theta^k,\overline{b}_{\theta^l}}, \ldots, \tilde{f}_{\theta^m}^{\theta^k,\overline{b}_{\theta^l}}\right)$. Therefore, it must hold $\sum_{j=1}^{l} \delta_{\theta^j} \leq 0$. By assumption, it holds that

$$\begin{aligned}&\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) \\ &\qquad > \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right)\end{aligned}$$

and due to Lemma **??**, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right)$$

from which follows that $\sum_{j=1}^{l} \delta_{\theta^j} > \sum_{j=1}^{k} \delta_{\theta^j} = 0$ which leads to a contradiction.

We conclude that in both cases the assumption that there exists a h with $1 \leq h \leq p^*$ such that the constraint with corresponding bid $\overline{b}_{\theta^h}$ is not binding in the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, leads to a contradiction.

Proof of step (3):

According to the first step, it holds that $f_j^{\theta^k,\bar{b}_{\theta^l}} > 0$ only for $1 \leq j \leq p^* + 1$ are greater than zero. According to step (2), this vector has to fulfill $p^* + 1$ equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) = \left(\sum_{j=1}^{h} f_{\theta^j}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h}\right) \quad \text{for all } h \in \{1, \ldots, p^*\}.$$

If we consider only roots which are real positive numbers, this is equivalent to

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l}) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^l}} = \left(\sum_{j=1}^{h} f_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^h}} \quad \text{for all } h \in \{1, \ldots, p^*\}.$$

We will show that this system of linear equations has a unique solution. In order to do so, we will show that the matrix corresponding to the system of equations has rank $p^* + 1$ by applying the Gauss elimination method and obtaining a row echelon form. The incentive constraints can be also summarized as

$$(f_{\theta^1} + \cdots + f_{\theta^h}) \left(\sqrt[n-1]{\theta^k - \bar{b}_{\theta^h}} - \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{h+1}}} \right) - f_{\theta^{h+1}} \sqrt[n-1]{\theta^k - \bar{b}_{\theta^{h+1}}} = 0$$

for all $h \in \{1, \ldots, p^* - 1\}$. In order to obtain an upper triangular matrix, we will successively eliminate the variables $f_{\theta^{p^*+1}}, f_{\theta^{p^*}}, \ldots, f_{\theta^2}$. We eliminate the variable $f_{\theta^{p^*+1}}$ by multiplying the equation

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

by $-\theta^{p^*+1}$ and adding it to

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu.$$

Multiplying the resulting equation by (-1) gives

$$\sum_{j=1}^{p^*} f_{\theta^j}\left(\theta^{p^*+1}-\theta^j\right)=\theta^{p^*+1}-\mu$$

which eliminates the variable $f_{\theta^{p^*+1}}$. Moreover, the coefficient $\left(\theta^{p^*+1}-\theta^j\right)$ is strictly positive. Now we subsequently use the transformed incentive constraints given by

$$(f_{\theta^1}+\cdots+f_{\theta^h})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^h}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}\right)-f_{\theta^{h+1}}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h+1}}}=0$$

for all $h\in\{1,\ldots,p^*-1\}$ in order to eliminate the variables $f_{\theta^{p^*}}, f_{\theta^{p^*-1}},\ldots f_{\theta^2}$. We show by induction that in every elimination step all coefficients are strictly positive. In particular, this implies that none of the coefficients is equal to zero and hence, we obtain an upper triangular matrix after applying the Gauss elimination method. We start the induction by showing that in the equation which is obtained after eliminating $f_{\theta^{p^*}}$ all coefficients are strictly positive. The variable $f_{\theta^{p^*}}$ is eliminated by multiplying the incentive constraint given by

$$(f_{\theta^1}+\cdots+f_{\theta^{p^*-1}})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*-1}}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}\right)-f_{\theta^{p^*}}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}=0.$$

by the factor

$$\frac{\theta^{p^*+1}-\theta^{p^*}}{\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}}$$

and adding it to the equation

$$\sum_{j=1}^{p^*} f_{\theta^j}\left(\theta^{p^*+1}-\theta^j\right)=\theta^{p^*+1}-\mu.$$

This gives the equation

$$\sum_{j=1}^{p^*-1} f_{\theta^j}\left(\theta^{p^*+1}-\theta^j+\frac{\left(\theta^{p^*+1}-\theta^{p^*}\right)\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*-1}}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}\right)}{\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{p^*}}}}\right)=\theta^{p^*+1}-\mu$$

where all coefficients are strictly positive. Now we turn our attention to the induction step and assume that the variables $f_{\theta^{p^*}}, f_{\theta^{p^*-1}},\ldots,f_{\theta^{h+1}}$ have been eliminated and in the resulting equation

$$\sum_{j=1}^{h} c_j f_{\theta^j}=c$$

all coefficients c and c_j for $1\leq j\leq h$ are strictly positive. Now we have to eliminate the variable f_{θ^h} using the incentive constraint

$$(f_{\theta^1}+\cdots+f_{\theta^{h-1}})\left(\sqrt[n-1]{\theta^k-\overline{b}_{\theta^{h-1}}}-\sqrt[n-1]{\theta^k-\overline{b}_{\theta^h}}\right)-f_{\theta^h}\sqrt[n-1]{\theta^k-\overline{b}_{\theta^h}}=0.$$

We multiply this equation by the factor

$$\frac{c_h}{\sqrt[n-1]{\theta^k - \bar{b}_{\theta^h}}}$$

and add it to the equation

$$\sum_{j=1}^{h} c_j f_{\theta^j} = c.$$

This gives the equation

$$\sum_{j=1}^{h-1} f_{\theta^j} \left(c_j + \frac{c_h \left(\sqrt[n-1]{\theta^k - \bar{b}_{\theta^{h-1}}} - \sqrt[n-1]{\theta^k - \bar{b}_{\theta^h}} \right)}{\sqrt[n-1]{\theta^k - \bar{b}_{\theta^h}}} \right) = c$$

in which all coefficients are strictly positive. We conclude that the system of equations given by

$$\sum_{j=1}^{p^*+1} f_{\theta^j} = 1$$

$$\sum_{j=1}^{p^*+1} f_{\theta^j} \theta^j = \mu$$

$$(f_{\theta^1} + \cdots + f_{\theta^l})^{n-1} \left(\theta^k - \bar{b}_{\theta^l} \right) = \left(\sum_{j=1}^{h} f_{\theta^j} \right)^{n-1} \left(\theta^k - \bar{b}_{\theta^h} \right) \quad \text{for all } h \in \{1, \ldots, p^*\}$$

can be rearranged to a system of linear equations such that the resulting matrix has rank p^*+1 and therefore this system of equations has a unique solution.

Since the vector $\left(\tilde{f}^{\theta^k, \bar{b}_{\theta^l}}_{\theta^1}, \ldots, \tilde{f}^{\theta^k, \bar{b}_{\theta^l}}_{\theta^{p^*+1}} \right)$ fulfills the same p^*+1 equations and the solution of the linear system of equations with p^*+1 equations and p^*+1 unknowns is unique, it holds that

$$\left(\tilde{f}^{\theta^k, \bar{b}_{\theta^l}}_{\theta^1}, \ldots, \tilde{f}^{\theta^k, \bar{b}_{\theta^l}}_{\theta^m} \right) = \left(f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^1}, \ldots, f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^m} \right).$$

Appendix H. Proof of Lemma ??

We have to show that for every pair of valuations θ^l and θ^k such that $\theta^z \leq \theta^l \leq \theta^{k-2}$ the minimum p for minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ is greater or equal then $l+1$. We will prove the claim by contradiction. Let $\left(f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^1}, \ldots, f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^m} \right)$ denote the solution of minimization problem $M^{\theta^k}_{\bar{b}_{\theta^l}}$ as constructed in Lemma ??. Assume that the minimum p is strictly smaller than $l+1$. Under this assumption, we will show the following steps:

(1) The minimum p is equal to $l-1$.

(2) Let $\delta_{\theta^1}, \ldots, \delta_{\theta^m}$ be real numbers such that

$$\left(f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^1}, \ldots, f^{\theta^k, \bar{b}_{\theta^l}}_{\theta^m} \right) = \left(f^{\theta^k, *}_{\theta^1} + \delta_{\theta^1}, \ldots, f^{\theta^k, *}_{\theta^m} + \delta_{\theta^m} \right).$$

Then for all $1 \leq j \leq l$ it holds $\delta_{\theta^j} > 0$, for all $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$ and for all $k+1 \leq j \leq m$ it holds that δ_{θ^j}=0.

(3) We use step (2) in order to show that the fact that $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ is the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ leads to a contradiction to the assumption $p^* < l+1$

Proof of step (1):

If the minimum p, denoted by $p*$, is strictly smaller than $l+1$, then the last equation added in the construction of Lemma **??** has a corresponding bid which is lower or equal than $\bar{b}_{\theta^{l-1}}$ because the incentive constraint corresponding to $\bar{b}_{\theta^l}$ given by

$$\left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right) \geq \left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right)$$

is fulfilled trivially. Therefore, it holds $p^* < l$. It cannot hold that $p^* < l-1$ because then according to Lemma **??** there would be no probability weight on types above θ^l. This would imply that $f_{\theta^1} + \cdots + f_{\theta^l}$ equals to 1 and therefore, the valuation of the objective function

$$\left(f_{\theta^1}^{\bar{b}_{\theta^l}} + \cdots + f_{\theta^l}^{\bar{b}_{\theta^l}}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^l}\right)$$

is maximized. This cannot be optimal because the worst-case belief of the θ^k-type is an element of the feasible set of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ and has a lower value of the objective function. We conclude that $p^* = l-1$.

Proof of step (2):

Let $\delta_{\theta^1}, \dots, \delta_{\theta^m}$ be real numbers such that it holds

$$\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}, \dots, f_{\theta^m}^{\theta^k,*} + \delta_{\theta^m}\right).$$

Since the minimum p equals to $l-1$, it follows from Lemma **??** that in the solution of minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$ there is no probability weight on types above θ^{l+1}. In the worst-case belief of the θ^k-type there is probability weight on types θ^j for $1 \leq j \leq k$ and there is no probability weight on types θ^j for $k+1 \leq j \leq m$. Therefore, for all j with $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$ and for $k+1 \leq j \leq m$ it holds that $\delta_{\theta^j} = 0$. Note that the set $\{j \mid l+2 \leq j \leq k\}$ is not empty because $l \leq k-2$. Since $\sum_{j=1}^{m} \delta_{\theta^j}$ has to be zero, it follows that $\sum_{j=1}^{l+1} \delta_{\theta^j} > 0$.

According to Lemma **??**, if plugging in the solution $\left(f_{\theta^1}^{\theta^k,\bar{b}_{\theta^l}}, \dots, f_{\theta^m}^{\theta^k,\bar{b}_{\theta^l}}\right)$ into minimization problem $M_{\bar{b}_{\theta^l}}^{\theta^k}$, all constraints with corresponding bid below $\bar{b}_{\theta^l}$ have to binding. We use this in order to show by induction that $\delta_{\theta^1}, \dots, \delta_{\theta^l}$ have to be strictly positive.

According to Lemma **??** it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right)$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{l+1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{l+1}}\right)$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^l}^{\theta^k,*}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{l+1}}^{\theta^k,*}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^{l+1}}\right)}. \tag{49}$$

It also holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right)$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{l+1}}^{\theta^k,*} + \delta_{\theta^{l+1}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{l+1}}\right)$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^l,*} + \delta_{\theta^l}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} \tag{50}$$
$$\geq \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{l+1}}^{\theta^k,*} + \delta_{\theta^{l+1}}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^{l+1}}\right)}.$$

Subtracting (**??**) from (**??**) gives

$$\delta_{\theta^1} + \cdots + \delta_{\theta^l} \geq \frac{\left(\delta_{\theta^1} + \cdots + \delta_{\theta^{l+1}}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^{l+1}}\right)}}{\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}} > 0.$$

We start the inductive proof by showing that δ_{θ^1} is strictly positive. According to Lemma **??**, it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}\right)^{n-1}\theta^k$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1}\right)\sqrt[n-1]{\theta^k}. \tag{51}$$

According to Lemma **??** it also holds that

$$\left(f_{\theta^1}^{\theta^k,*}\delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^k}\right) = \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^l}\right)$$

$$\Leftrightarrow \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^k}^{\theta^k,*} + \delta_{\theta^k}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)} \tag{52}$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^l}^{\theta^k,*} + \delta_{\theta^l}\right)\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}.$$

Subtracting (**??**) from (**??**) gives

$$\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}\left(\delta_{\theta^1} + \cdots + \delta_{\theta^l}\right) = \sqrt[n-1]{\theta^k}\delta_{\theta^1}$$

$$\Leftrightarrow \delta_{\theta^1} = \frac{\sqrt[n-1]{\left(\theta^k - \overline{b}_{\theta^l}\right)}\,(\delta_{\theta^1} + \cdots + \delta_{\theta^l})}{\sqrt[n-1]{\theta^k}} > 0.$$

Assume that we have shown that $\delta_{\theta^j} > 0$ for all $1 \leq j < h$ for some $1 < h < l$. Then we can show that $\delta_{\theta^{h+1}} > 0$. According to Lemma **??** it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^{h+1}}^{\theta^k,*} + \delta_{\theta^{h+1}}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{h+1}}\right)$$
$$= \left(f_{\theta^1}^{\theta^k,*} + \delta_{\theta^1} + \cdots + f_{\theta^h}^{\theta^k,*} + \delta_{\theta^h}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right)$$

and according to Lemma **??** it holds that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{h+1}}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{h+1}}\right) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^h}^{\theta^k,*}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^h}\right)$$

from which follows that

$$(\delta_{\theta^1} + \cdots + \delta_{\theta^h})\ \sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} = (\delta_{\theta^1} + \cdots + \delta_{\theta^h} + \delta_{\theta^{h+1}})\ \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}$$
$$\Leftrightarrow \delta_{\theta^{h+1}} = \frac{(\delta_{\theta^1} + \cdots + \delta_{\theta^h})\left(\sqrt[n-1]{\theta^k - \overline{b}_{\theta^h}} - \sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}\right)}{\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{h+1}}}} > 0.$$

We conclude that for all j with $1 \leq j \leq l$ it holds $\delta_{\theta^j} > 0$.

Proof of step (3):
Let α and β be strictly positive be real numbers such that $\sum_{j=1}^{l+1} \delta_{\theta^j} = \alpha$ and $\sum_{j=l+2}^{k} \delta_{\theta^j} = -\beta$. Due to the two probability constraints it must hold that

$$\alpha - \beta = 0 \tag{53}$$

$$\sum_{j=1}^{l+1} \delta_{\theta^j}\theta^j + \sum_{j=l+2}^{k} \delta_{\theta^j}\theta^j = 0. \tag{54}$$

Since for $l+2 \leq j \leq k$ it holds that $\delta_{\theta^j} < 0$, it holds that $\sum_{j=l+2}^{k} \delta_{\theta^j}\theta^j < \sum_{j=l+2}^{k} \delta_{\theta^j}\theta^{l+2} = -\beta\theta^{l+2}$. It follows from step (2) that $\sum_{j=1}^{l+1} \delta_{\theta^j}\theta^j < \sum_{j=1}^{l+1} \delta_{\theta^j}\theta^{l+1} = \alpha\theta^{l+1}$. According to (**??**), it holds that $\alpha = \beta$ and it follows that

$$\sum_{j=1}^{l+1} \delta_{\theta^j}\theta^j + \sum_{j=l+2}^{k} \delta_{\theta^j}\theta^j < \alpha\theta^{l+1} - \beta\theta^{l+2} = \beta\theta^{l+1} - \beta\theta^{l+2} < 0$$

which is a contradiction to (**??**). Hence, we have found a contradiction to the assumption that the minimum p is strictly smaller than $l+1$.

Appendix I. Proof of Lemma ??

Proof. For $b \in [\bar{b}_{\theta^{k-1}}, \bar{b}_{\theta^k}]$ let $\left(f_{\theta^1}^{\theta^k,b}, \ldots, f_{\theta^m}^{\theta^k,b}\right)$ denote a solution of minimization problem $M_b^{\theta^k}$. Let $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ be real numbers such that

$$\left(f_{\theta^1}^{\theta^k,*}, \ldots, f_{\theta^m}^{\theta^k,*}\right) = \left(f_{\theta^1}^{\theta^k,b} + \delta_1, \ldots, f_{\theta^m}^{\theta^k,b} + \delta_m\right).$$

Assume that $f^{\theta^k,b} \neq f^{\theta^k,*}$. Then there exists $1 \leq j \leq m$ such that $\delta_j \neq 0$. Therefore, one can decompose the vector $(\delta_{\theta^1}, \ldots, \delta_{\theta^m})$ into δ-sequences and if there does not exist a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences end with a δ-sequence of length 2, i.e. with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$.

Assume there exists a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. We consider two cases: $t \leq k$ and $t > k$.

Case 1: $t \leq k$.

Following the steps in the proof of Lemma ??, one can show that it either holds

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} \left(\theta^k - b\right) = \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^k}^{\theta^k,b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^k}\right) \tag{55}$$

or

$$\begin{aligned} &\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} \left(\theta^k - b\right) \\ &\qquad = \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^{k-1},b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^{k-1}}\right). \end{aligned} \tag{56}$$

Thus, we consider two subcases.

Case 1.1: (??) holds. It follows from the definition of $\bar{b}_{\theta^k}$ and from Lemma ?? that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^k}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^k}) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}). \tag{57}$$

Since $f^{\theta^k,b}$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it holds that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b} G_{\theta^k}(b)\right)^{n-1} (\theta^k - b) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t})$$

It follows from (??) that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^k}^{\theta^k,b}\right)^{n-1} \left(\theta^k - \bar{b}_{\theta^k}\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1} (\theta^k - \bar{b}_{\theta^t}).$$

Subtracting equation (??) gives

$$\left(\sum_{j=1}^{k} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^k}} \geq \left(\sum_{j=1}^{t} \delta_{\theta^j}\right) \sqrt[n-1]{\theta^k - \bar{b}_{\theta^t}}.$$

Thus, it holds that $\sum_{j=1}^{k}\delta_{\theta^j} > 0$. Due to the first probability constraint, it follows that $\sum_{j=k+1}^{m}\delta_{\theta^j} < 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, this leads to a contradiction to the constraint $f_{\theta^j}^{\theta^k,b} \geq 0$ for all $1 \leq j \leq m$.

Case 1.2: (**??**) holds. It follows from Lemma **??** that

$$\left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^{k-1}}^{\theta^k,*}\right)^{n-1}(\theta^k - \overline{b}_{\theta^{k-1}}) = \left(f_{\theta^1}^{\theta^k,*} + \cdots + f_{\theta^t}^{\theta^k,*}\right)^{n-1}(\theta^k - \overline{b}_{\theta^t}). \tag{58}$$

Since $f^{\theta^k,b}$ is an element of the feasible set of minimization problem $M_b^{\theta^k}$, it holds that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b} + f_{\theta^k}^{\theta^k,b}G_{\theta^k}(b)\right)^{n-1}(\theta^k - b) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1}(\theta^k - \overline{b}_{\theta^t}).$$

It follows from (**??**) that

$$\left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^{k-1}}^{\theta^k,b}\right)^{n-1}\left(\theta^k - \overline{b}_{\theta^{k-1}}\right) \geq \left(f_{\theta^1}^{\theta^k,b} + \cdots + f_{\theta^t}^{\theta^k,b}\right)^{n-1}(\theta^k - \overline{b}_{\theta^t}).$$

Subtracting equation (**??**) gives

$$\left(\sum_{j=1}^{k-1}\delta_{\theta^j}\right)\sqrt[n-1]{\theta^k - \overline{b}_{\theta^{k-1}}} \geq \left(\sum_{j=1}^{t}\delta_{\theta^j}\right)\sqrt[n-1]{\theta^k - \overline{b}_{\theta^t}}.$$

Thus, it holds that $\sum_{j=1}^{k-1}\delta_{\theta^j} > 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, it holds due to the constraint

$$f_{\theta^j}^{\theta^k,b} \geq 0 \text{ for all } 1 \leq j \leq m$$

that $\delta_{\theta^j} \geq 0$ for all $k+1 \leq j \leq m$. Since $\sum_{j=1}^{m}\delta_{\theta^j} = 0$, it follows that $\delta_k < 0$ which is a contradiction to (**??**).

Case 2: $t > k$.
Due to the first probability constraint, it follows from $\sum_{j=1}^{t}\delta_{\theta^j} > 0$ that $\sum_{j=t+1}^{m}\delta_{\theta^j} < 0$. Since $f_{\theta^j}^{\theta^k,*} = 0$ for all $j > k$, this leads to a contradiction to the constraint $f_{\theta^j}^{\theta^k,b} \geq 0$ for all $1 \leq j \leq m$.

We conclude that in both cases the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$ and there exists a θ^{final} such that

$$\sum_{j=1}^{m}\delta_{\theta^j}\theta^j > \sum_{j=1}^{m}\delta_{\theta^j}\theta^{final} = 0.$$

Since this is a contradiction to the fact that the vector $f^{\theta^k,b}$ fulfills the constraint

$$\sum_{j=1}^{m} f_{\theta^j}^{\theta^k,b}\theta^j = \mu,$$

the assumption that $f^{\theta^k,b} \neq f^{\theta^k,*}$, leads to a contradiction.

□

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