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# Non-technical summary

Basic models of economic dynamics are used to analyse how capital accumulation and technology influence economic growth and income distribution. A central element of such a model is the production function. It relates the economy's input of capital and labour to its total output. The production function with a constant elasticity of substitution (CES) represents a commonly used functional form. The elasticity of substitution is a parameter that can be thought to reflect an economy's overall flexibility. It has been estimated in a number of empirical studies. The CES function has two more parameters. Current practice of choosing them in applications of dynamic models can lead to arbitrary and inconsistent results. Based on the concept of normalisation introduced by Klump and de La Grandville (2000), we develop a method that chooses them using empirical values of the income share of capital, the ratio of capital to output, and the elasticity of substitution. We illustrate the method with an example from the Ramsey growth model.

# Calibration of Normalised CES Production Functions in Dynamic Models*

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**Abstract**

Normalising CES production functions in the calibration of basic dynamic models allows to choose technology parameters in an economically plausible way. When variations in the elasticity of substitution are considered, normalisation is necessary in order to exclude arbitrary effects. As an illustration, the effect of the elasticity of substitution on the speed of convergence in the Ramsey model is computed with different normalisations.

**Keywords:** CES production functions, normalisation, calibration, Ramsey model.
**JEL classification:** *E24; E27; O41.*

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# 1 Introduction

Production functions with a constant elasticity of substitution (CES) have been used extensively in recent macroeconomic research on the dynamics of production and income distribution. In the simulation of dynamic models with CES functions, variations in its central parameter, the elasticity of substitution between capital and labour, are considered in a number of works. Some contributions take an interest in the economic determinants and effects of differences in the elasticity of substitution $\sigma$ (Klump 2005, Miyagiwa and Papageorgiou forthcoming), others vary it in the course of sensitivity analysis (King and Rebelo 1993, Turnovsky 2002).

From a mathematical point of view a CES production function with $n$ factors is a general mean of order $\frac{\sigma-1}{\sigma}$ in which inputs and output are all measured as dimensionless index numbers. In economic applications this characteristic is taken into account by (explicitly or implicity) normalising the function. Klump and de La Grandville (2000) introduce the normalisation in an analytical way. They do not indicate how it should be used for calibration. Rutherford (2002) considers normalisation in computable general equilibrium models. However, he does not discuss the effects of changes in the elasticity of substitution.

This note aims to provide a guide for the calibration of normalised CES production functions in basic dynamic models. Normalisation allows to deal with two important issues. First, it allows to calibrate the parameters of a CES production function in an economically meaningful way. Second, when the effect of a change in the elasticity of substitution is calculated in dynamic models, using normalised CES functions helps to avoid arbitrary and inconsistent results. We illustrate our findings by computing the speed of convergence in the Ramsey model.

## 2 The meaning of the baseline point

A neoclassical production function with a constant elasticity of substitution between capital and labour has three parameters. The most popular variant to choose them goes back to Arrow et al. (1961, henceforth ACMS). With $y$ as output and $k$ as capital in per capita notation, they write the CES function as:

$$y = f(k) = A[\alpha k^{\psi} + (1-\alpha)]^{\frac{1}{\psi}}, \tag{1}$$

where $A$ and $a$ are usually termed the efficiency and the distribution "parameter".

Although two early contributions by Kamien and Schwartz (1968) and Kmenta (1967) had already alluded to it, it often went unnoticed that this parametrisation of the function, as any other parametrisation, has an implicit baseline point. The baseline point is the point in which two CES functions with different elasticities of substitution $\sigma = \frac{1}{1-\psi}$ and otherwise equal parameters are tangent. With the ACMS parametrisation it lies at $k_0 = 1$. Choosing another baseline point requires to change $A$ and $\alpha$ when varying the elasticity of substitution. Klump and de La Grandville (2000) show that choosing a particular baseline point $k_0$ corresponds to the following normalisation of the ACMS parameters, with $y_0$ as output per capita at $k_0$ and $\pi_0$ as income share of capital under remuneration at marginal product at $k_0$[1]:

$$A = y_0 \left[\pi_0 k_0^{-\psi} + (1-\pi_0)\right]^{\frac{1}{\psi}}, \tag{2}$$

$$\alpha = \frac{\pi_0 k_0^{-\psi}}{\pi_0 k_0^{-\psi} + (1-\pi_0)}. \tag{3}$$

To clarify the meaning of the baseline point, we consider absolute output $Y$ in the "calibrated share form" (Rutherford 2002). It is obtained from (1) using (2) and (3):

$$Y = Y_0 \left[\pi_0 \left(\frac{K}{K_0}\right)^{\psi} + (1-\pi_0)\left(\frac{L}{L_0}\right)^{\psi}\right]^{\frac{1}{\psi}}. \tag{4}$$

Formally $Y/Y_0$ represents normalised output as a weighted mean of order $\psi$ taken over normalised inputs of capital $K/K_0$ and labour $L/L_0$. The mean is independent

[1]Klump and de La Grandville (2000) write the normalisation with the factor price ratio instead of the capital share, but the capital share is more straightforward to calibrate. See Appendix.

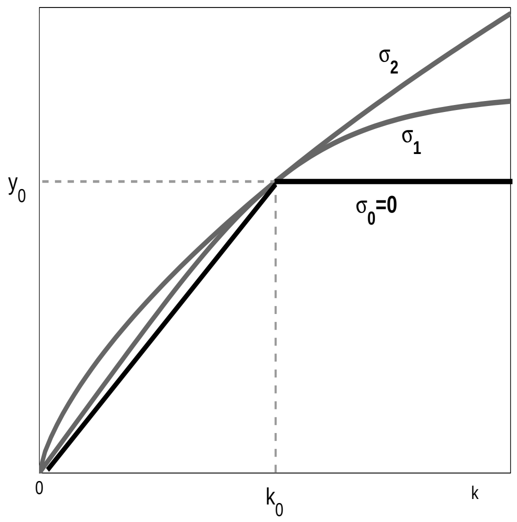


Figure 1: Two CES functions with $\sigma > 0$ and Leontief function of the same family

of $\psi$ if the normalised input values are equal. But what does equality of normalised inputs imply from an economic point of view? A look at the Leontief case in which $\sigma = 0$ sheds light on the economic meaning of normalised input values:

$$\frac{Y}{Y_0} = \min\left[\frac{K}{K_0}, \frac{L}{L_0}\right]. \tag{5}$$

If normalised input values are equal, that is if the capital intensity $k$ is equal to its baseline value $k_0$, both inputs are fully employed. In any other case, part of one input is unemployed. If the elasticity of substitution is very low yet positive, competitive markets bring about full employment (Solow 1956). For $k < k_0$ the economy's relative bottleneck still resides in its capacity to make productive use of additional labor. If $k > k_0$ the same is true for capital. The baseline capital intensity $k_0$ therefore corresponds to the capital intensity that would be efficient if the economy's elasticity of substitution were zero.

# 3 How to calibrate normalised CES functions

Calibrating normalised CES production functions in basic dynamic models involves two steps: calibrating an economically relevant point and normalising in the baseline point of a family of CES functions. While the first step applies to any calibration of CES production functions, the second is only necessary if the elasticity of substitution will be varied. Current practice is to calibrate directly the parameters $A$ and $\alpha$ of the CES function. While $\alpha$ equals the capital share in the Cobb-Douglas case with $\sigma = 1$, it has no straightforward interpretation in the general case. As Rutherford (2002) we argue that the most intuitive way to calibrate the CES function is based on values for inputs and factor shares.

In the first step one point, indexed with $i$, is calibrated with plausible values of these variables. In the simulation of a dynamic model it will often correspond to the initial point. Alternatively one can calibrate the steady state, as we will do in the next section. We suggest that the capital intensity $k_i$, the capital-output ratio $k_i/y_i$, the capital share $\pi_i$, and the elasticity of substitution between capital and labour $\sigma_i$ in this point be used for the calibration of the CES function. Choosing a capital intensity corresponds just to a choice of units, so it can be done under the aspect of numerical convenience. The remaining magnitudes can be calibrated using values from the empirical literature.

The calibration exactly determines the parameters $A$ and $\alpha$ of the CES function:

$$A_i = y_i \left[\pi_i k_i^{-\psi_i} + (1 - \pi_i)\right]^{\frac{1}{\psi_i}} , \tag{6}$$

$$\alpha_i = \frac{\pi_i k_i^{-\psi_i}}{\pi_i k_i^{-\psi_i} + (1 - \pi_i)}. \tag{7}$$

The substitution parameter is equal to $\psi_i = \frac{\sigma_i - 1}{\sigma_i}$.

If changes in the elasticity of substitution are to be considered, one has to choose in a second step the point of tangency of this production function with others that differ only in their elasticity of substitution. This point represents the baseline point of the relevant family of CES functions. From a formal point of view, any capital intensity can be a baseline capital intensity $k_0$ of a given CES function. The corresponding values of output per capita and the capital share are:

$$y_0 = f(k_0) = A_i \left[\alpha_i k_0^{\psi_i} + (1 - \alpha_i)\right]^{\frac{1}{\psi_i}} \tag{8}$$

and

$$\pi_0 = \frac{f'(k_0)k_0}{y_0} = \frac{\alpha_i k_0^{\psi_i}}{\alpha_i k_0^{\psi_i} + (1 - \alpha_i)}. \tag{9}$$

The parameters for the new elasticity of substitution $\sigma_j$ are obtained from plugging these values into (2) and (3):

$$A_j = y_0 \left[\pi_0 k_0^{-\psi_j} + (1 - \pi_0)\right]^{\frac{1}{\psi_j}}, \tag{10}$$

$$\alpha_j = \frac{\pi_0 k_0^{-\psi_j}}{\pi_0 k_j^{-\psi_j} + (1 - \pi_0)}. \tag{11}$$

In simulations one can either use the calibrated share form of the production function, or one can use the ACMS form with the parameters given in (11) and (10).

Normalisation requires to choose a particular baseline capital intensity $k_0$. As $\sigma = 0$ is an unrealistic situation for modern economies, one has to discuss on a theoretical level how this point is understood. If output $y_i$ is currently produced with inputs $k_i$ and if this remains possible independently of changes in the elasticity of substitution, the baseline capital intensity $k_0$ equals $k_i$. If on the other hand the current production method could only be attained thanks to a positive elasticity of substitution and if it would not be available anymore if the elasticity of substitution fell to zero, then one has to assume $k_0 \neq k_i$. As basic growth models are concerned with the economy's limited capacity to absorb capital in a productive way, $k_i > k_0$ is an appropriate assumption in this case. The more the steady state technique is thought to depend on the possibility of substituting capital for labour, the lower $k_0$ will be chosen.

# 4 An example: the speed of convergence in the Ramsey model

We illustrate the use of normalisation in the calibration of the Ramsey model. Researchers have been interested in the magnitude of the speed of convergence in the Ramsey growth model because it reveals the relative importance of transitional dynamics versus the steady state (see Turnovsky 2002 for an extensive simulation study). If heterogeneity of consumers is introduced into the Ramsey model, it also has a critical impact on distributional effects of growth (Caselli and Ventura 2000, Glachant and Vellutini 2002). We consider how the baseline point influences the effect of the elasticity of substitution on the speed of convergence.

In the Ramsey model, one could calibrate the initial point of an economy from which it converges to the steady state or the steady state itself. As we are interested in the speed of convergence near the steady state, we calibrate the latter. We follow Garcia-Penalosa and Turnovsky (2006) in the choice of values for the rate of time preference $\rho$, the rates of depreciation $\delta$ and population growth $n$, the intertemporal elasticity of substitution $\theta$, and the capital share $\pi_i$. The capital-output ratio is not calibrated directly but obtained using the steady state interest rate $r_i = r^* = \rho + n + \delta$, with $\frac{k_i}{y_i} = \frac{\pi_i}{r_i}$. For direct calibration the international data by King and Levine (1994) could be used. Compared to these data the ratio of about 4 obtained here lies in the upper range. The baseline capital intensity, which by definition equals the steady state capital intensity, is set to 10.

| | |
|---|---|
| Calibrated point of initial production function: | $\pi_i = 0.4$, $r_i = r^*$, $k_i = k^* = 10$ |
| Other parameters of the economy: | $\rho = 0.04$, $\theta = 0.4$, $n = 0.015$, $\delta = 0.04$ |

The asterisk (*) denotes values in the steady state. The dynamics of capital accumulation and consumption per capital $c$ are characterised by the following usual equations:

$$\dot{k} = f(k) - (n + \delta)k, \tag{12}$$

$$\dot{c} = \frac{c}{\theta} \left( f'(k) - \rho - n - \delta \right). \tag{13}$$

If a positive and finite steady state exists, the speed of convergence $\lambda$ is obtained from linearising around it:

$$\lambda = -\frac{\rho}{2} + \left( \frac{\rho^2}{4} + \frac{\rho + n + \delta}{\theta \sigma} \left( 1 - \pi^* \right) \frac{c^*}{k^*} \right)^{\frac{1}{2}} . \tag{14}$$

We compute it for an elasticity of substitution of 0.8 and study with five different baseline points how it changes when the elasticity of substitution rises to 1.2.

| | $A$ | $\alpha$ | $k^*$ | $\pi^*$ | $\lambda$ |
|---|---|---|---|---|---|
| $k_i = 10$, $\sigma = 0.8$ | *0.80* | *0.54* | *10* | *0.40* | *0.1614* |
| $k_0 = 1$, $\sigma = 1.2$ | 0.8 | 0.54 | 167.10 | 0.74 | 0.0454 |
| $k_0 = 5$, $\sigma = 1.2$ | 0.92 | 0.38 | 15.42 | 0.49 | 0.1004 |
| $k_0 = 10$, $\sigma = 1.2$ | 1.05 | 0.31 | 10 | 0.40 | 0.1286 |
| $k_0 = 20$, $\sigma = 1.2$ | 1.24 | 0.25 | 7.52 | 0.32 | 0.1603 |
| $k_0 = 100$, $\sigma = 1.2$ | 1.99 | 0.15 | 5.10 | 0.19 | 0.2518 |

The effect a given rise in the elasticity of substitution has on the speed of convergence depends on the relative magnitude of baseline and steady state capital intensity. If both are equal, we compare two economies with different elasticities of substitution converging to the same steady state. The different speeds of convergence reflect only the moderate direct effect of $\sigma$ visible in (14), as the indirect effect via the steady state values is zero.

If we compare the speeds of convergence of two economies that have different steady states depending on their elasticity of substitution, $k_0 = 1$ and $k_0 = 5$ are possible assumptions. With $k_0 = 5$ the effect of a higher elasticity of substitution on the speed of convergence is only half as large as with $k_0 = 1$.

In the previous section we argued that $k_i \geq k_0$ is a plausible assumption when considering long term growth. We see here that $k_i < k_0$ yields counterintuitive results, $k^*$ declines with higher $\sigma$. As a consequence the speed of convergence may even rise with a higher elasticity of substitution (see also Klump 2001).

Using the ACMS function ($k_0 = 1$) would thus not lead to "false" results, but the underlying interpretation of differences in $\sigma$ and the sensitivity of results with respect to the baseline point should be discussed. Normalisation is a helpful tool in making the calibration and its sensitivity to parameter changes as transparent as possible.

## 5 Conclusion

Calibrating normalised CES production functions proceeds in two steps: first, calibrate an economically meaningful point, second, decide where the baseline point of the family of CES functions lies relatively to the calibrated point. Normalisation grounds the parametrisation of the production function more firmly on economic reasoning and eliminates arbitrary effects.

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# A Appendix

Klump and de La Grandville (2000) show that the parameters of the ACMS variant of the CES function can be normalised in the following way:

$$A = y_0 \left( \frac{k_0^{1-\psi} + \mu_0}{k_0 + \mu_0} \right)^{\frac{1}{\psi}}, \tag{15}$$

$$\alpha = \frac{k_0^{1-\psi}}{k_0^{1-\psi} + \mu_0}, \tag{16}$$

with $\mu_0 = \frac{w_0}{r_0}$ the ratio between the baseline wage rate and the baseline interest rate under remuneration at marginal product. Equation (2) immediately follows from

$$\pi_0 = \frac{k_0}{k_0 + \mu_0}. \tag{17}$$

Equation (3) is obtained in the following way:

$$\begin{aligned} \frac{1}{\alpha} &= 1 + \frac{\mu_0}{k_0^{1-\psi}} \\ \Leftrightarrow \frac{1}{\alpha} &= 1 + \frac{(1-\pi_0) y_0 k_0}{\pi_0 y_0 k_0^{1-\psi}} \\ \Leftrightarrow \alpha &= \left( \frac{\pi_0 k_0^{-\psi} + (1-\pi_0)}{\pi_0 k_0^{-\psi}} \right)^{-1}. \end{aligned} \tag{18}$$