Entry and Word-of-Mouth Diffusion

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Abstract

We consider the entry and information diffusion process into a (telecoms) market where only informed consumers consider to buy the new product, and only actual customers of the entrant spread information. Diffusion is slower (faster) than socially optimal if the incumbent has few (many) captive customers. Advertising by the entrant, though it accelerates diffusion, has ambiguous welfare effects. Regulatory intervention may impose time-varying mandatory prices on the incumbent, while imposing uniform pricing can lead to high prices and too fast diffusion. If consumers can search, or wait until they are informed, all search occurs at the moment of entry.

Keywords: Entry, Diffusion, Word-of-mouth, Differential Games
JEL: C73, L11, L51

1 Introduction

This paper was motivated by the observation that market the penetration of new telephone operators on the fixed network continues to remain disappointing. This is quite surprising given that savings in call charges can be substantial as compared to the traditional incumbent operator: One should expect many more users to change providers than is actually observed. This is true for call-by-call access, with choice of a possibly different operator for each call, with the inconvenience of having to dial an access code, as much as for an actual change of provider.

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Switching costs are only part of the explanation. Enough savings could be realized by consulting just one alternative operator, and still many consumers do not even consider this possibility. What is most common is a lack of trust in the reliability and the business practices of new operators, as well as lack of information about their offers. For most consumers the cost of acquiring this information seems to be rather high.

Most models of entry presume that all consumers have instant knowledge about the existence of the new product or service, and that consumers are willing buy the new product if its price is low enough to compensate for eventual switching costs.

The model we have in mind assumes that "uninformed" consumers randomly meet current customers of the entrant, and the ensuing exchange of information and opinions leaves the former "informed". This does not mean that they will necessarily become customers of the entrant, rather they take notice of its offer in the first place, which then they may take up or not. We thus introduce a distinction between consumers that are informed about a product, and customers who actually buy it. Lastly, we assume that an additional part of the population remains completely impenetrable to whatever information they may receive about the new product. Therefore, there will be inertia not only due to a slow increase in potential customers, but also due to customers who will never switch. (Armstrong, Shapiro/Varian: described but not modelled). The latter, as they remain captive to the incumbent, can strongly influence the incumbent's pricing decisions: He must trade off gains from competing with the entrant with profits obtained by milking his captive customers.

Modelled as described above the number of informed customers increases over time following a diffusion process. What makes our approach different from a significant part of the existing literature on diffusion is that the speed of diffusion is endogenous. From the point of view of both firms involved there is strategic value in influencing the speed of diffusion. The entrant wants to accelerate diffusion to be able to sell to more customers in the future, and thus sets low prices to gain market share today. On the other hand, the incumbent has an interest in limiting the speed of diffusion to maintain his share of the total market, and his most effective instrument is to lower prices. This latter effect is mitigated because of the "fat cat effect": Lowering price too much leads to large losses on captive consumers.

Though not directly related to our model, we mention the work of Peitz and de Bijl (2000, 2002). They take the innovative approach of considering entry under a dynamic process whereby the consumers' utility derived from the entrant's product increases over time, so that entry is again described by a dynamic process. The main difference to our approach is that this dynamic process is completely exogenous. In fact, in their models both entrant and incumbent play static Nash equilibria since there are no intertemporal strategic effects present. Our paper, on the other hand, concentrates precisely on these effects.

Discuss literature on diffusion models here.

A second channel through which the entrant can "spread the news" is advertising its presence, the type of service it offers, and its pricing schemes. If advertising is assumed similarly "informative" as word-of-mouth transmission then it serves as a substitute for lower prices by the entrant. The likely result should be higher prices in the short run, but possibly a faster diffusion. Thus even before taking into account the costs of advertising the welfare consequences of advertising are ambiguous.

Telecoms regulators have been trying to facilitate entry through imposing restrictions on the incumbent's pricing, among other instruments. Universal service obligations such as uniform pricing, even though their original motivation was primarily egalitarian and not based on arguments of competitive entry, can have a positive influence on entry. In our model the incumbent's uniform price for both "inert" and "active" consumers creates a price umbrella that the entrant benefits from. The additional dimension revealed by our model is that this price umbrella not only protects the entrant's profits in the short run, but also has dynamic effects in the form of influencing the speed of diffusion. We show below that imposing uniform pricing may raise welfare, through faster diffusion, if the incumbent's captive customers are not too many, while may have ambiguous effects if the number of captive consumers is large.

In the following section we will introduce the model, and section 3 characterizes the socially optimal rate of diffusion. Section 4 considers competition in prices only, while section 5 adds informative advertising by the entrant. Section 5 considers simple regulatory interventions such as fixing the incumbent's price or imposing a price cap, and section 6 concludes.

2 The Model

Time is continuous, $t \geq 0$, and firms discount the future with factor $\delta \in (0,1)$. We assume the total mass of consumers is $1+L$, of which $L \geq 0$ are captive with the incumbent (firm 1), and the rest of mass 1 may consider switching to the entrant (firm 2) once "informed". The share of informed consumers at any moment t is $S \in [0,1]$ (we omit the time indext to not crowd notation). We measure the entrant's market share at any point in time β as relative to informed consumers, so that the mass of his customers is $M_2 =$

βS, while the customers of the incumbent have mass $M_1 = (1 - \beta S + L)$. The fundamental assumption of this paper is that at each point in time the entrant's customers βS meet randomly with the rest of the population and transmit the knowledge about the entrant. The $(1-\beta) S$ customers of the incumbent that are informed, but have chosen to stay with the incumbent, are not seen to consume the entrant's product, and therefore do not transmit any information. Of these encounters only meetings with the $(1 - S)$ customers who do not yet know the entrant, but might consider switching once they have that information, have any effect. We assume that in a proportion $\alpha \in (0, 1)$ of these meetings useful information is transmitted, so that the diffusion of knowledge is described by the following equation:

$$\dot{S} = \alpha\beta S\,(1 - S)\,. \tag{1}$$

Contrary to what it seems, this is not a logistic diffusion equation (though it is similar) because the market share β is endogenous and will depend, among other variables, on S. This diffusion equation will later be augmented by a term that models the effect of advertising (see Section 5).

Unfortunately this formulation implies that we must assume an initial level of $S_0 > 0$ of informed customers (which can be the CEO of the entrant, for example), otherwise no diffusion would take place. This problem will be resolved in section 7 when consumers can decide to become informed, as immediately some consumers acquire the necessary information.

In principle competition between the entrant and the incumbent could take any form. Since we are considering the market for telecommunication services, we are interested in modelling the (discrete) choice between the entrant's and the incumbent's variety. Any discrete choice model described for example in Anderson *et al.* (1992) or the Hotelling model would be appropriate, but not models like differentiated goods Cournot or Bertrand duopolies. There has been a recent tradition of using the Hotelling model to represent telecommunications markets, for example Armstrong (1998) and Laffont, Rey and Tirole (1998), so we adopt it as well. A remark on the side: It would also be interesting to consider a model with homogeneous goods, but the presence of captive customers means that at each point in time equilibrium would be in mixed strategies (this goes back at least to), which leads to technical problems that a yet are unresolved.

Assume thus that all $(1 + L)$ consumers are distributed uniformly on a Hotelling line $[0, 1]$, with the entrant situated at zero and the incumbent at 1. Consumers' location on this line is independent of whether they are captive to the incumbent, or willing to switch once they are informed. One could of course argue that the captive consumers should be the ones closest to the incumbent, but this would take the bite out of information transmission. "Transport costs" are $z|x-a|$, where $z \geq 0$ measures product differentiation, x is the location of each consumer, and a the supplier's location. Captive and uninformed customers buy at the incumbent's, while informed customers choose between both firms; the indifferent informed customer β is situated at

$$v_2 - z\beta = v_1 - z(1-\beta),$$

where v_i is the consumer surplus derived from firm i's offer. For now we assume that consumers buy exactly one unit to which they attribute a valuation v, so $v_i = v - p_i$.[1] The entrant's market share will be

$$\beta = \frac{1}{2} + \frac{p_1 - p_2}{2z} \tag{2}$$

while $p_2 \geq p_1 - z$, and $\beta = 1$ otherwise.

The incumbent and the entrant have constant marginal cost of production c_1 and c_2, respectively, and the difference is $\Delta = c_1 - c_2$, which we will assume is positive (entrant more efficient than incumbent, for example due to modern technology). Instantaneous profits, given the mass S of informed consumers, are then given by $\pi_i = M_i(p_i - c_i)$.

In the following we will first characterize the socially optimal pattern of diffusion, and then in section 4 we determine the market equilibrium.

3 The Social Optimum

At each instant consumers buy from one of the firms and support transport cost. The average instantaneous consumer surplus of the mass S of informed customers is given by

$$\begin{aligned} s_{12} &= \beta v_2 - z\int_0^\beta x\,dx + (1-\beta)v_1 - z\int_\beta^1 (1-x)\,dx \\ &= v - \beta p_2 - (1-\beta)p_1 - \frac{1}{2}z\left(1 - 2\beta + 2\beta^2\right) \end{aligned}$$

while the average surplus of uninformed and captive consumers is

$$s_1 = v_1 - z\int_0^1 (1-x)\,dx = v - p_1 - \frac{1}{2}z.$$

[1] In later versions we will consider variable demand, which is indispensable if we wish to consider the efficiency effects of lower prices: With inelastic demand different prices only lead to transfers between consumers and firms but do not change total welfare. A further reason is that with inelastic demand we cannot model non-linear tariffs.

We will first determine the static social optimum, for a given level of diffusion S. This serves as a benchmark against which to contrast the socially optimal market shares, or price difference. This static social optimum is found by solving

$$\max_{p_1,p_2} s_{12}S + s_1(1 - S + L) + \pi_1 + \pi_2, \tag{3}$$

which can be expressed equivalently as

$$\max_{\beta} S\beta(\Delta + z(1-\beta)) + K, \tag{4}$$

where K is a constant term that contains neither S nor β. The socially optimal market static share would be independent of S, with

$$\beta^{st} = \frac{1}{2} + \frac{\Delta}{2z}. \tag{5}$$

Here we assume that $\beta^{st} < 1$, or $\Delta < z$, otherwise the problem is trivial. If this were to be implemented using prices, it corresponds to prices related by

$$p_2 = p_1 - \Delta, \tag{6}$$

while the level irrelevant due to the assumption of inelastic demand.

Assume that the social planner also has discount factor δ, and maximizes the sum of discounted instantaneous surplus. Since the constant K mentioned above does not contain S and β it does not change over time and thus can be dropped:

$$\begin{aligned} &\max_{\beta} \int_0^{\infty} e^{-\delta t}\beta S(\Delta + z(1-\beta))\, dt \\ &s.t.\ \dot{S} = \alpha\beta S(1-S), \end{aligned} \tag{7}$$

with initial condition $S(0) = S_0$ and transversality condition $\lim_{t\to\infty} e^{-\delta t}\lambda = 0$ on the shadow price of the equation of motion λ. This is an optimal control problem with current value Hamiltonian, with costate λ,

$$H = \beta S(\Delta + t(1-\beta)) + \lambda\alpha\beta S(1-S) \tag{8}$$

Applying the maximum principle lets to the necessary conditions

$$\frac{\partial H}{\partial \beta} = S(\Delta + t(1-2\beta)) + \lambda\alpha S(1-S) = 0, \tag{9}$$

$$\frac{\partial H}{\partial S} = \beta(\Delta + t(1-\beta)) + \lambda\alpha\beta(1-2S) = \delta\lambda - \dot{\lambda}, \tag{10}$$

where we implicitly assume an interior solution for β. The boundary solution $\beta = 1$ will be considered below.

Solving (9) for β leads to

$$\beta = \frac{1}{2} + \frac{\Delta}{2z} + \frac{\alpha}{2z}\lambda\left(1 - S\right). \tag{11}$$

The dynamically optimal market share, as compared to the static optimum (5), corrects for the effect of current market share on the diffusion of knowledge about the entrant. If (what is likely) the shadow social value λ of an additional informed consumer is positive then the dynamically optimal market share β is larger than the static one. As all non-captive consumers become informed ($S \to 1$) this market share converges to the statically optimal one.

Substituting β into the dynamic equations yields the system

$$\begin{aligned} \dot{\lambda} &= \delta\lambda - \frac{1}{4z}\left(z + \Delta + \alpha\lambda\left(1 - S\right)\right)\left(\Delta + z + \alpha\lambda\left(1 - 3S\right)\right), \\ \dot{S} &= \frac{\alpha}{2z}\left(z + \Delta + \alpha\lambda\left(1 - S\right)\right)S\left(1 - S\right), \\ S\left(0\right) &= S_0,\ \lim_{t\to\infty} e^{-\delta t}\lambda = 0, \end{aligned} \tag{12}$$

which describes the dynamics of diffusion in the social optimum. This cannot be solved explicitly, but nevertheless its solution can be completely characterized.

First notice that it has steady states

$$S^* = 1,\ \lambda^* = \frac{1}{2}\frac{\left(z + \Delta\right)^2}{2\delta z + \alpha\left(z + \Delta\right)} > 0, \tag{13}$$

which is the one that we would like to converge to, and, if and only if $\delta \geq \alpha\left(1 + \Delta/z\right)$,

$$\hat{S} = 0,\ \hat{\lambda} = \frac{2z\delta - \alpha\left(z + \Delta\right) \pm 2\sqrt{z\delta\left(z\delta - \alpha\left(z + \Delta\right)\right)}}{\alpha^2} > 0. \tag{14}$$

In Appendix 1 we show that the steady state with $S = 1$ is saddle-path stable, while the other two (if they exist) are unstable. The solution to the system (12) we be on this saddle path, where S is monotonically increasing, and λ and β are monotonically decreasing. We can already compare the dynamically optimal market shares to the static optimum:

Proposition 1 *The dynamically optimal market share of the entrant is always higher than at the static social optimum:* $\beta > \beta^{st}$.

Proof. From the equation of movement for the shadow price λ in 12 we can infer that for $S \in [0,1]$ the shadow-price never crosses over from negative to positive values as

$$\dot{\lambda}\Big|_{\lambda=0} = -\frac{1}{4z}(z+\Delta)^2 < 0.$$

Since in the stable steady state λ is positive, it will be so on the whole saddle path, and therefore the last term in (11) is strictly positive.[2] ■

As concerns the cost of inert consumers, given by λ^*, it is decreasing in patience δ and speed of diffusion α, but increasing in the cost difference Δ; it is decreasing (increasing) in transport cost depending on whether it is small (large):

$$\frac{d\lambda^*}{d\delta} < 0, \ \frac{d\lambda^*}{d\alpha} < 0, \ \frac{d\lambda^*}{d\Delta} > 0; \ \frac{d\lambda^*}{dz} > (<) \ 0 \text{ if } z > (<) \frac{\Delta(2\delta-\alpha)}{2\delta+\alpha}.$$

If the social planner is impatient ($\delta \geq \alpha(1+\Delta/z)$) then the saddle path emanates from the lower one of the other two steady states, as shown in Figure 1.[3]

[2] If we are in the boundary case $\beta = 1$, which is considered below, then naturally the same result holds.

[3] All numerical calculations were done by programs written for Matlab 5 which are available from the author. In particular, the saddle paths were found by deviating a bit from the steady state along the stable eigenvector and then running the equations in inverse time. These numerical results use specific values for the parameters and are merely illustrative, while the qualitative analysis performed in the main text applies to all (reasonable) parameter values.

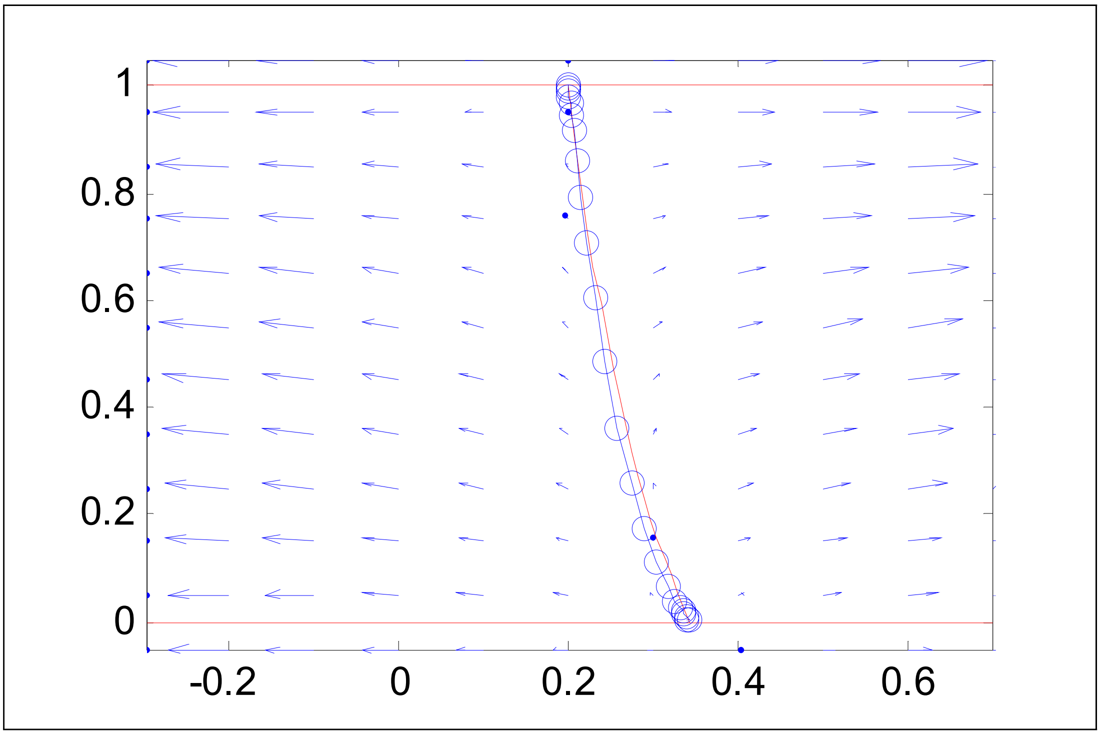

Figure 1: Phase portrait and saddle path at the social optimum with impatient social planner ($a = 0.5$, $\delta = 1$, $\Delta = 0$, $z = 1$).

In this case the highest value that β can take on is

$$\hat{\beta} = \frac{\delta}{\alpha} - \sqrt{\frac{\delta}{\alpha}\left(\frac{\delta}{\alpha} - (1 + \Delta/z)\right)}.$$

If on the other hand he is patient ($\delta < \alpha\,(1 + \Delta/z)$), then these other steady states do not exist. The saddle path is as shown in Figure 2.

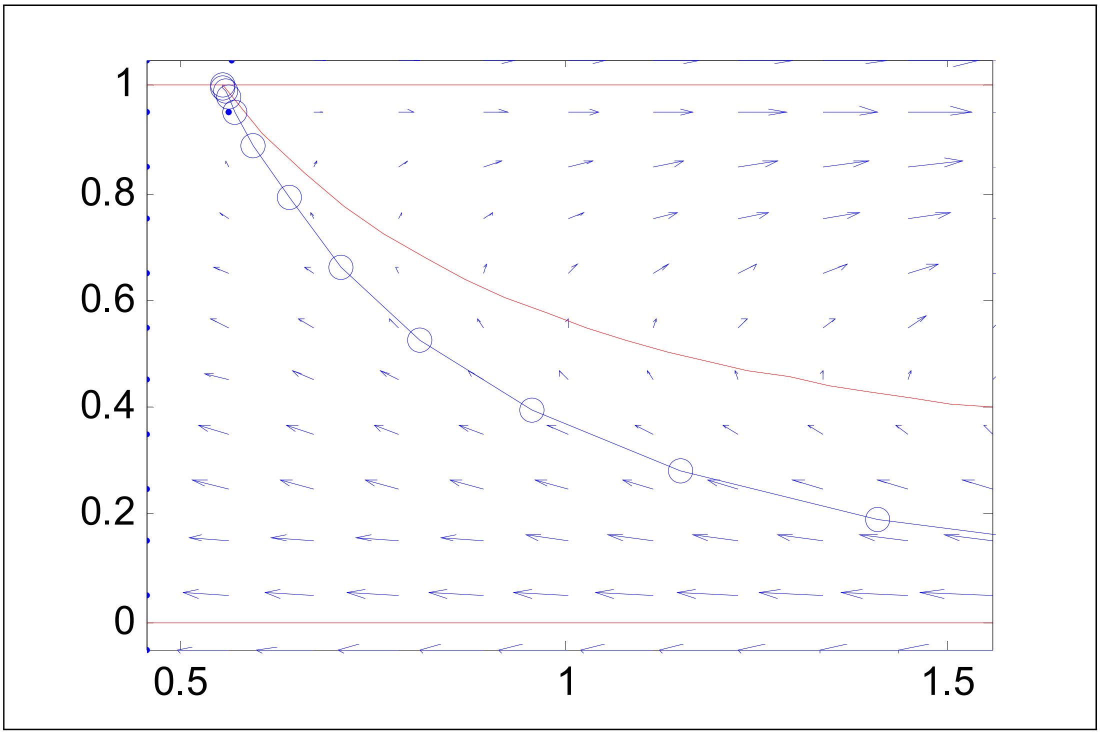

Figure 2: Phase portrait and saddle path at the social optimum with patient social planner ($a = 0.5$, $\delta = 0.2$, $\Delta = 0$, $z = 1$).

The main difference is that it emanates from a zone on the lower right where depending on the initial value S_0 the optimal market share is $\beta = 1$, as for smaller and smaller initial values λ increases without bound on the saddle path. This zone is delineated by the curve $\beta = 1$ or $\lambda (1 - S) = (z - \Delta) / \alpha$. By the principle of optimality, what happens below this curve does not influence the location of the saddle path above it: Our above results continue to be valid without qualification up until the point where the saddle path "hits" this boundary.

Below this curve the entrant's market share will be $\beta = 1$ in equilibrium. The adjoint equation in this case is

$$\frac{\partial H}{\partial S} = \Delta + \lambda\alpha (1 - 2S) = \delta\lambda - \dot{\lambda}, \tag{15}$$

leading to the equations of motion

$$\begin{aligned} \dot{\lambda} &= \delta\lambda - \Delta - \lambda\alpha (1 - 2S) \\ \dot{S} &= \alpha S (1 - S) \end{aligned} \tag{16}$$

These have the exact solutions, with constants of integration C_1 and C_2,

$$
\begin{aligned}
\lambda(t) &= \left(C_1 - \Delta \int_{t_0}^{t} \frac{1}{e^{(\delta+\alpha)u}\left(1+C_2 e^{-\alpha u}\right)^2} du\right) e^{(\delta+\alpha)t}\left(1+C_2 e^{-\alpha t}\right)^2, \\
S(t) &= \frac{1}{1+C_2 e^{-\alpha t}},
\end{aligned} \tag{17}
$$

and

$$
C_1 = \lambda(0) S(0)^2, \; C_2 = \frac{1-S(0)}{S(0)}.
$$

Here it is useful to normalize to $t = 0$ the point in time when the saddle path crosses the boundary

$$
\lambda(0)(1-S(0)) = \frac{z-\Delta}{\alpha}.
$$

For a patient social planner the optimal path of diffusion (17) is such that for small initial values S_0 diffusion at first occurs along a traditional logistic S-curve. In this phase the social planner sacrifices a short-term rise in transport cost for faster diffusion. After a while the trade-off between faster diffusion and high transport cost for customers close to the incumbent is reversed, and the optimal market share decreases continuously towards the static optimum, implying slower diffusion than on an exogenously imposed S-curve.

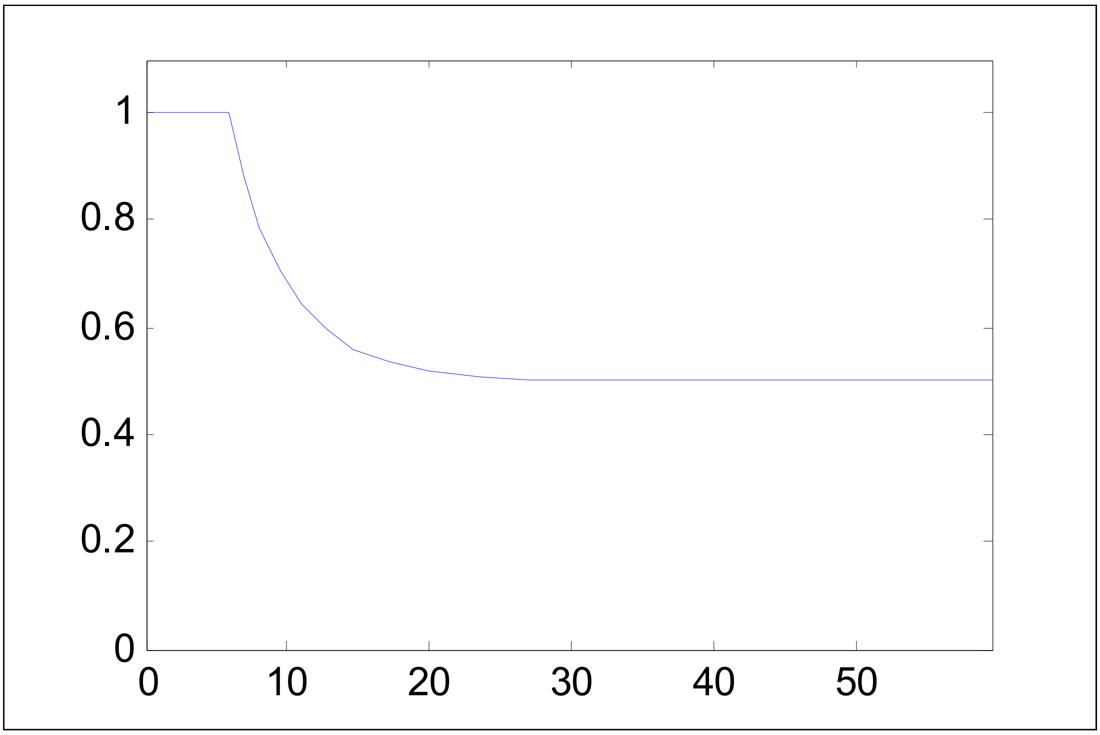

Figure 3: The entrant's market share β with a patient social planner ($a = 0.5$, $\delta = 0.2$, $\Delta = 0$, $z = 1$).

We will now determine the market equilibrium, and compare this with the social optimum.

4 Market Equilibrium

The incumbent (firm 1) and the entrant (firm 2) play the following differential game:

$$\begin{aligned} \max_{p_1(.)} \Pi_1 &= \int_0^{\infty} e^{-\delta t} (1 - \beta S + L)(p_1 - c_1)\, dt \\ \max_{p_2(.)} \Pi_2 &= \int_0^{\infty} e^{-\delta t} \beta S (p_2 - c_2)\, dt \\ s.t.\ \dot{S} &= \alpha \beta S (1 - S) \\ S(0) &= S_0 \end{aligned} \tag{18}$$

Here $L \geq 0$ is the mass of captive customers of the incumbent; for $L > 0$ the above expression implicitly assumes uniform pricing by the incumbent, while setting $L = 0$ includes the case of non-uniform pricing. In the latter the captive customers will have to pay as high a price as the incumbent is able to set, at the same time that their presence and number does not influence pricing for the other customers.

We will solve this game in open-loop strategies, i.e. pricing strategies that only depend on the initial state and time (see Basar and Olsder 1999 , Dockner *et al.* 2000). It would clearly be preferrable to find a solution in closed-loop strategies, but at the current state of economics this is not feasible.

Current-value Hamiltonians are, with shadow values for an additional informed customer μ_1 and μ_2,

$$\begin{aligned} H_1 &= (1 - \beta S + L)(p_1 - c_1) + \mu_1 \alpha \beta S (1 - S), \\ H_2 &= \beta S (p_2 - c_2) + \mu_2 \alpha \beta S (1 - S), \end{aligned} \tag{19}$$

and the necessary conditions for the controls p_1 and p_2 are

$$\begin{aligned} \frac{\partial H_1}{\partial p_1} &= -\frac{1}{2z} S (p_1 - c_1) + (1 - \beta S + L) + \mu_1 \alpha \frac{1}{2z} S (1 - S) = 0, \\ \frac{\partial H_2}{\partial p_2} &= -\frac{1}{2z} S (p_2 - c_2) + \beta S - \mu_2 \alpha \frac{1}{2z} S (1 - S) = 0. \end{aligned}$$

Substituting β and solving for p_1 and p_2 leads to equilibrium prices

$$\begin{aligned} p_1 &= c_1 + \frac{1}{3}\left(z\left[4\frac{1+L}{S}-1\right] - \Delta + \alpha\left(1-S\right)\left(2\mu_1-\mu_2\right)\right) \\ p_2 &= c_2 + \frac{1}{3}\left(z\left[2\frac{1+L}{S}+1\right] + \Delta + \alpha\left(1-S\right)\left(\mu_1-2\mu_2\right)\right) \end{aligned} \tag{20}$$

Letting $\mu = (\mu_1 + \mu_2)$, equilibrium market shares will be

$$\beta = \frac{1}{6z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\mu\left(1-S\right)\right). \tag{21}$$

This equilibrium market share will be larger or smaller than the static equilibrium market share (which corresponds to $\alpha = 0$ i.e. no diffusion) depending on whether μ is positive or negative.

The two adjoint equations are

$$\begin{aligned} \frac{\partial H_1}{\partial S} &= -\beta\left(p_1-c_1\right) + \mu_1\alpha\beta\left(1-2S\right) = \delta\mu_1 - \dot{\mu}_1, \\ \frac{\partial H_2}{\partial S} &= \beta\left(p_2-c_2\right) + \mu_2\alpha\beta\left(1-2S\right) = \delta\mu_2 - \dot{\mu}_2, \end{aligned} \tag{22}$$

and after substituting β we obtain

$$\begin{aligned} \dot{\mu}_1 &= \delta\mu_1 - \frac{1}{18z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu\right) \\ &\quad\times\left(z\left[1-4\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu - 3\alpha S\mu_1\right), \\ \dot{\mu}_2 &= \delta\mu_2 - \frac{1}{18z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu\right) \\ &\quad\times\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu - 3\alpha S\mu_2\right). \end{aligned} \tag{23}$$

The sum of both equations is

$$\begin{aligned} \dot{\mu} &= \delta\mu - \frac{1}{9z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu\right) \\ &\quad\times\left(z\left[1-\frac{1+L}{S}\right] + \Delta + \alpha\left(1-\frac{5}{2}S\right)\mu\right), \end{aligned} \tag{24}$$

and the evolution of market shares follows

$$\dot{S} = \frac{\alpha}{6z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\left(1-S\right)\mu\right)S\left(1-S\right) \tag{25}$$

The steady states of the system in (μ_1, μ_2, S) are

$$\begin{aligned}
S^* &= 1,\ \mu_1^* = -\frac{1}{3}\frac{(3z+4Lz-\Delta)}{\alpha+6\delta z/\,(3z+2zL+\Delta)}, \\
\mu_2^* &= \frac{1}{3}\frac{(3z+2Lz+\Delta)}{\alpha+6\delta z/\,(3z+2zL+\Delta)}; \\
\hat{S} &= -2z\frac{1+L}{z+\Delta},\ \hat{\mu}_1 = 0,\ \hat{\mu}_2 = 0,
\end{aligned} \tag{26}$$

and of the simpler system (S, μ),

$$\begin{aligned}
S^* &= 1,\ \mu^* = \frac{2}{3}\frac{\Delta - zL}{\alpha+6\delta z/\,(3z+2zL+\Delta)}; \\
\hat{S} &= -2z\frac{1+L}{z+\Delta},\ \hat{\mu} = 0.
\end{aligned} \tag{27}$$

That is, if the number of captive consumers is large, $L > \Delta/z$, then the steady state shadow value μ^* is negative, while it remains positive if $L < \Delta/z$. This time the saddle path cannot "emanate" from the vicinity of the other steady state because it cannot cross the line $S = 0$. In fact, close to this line $\dot{\mu}$ tends to infinity while $\dot{S}$ becomes zero.

In appendix 2 we show that the steady state (S^*, μ^*) is again saddle-path stable, and that on the saddle path S and μ are increasing. We also show that the market share β of the entrant is monotonically decreasing. Market shares on this saddle path, as compared to the static equilibrium market shares, depend on the sign of μ:

Proposition 2 *If the number of captive consumers L is large ($L \geq \Delta/z$) then the entrant's market share in the dynamic equilibrium is smaller than in the static equilibrium for the corresponding number of informed consumers S. If on the other hand the number of captive consumers is small ($L < \Delta/z$) then for small S the entrant's market share is smaller than in the static equilibrium, but exceeds this value as the number of informed customers becomes sufficiently large.*

Proof. From (24) we can deduce that (see also Figures 4 and 5 below)

$$\begin{aligned}
\dot{\mu}|_{\mu=0} &> 0 \text{ if } 0 < S < \frac{1+L}{1+\Delta/z} \\
\dot{\mu}|_{\mu=0} &< 0 \text{ if } S > \frac{1+L}{1+\Delta/z}.
\end{aligned}$$

This means that for $L \geq \Delta/z$ only the first case is relevant, thus the saddle path cannot cross the axis $\mu = 0$ from the right. Since $\mu^* \leq 0$ in this case this means that for $L \geq \Delta/z$ we must have $\mu < 0$ on the whole saddle path. On the other hand, for $L < \Delta/z$ the steady state value μ^* is positive, so that at some point in time t' the saddle path must have crossed the vertical axis from the left ($\dot{\mu}$ is so large for small values of S that the saddle path must begin with negative values of μ). After this point it cannot get back to negative values of μ because otherwise it would not be able to reach $\mu^* > 0$. ■

This result may appear counter-intuitive. One would expect competition to be less intensive when the number of captive consumers is large, this would translate into a larger (not smaller) market share by the entrant. In fact, we are making the wrong comparison here. The correct one would be with a static situation with the same number of captive customers, and where the incumbent knows that he will not lose customers to the entrant. Now it seems natural that there will be a more aggressive equilibrium price by the incumbent, and therefore a smaller market share for the entrant. Even more interesting is therefore the case of few captive customers and S approaching 1: Almost all non-captive customers are now informed. In this situation there is no point in being aggressive, also given the fact that the incumbent is at a cost disadvantage.

The phase portrait for the case of large $L > \Delta/z$ is shown in Figure 4:

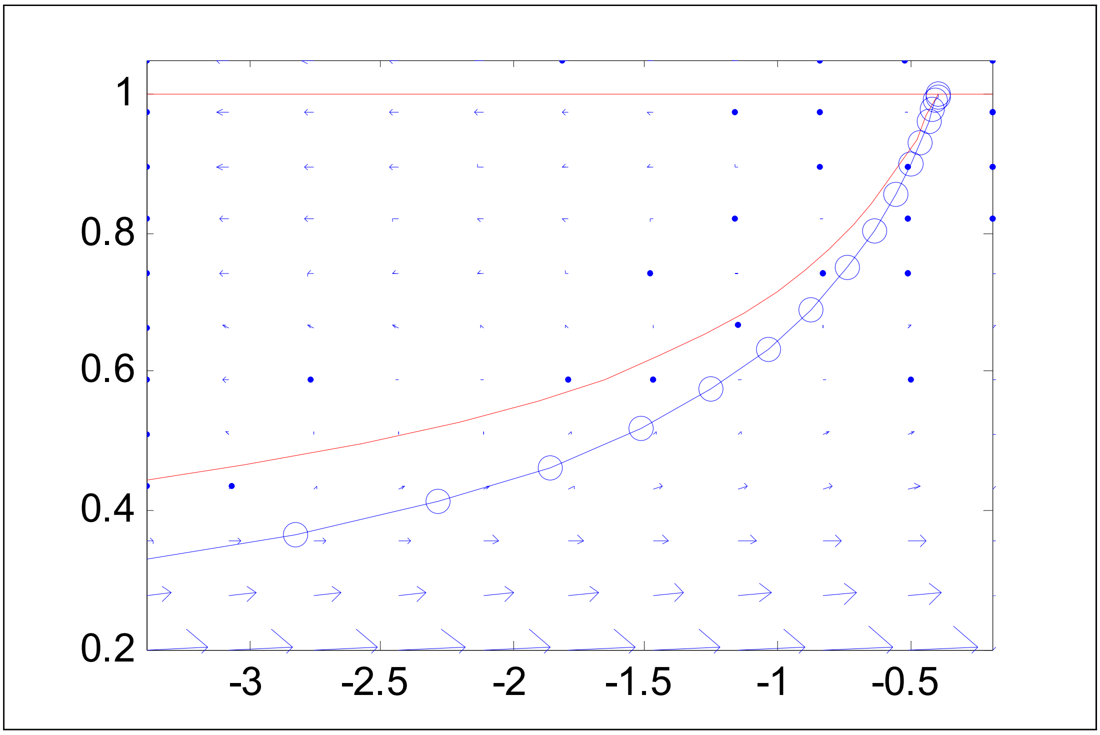

Figure 4: Phase portrait and saddle path in market equilibrium when L is large.

If on the other hand L is less than Δ/z we obtain the following Figure 5:

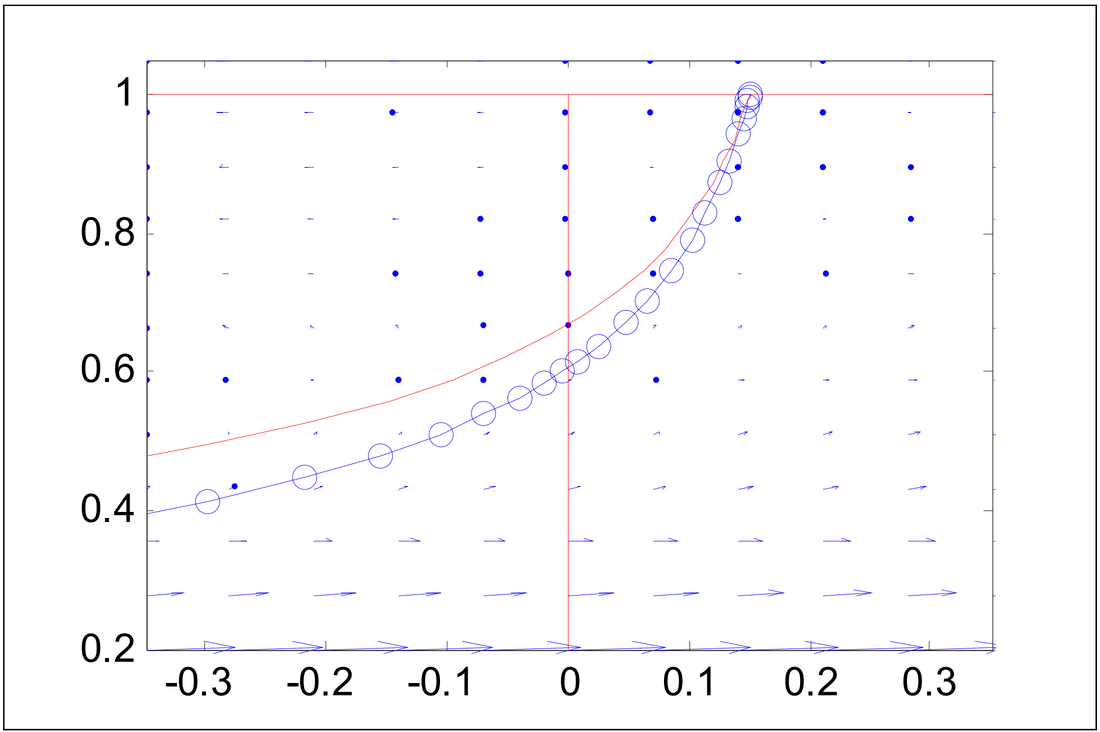

Figure 5: Phase portrait and saddle path in market equilibrium when L is small or zero.

The most important issue in this section is the comparison between the dynamic equilibrium market share and the socially optimal market share. Since in both cases β is monotonically decreasing over time it is difficult to establish direct comparisons. A first attack is to compare the steady states:

Proposition 3 *If the number of informed customers is sufficiently large then if the number of captive consumers is large (small), i.e. $L > (<) \Delta/z$, the entrant's market share β in market equilibrium is higher (lower) than the socially optimal market share.*

Proof. We have $\beta = \frac{1}{2} + \frac{1}{6z}\Delta + \frac{1}{3}L$ in the market steady state, and $\beta = \frac{1}{2} + \frac{\Delta}{2z}$ in the socially optimal steady state. The statement follows from continuity. ■

We find the expected: If the incumbent has a large group of captive consumers then he prefers to set high prices instead of fighting the entrant (remember that $L > 0$ if the incumbent is subject to uniform pricing). As result, diffusion is even more rapid than in the social optimum, but this is more than counterbalanced by the additional transport cost of the entrant's customers who are close to the incumbent on the Hotelling line.[4] If on the other hand there are only few captive customers, or if the incumbent is not subject to uniform pricing, then the incumbent prices more aggressively than would be socially optimal to achieve the optimal path of diffusion. This not a phenomenon of limit pricing, but rather a competitive response that arises from the incumbent realizing that preserving market share today will slow down the future cost of customers.

(Do some numerical simulations here).

5 Advertising

Assume the entrant can spend an amount $a^2/2k$ at each point in time on advertising, which influences the diffusion process in the following way:

$$\dot{S} = (\alpha\beta S + a)(1 - S).$$

The variable a measures the percentage of uninformed non-captive customers that are effectively informed when they are reached by an advertisement. How does this possibility change the pattern of diffusion?

With shadow values η_i, the entrant's current-value Hamiltonian becomes

$$H_2 = \beta S (p_2 - c_2) - a^2/2k + \eta_2 (\alpha\beta S + a)(1 - S), \tag{28}$$

and the optimal values of his controls p_2 and a are described by

$$\begin{aligned} \frac{\partial H_2}{\partial p_2} &= -\frac{1}{2z} S (p_2 - c_2) + \beta S - \eta_2 \alpha \frac{1}{2z} S (1 - S) = 0. \qquad (29) \\ \frac{\partial H_2}{\partial a} &= -a/k + \eta_2 (1 - S) = 0. \end{aligned}$$

That is, the optimal advertising intensity is $a = k\eta_2 (1 - S)$, and equilibrium prices are given by (20), where the μ_i are substituted by η_i. These prices do therefore only indirectly depend on a, through S and η_i, and the optimal market share is as above,

$$\beta = \frac{1}{6z} \left(z \left[1 + 2\frac{1 + L}{S} \right] + \Delta + \alpha\eta (1 - S) \right), \tag{30}$$

[4] If demand were elastic we would encounter the additional effect of less than efficient quantities due to highe prices.

where $\eta = \eta_1 + \eta_2$. The question about how advertising affects diffusion boils down to asking how it influences the shadow value η.

The co-state equations are

$$\begin{aligned}\frac{\partial H_1}{\partial S} &= -\beta\,(p_1 - c_1) + \eta_1 \alpha\beta\,(1-2S) - a\eta_1 = \delta\eta_1 - \dot{\eta}_1, \\ \frac{\partial H_2}{\partial S} &= \beta\,(p_2 - c_2) + \eta_2 \alpha\beta\,(1-2S) - a\eta_2 = \delta\eta_2 - \dot{\eta}_2,\end{aligned} \tag{31}$$

so that the equations of motion for (η, S) become

$$\begin{aligned}\dot{\eta}_2 &= (\delta + k\eta_2\,(1-S))\,\eta_2 - \frac{1}{18z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\,(1-S)\,\eta\right) \\ &\quad \times \left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\,(1-S)\,\eta - 3\alpha S\eta_2\right) \\ \dot{\eta} &= (\delta + k\eta_2\,(1-S))\,\eta - \frac{1}{9z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\,(1-S)\,\eta\right) \\ &\quad \times \left(z\left[1-\frac{1+L}{S}\right] + \Delta + \alpha\left(1-\frac{5}{2}S\right)\eta\right) \\ \dot{S} &= \left(\frac{\alpha}{6z}\left(z\left[1+2\frac{1+L}{S}\right] + \Delta + \alpha\eta\,(1-S)\right)S + k\eta_2\,(1-S)\right)(1-S)\end{aligned} \tag{32}$$

The steady states are the same as before: Advertising has no influence since in the limit $S \to 1$ it is ineffective. On the other hand, in both co-state equations the effect of advertising is to raise the effective discount rate from δ to $\delta + a$, which means that the industry behaves as if it had become more impatient. This effect should lead to a reduction in competitive intensity because making profits now gains in relative importance as compared to increasing S. Still, S tends to increase more due to the direct effect of advertising, therefore it is ambiguous where diffusion is slower or faster.

Our intuition would be that advertising crowds out low prices as a means of gaining market share, therefore one would expect to see higher prices, a lower β and a smaller η on the equilibrium path.

This analysis is obviously not yet complete (e.g. the welfare effects of advertising should be discussed), and will be pursued further, at least numerically.

6 Regulatory Constraints on Pricing

We have analyzed above the market equilibrium assuming that the incumbent has $L \geq 0$ captive customers and that uniform pricing is imposed on him. For

$L = 0$ the above considerations can be interpreted as not involving uniform pricing: In the latter case the captive consumers are irrelevant for pricing under competition. Thus the above finding applies that in this case diffusion is slower than socially optimal because the incumbent prices too aggressively. The uniform pricing constraint makes the incumbent less aggressive, but, as shown above, may lead to "overshooting" in that for large L diffusion may be too fast as compared to the social optimum.

Assume now that a regulator can impose on the incumbent that his price should be equal to p_0. This restriction turns the incumbent into a passive by-stander, and only the entrant makes strategic decisions. His current value Hamiltonian is

$$H_2 = \beta S (p_2 - c_2) + \nu\alpha\beta S (1 - S) \tag{33}$$

and the optimal prices and market shares are

$$\begin{aligned} p_2 &= \frac{1}{2}(c_2 + z + p_0 - \nu\alpha (1 - S)) \\ \beta &= \frac{1}{4}\frac{z + p_0 - c_2 + \nu\alpha (1 - S)}{z} \end{aligned} \tag{34}$$

If we compare with the socially optimal market share (11) is becomes immediately clear that if p_0 were to be chosed to reach the social optimum it cannot be fixed but must be a function of time.

The adjoint equation is

$$\frac{\partial H_2}{\partial S} = \beta (p_2 - c_2) + \nu\alpha\beta (1 - 2S) = \delta\nu - \dot{\nu}, \tag{35}$$

and both equations of motion are, with $P = p_0 - c_2$,

$$\begin{aligned} \dot{\nu} &= \delta\nu - \frac{1}{8z}(z + P + \nu\alpha (1 - S))(z + P + \nu\alpha (1 - 3S)) \\ \dot{S} &= \frac{\alpha}{4z}(z + P + \nu\alpha (1 - S)) S (1 - S) . \end{aligned} \tag{36}$$

The steady state is

$$\begin{aligned} S^* &= 1, \nu^* = \frac{1}{2}\frac{(z + p_0 - c_2)^2}{4\delta z + \alpha (z + p_0 - c_2)} \\ \beta^* &= \frac{1}{4} + \frac{1}{4z}(p_0 - c_2) . \end{aligned}$$

To achieve the socially optimal market share in steady state p_0 must be equal to

$$p_0 = c_1 + (\Delta + z) .$$

That is, the socially optimal price imposed on the incumbent must be high enough.

(in future, more numerical results here)

7 Consumer search

In this section we will give a more active role to consumers, at least to the ones who are not captive to the incumbent: These consumers can decide whether to pay a search cost k and get informed immediately, or wait until they meet someone who tells them.

A fundamental question in this approach is: What do consumers know, and what do they believe? We see our analysis as a starting point, and will assume rational expectations over all relevant variables. That is, even though consumers do not "know" the entrant's prices or market share, or even the number of informed customers, they have correct expectations about these variables. Obviously this is a very strong assumption, but is needed here to make the question well-defined since without this information consumers cannot assign values to waiting or searching.[5]

7.1 Market Equilibrium

We state our result first, then discuss the intuition, and only then introduce the formal model.

Proposition 4 *For moderate search cost all consumers on some interval $[0, x^*]$ search, and all search occurs at time 0. That is, this group of consumers becomes informed immediately, while all other consumers prefer to wait until someone else informs them.*

It is not surprising that all consumers situated close to the entrant search, because these have higher transport cost when they buy from the incumbent. All else equal, they have more to gain from switching to the entrant. The interesting result is that all searching occurs right at the beginning. This happens for two reasons, which depend on the paths of prices discussed above.[6]

[5]An alternative would be to assume that consumers only expect that the entrant will not be more expensive than the incumbent (but have no beliefs over the price difference). In this case a consumer searches if he is close enough to the entrant: $zx + \delta k \leq z(1-x)$, or $x \leq 1/2 - \delta k/2z$. This searching occurs immediately, so that qualitatively we arrive at the same result as below.

[6]Strictly speaking, it is still necessary to prove that the qualitative properties of price paths are the same under the new laws of motion under consumer search. This is indeed the case.

First, the value of searching decreases over time since the entrant's price rises, that is the best deals are to made right after entry. Second, the value of waiting increases as the incumbent's price decreases, and the number of informed consumers increases: Waiting costs less, and the expected time until meeting an informed customers becomes shorter.

Let us now give the details of the model. We take the paths of prices p_1, p_2, informed customers S, market shares β, and number of searchers x^* as given. Of the latter we assume that only consumers search who expect to adhere to the entrant.

The surplus of a consumer located at x if he searches at time t and then buys from the entrant is:

$$\begin{aligned} V_s(t) &= \int_t^\infty (v - p_2 - zx)\, e^{-\delta(\tau - t)} d\tau \\ &= \int_0^\infty (v - zx)\, e^{-\delta\tau} d\tau - \int_t^\infty p_2(\tau)\, e^{-\delta(\tau-t)} d\tau \\ &= \frac{v - zx}{\delta} - P_2(t)\,. \end{aligned} \tag{37}$$

Rational expectations are incorporated here through the assumption that the consumer correctly anticipates the whole path of prices $p_2(.)$. Over time V_s decreases if the entrant's price increases:

$$\dot{V}_s = -\int_0^\infty \dot{p}_2(\tau + t)\, e^{-\delta\tau} d\tau. \tag{38}$$

The value of waiting is given by the following HJB equation, with $v_1 = v - p_1 - z(1-x)$,

$$\delta V_w - \dot{V}_w = v_1 + r(V_s - V_w)\,, \tag{39}$$

where the customer gets informed at the variable rate $r = \alpha(x^*(1-S) + \beta S)$. We assume that the consumer has correct expectations about the paths of p_1, β, S and x^*, where the latter are the consumers who already have searched. We can treat the HJB equation as a linear ordinary differential equation

$$\dot{V}_w = (\delta + r)\, V_w - (v_1 + rV_s)\,. \tag{40}$$

Since $(\delta + r) > 0$ while the steady state must be finite, the solution is

$$V_w(t) = \int_t^\infty e^{-\int_t^\tau (\delta + r) dy} (v_1 + rV_s)\, d\tau. \tag{41}$$

After substituting V_s and v_1 and some further transformations we obtain

$$\begin{aligned} V_w(t) &= \frac{v - zx}{\delta} + z(2x-1)\int_t^\infty e^{-\int_t^\tau (\delta + r) dy} d\tau \\ &\quad - \int_t^\infty e^{-\int_t^\tau (\delta + r) dy} (p_1 + rP_2)\, d\tau \\ &= \frac{v - zx}{\delta} + z(2x-1) T_1(t) - T_2(t), \end{aligned} \tag{42}$$

with $T_i(.) > 0$.[7]

Comparing V_s and V_w we obtain the result that a consumer at x searches at time t if and only if

$$x \leq \tilde{x}(t) = \frac{1}{2} - \frac{k - (T_2(t) - P_2(t))}{2zT_1(t)}. \tag{43}$$

The right-hand side decreases in t, at least if $x^*(t) < 1/2$, since T_1 and T_2 are decreasing, and P_2 is increasing over time. This means that $\tilde{x}(0) > \tilde{x}(t)$ for all $t > 0$, and all customers that are ever willing to search do so at time zero: $x^*(t) = x_0$ for all t.

The number of searching customers is given by the implicit equation

$$x_0 = \tilde{x}(0) = \frac{1}{2} - \frac{k - (T_2(0) - P_2(0))}{2zT_1(0)}, \tag{44}$$

where the implicit dependence is directly through r, and more hidden through the equilibrium paths of p_1, p_2, β and S.[8] While this means that x_0 can only be found through simulations, we can finally state the new law of motion under consumer search, given x_0: The number of uninformed customers that are not captive of the incumbent are $(1 - x_0)(1 - S)$, and these customers can meet the x_0 customers who have searched and switched to the entrant, or the $(\beta - x_0) S$ customers who became informed through meetings. Therefore the new law of motion is

$$\dot{S} = \alpha (x_0 + (\beta - x_0) S)(1 - x_0)(1 - S). \tag{45}$$

With this law of motion, and the corresponding versions of each firm's objective function, for each x_0 the equilibrium can be found, and one can iterate on (44) until x_0 is determined.

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[7] It may help intuition that if r, p_1 and p_2 were constant we would have $T_1 = 1/(\delta + r)$, $P_2 = p_2/\delta$, $T_2 = (p_1 + rp_2/\delta)(\delta + r)$.

[8] As of yet we cannot show that there is a unique solution.

7.2 Social Optimum

It would be useful to determine the socially optimal number of searching consumers x_s. It is clear that all search should again take place in the beginning since there is no social gain from waiting. On the other hand, there is a trade-off between faster spread of information and higher search costs, so the optimal value of search will be finite.

The problem to solve is of the type

$$\begin{aligned} &\max_{\beta(.),x_s} \int_0^{\infty} e^{-\delta t} F\left(S, \beta, x_s\right) \, dt - kx_s \\ &s.t.\ \dot{S} = f\left(S, \beta, x_s\right) \end{aligned} \tag{46}$$

with initial condition $S(0) = S_0$. In this problem x_s acts as a constant control variable. Alternatively, we can interpret x_s as a constant state variable:

$$\begin{aligned} &\max_{\beta(.),x_s} \int_0^{\infty} e^{-\delta t} F\left(S, \beta, x_s\right) \, dt - kx_s \\ &s.t.\ \dot{S} = f\left(S, \beta, x_s\right) \\ &\quad \dot{x}_s = 0 \end{aligned} \tag{47}$$

Here x_s has associated shadow price μ, and the transversality condition corresponding to the free initial value of x_s is $\mu(0) = k$.

(to be continued)

8 Conclusions

We have described the socially optimal and market equilibrium processes of diffusion of a new (telecoms) service when only informed customers consider switching to the entrant, and when knowledge spreads mainly through interactions between customers.

Competitive pricing leads to slower diffusion of the entrant's new service than is socially optimal if no uniform pricing is imposed on the incumbent, or if only a small part of the population is captive to the incumbent. The opposite is true if there is a large number of captive consumers who will never consider switching to the entrant. This poses a dilemma for the regulator: On the one hand, imposing uniform pricing on the incumbent accelerates the process of entry and diffusion, but on the other leads to higher prices and makes some customers buy from the entrant who in a first-best situation should buy from the incumbent.

Advertising may accelerate diffusion, but lead to higher equilibrium prices. This is so because these prices have a strategic in accelerating (entrant) or slowing down (incumbent) diffusion, both of which will leads to lower prices. They are crowded out in this function by advertising and therefore prices rise.

As concerns the imposition of mandatory prices on the incumbent by a regulatory entity, we found that the optimal price will be time-dependent, and that it must be high enough make diffusion socially optimal.

Future research will be concerned with analyzing interconnection and price discrimination; market models other than Hotelling (which has its weaknesses); and consumer choice: would consumers passively wait until they meet some friend, or will they actively search for information? If so, what are the welfare consequences?

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Appendices

Appendix 1: The Social Optimum
The stability matrix around the steady state (S^*, λ^*) is

$$\begin{bmatrix} \delta + \frac{\alpha}{2}(1+\Delta/z) & \frac{z\alpha(8\delta+3\alpha(1+\Delta/z))(1+\Delta/z)^3}{8(2\delta+\alpha(1+\Delta/z))^2} \\ 0 & -\frac{1}{2}\alpha(1+\Delta/z) \end{bmatrix},$$

with positive eigenvalue $\delta + \frac{\alpha}{2}(1+\Delta/z)$ and negative eigenvalue $-\frac{1}{2}\alpha(1+\Delta/z)$. This means that this steady state is saddlepoint stable. Furthermore, the stable eigenvector points upwards and left, which means that the saddlepath approaches the steady state from the right. On the other hand, the eigenvalues of the stability matrices at the other two steady states can be shown to be all positive, thus these are instable.

The curve $\dot{\lambda} = 0$ is given by

$$S(\lambda) = \frac{1}{3\lambda\alpha}\left(2(z+\Delta+\lambda\alpha) - \sqrt{(z+\Delta+\lambda\alpha)^2 + 12z\delta\lambda}\right),$$

which is always well-defined and approaches the steady state from the right. This curve also divides the space (λ, S) into two halves, with $\dot{\lambda} < 0$ below it, and $\dot{\lambda} > 0$ above it. Since the saddlepath approaches the steady state from the right, at least close to it it must lie below this curve. But since S is always increasing and the curve $\dot{\lambda} = 0$ does not bend backwards to lower values of λ, the whole of the saddle path lies below the curve $\dot{\lambda}$. In other words, $\dot{\lambda} < 0$ on the whole saddle path.

If $\delta \geq \alpha(1+\Delta/z)$ then the curve $\dot{\lambda} = 0$ cuts the horizontal axis at the two additional steady states, while for $\delta < \alpha(1+\Delta/z)$ it either has a minimum somewhere or continues to decrease monotonically; in both cases in the limit $\lambda \to \infty$ it approaches $S = 1/3$.

Therefore for $\delta \geq \alpha(1+\Delta/z)$ the saddle path emanates from the vicinity of the steady state

$$S = 0, \lambda_0 = \frac{2z\delta - \alpha(z+\Delta) - 2\sqrt{(z\delta - \alpha(z+\Delta))z\delta}}{\alpha^2},$$

where the market share is

$$\beta_0 = \frac{\delta}{\alpha} - \sqrt{\frac{\delta}{\alpha}\left(\frac{\delta}{\alpha} - (1+\Delta/z)\right)} \leq 1$$

For $\delta < \alpha(1+\Delta/z)$ it simply follow below the curve $\dot{\lambda} = 0$. Here must take into account that the curve $\beta = 1$ cuts off this part of the saddle path, and that to its right it takes on a different form.

Appendix 2: The Market Equilibrium

First we will show that the steady state (S^*, μ^*) is saddle-path stable. The stability matrix at the steady state has shape

$$\begin{bmatrix} \frac{\partial \dot{\mu}}{\partial \lambda} & \frac{\partial \dot{\mu}}{\partial S} \\ \frac{\partial \dot{S}}{\partial \lambda} & \frac{\partial \dot{S}}{\partial S} \end{bmatrix} = \begin{bmatrix} \delta + \frac{\alpha}{6}\left(3 + 2L + \Delta/z\right) & T \\ 0 & -\frac{\alpha}{6}\left(3 + 2L + \Delta/z\right) \end{bmatrix},$$

where

$$T = -\frac{z}{27}\frac{108(1+L)(4L+3-\Delta/z)\delta^2 + 6\alpha(13L+6-7\Delta/z)(3+2L+\Delta/z)^2\delta + \alpha^2(8L+3-5\Delta/z)(3+2L+\Delta/z)^3}{(6\delta+\alpha(3+2L+\Delta/z))^2}$$

with positive eigenvalue $\delta + \alpha\left(\frac{1}{2} + \frac{1}{3}L + \frac{1}{6}\Delta/z\right)$ and negative eigenvalue $-\alpha\left(\frac{1}{2} + \frac{1}{3}L + \frac{1}{6}\Delta/z\right)$. Therefore this steady state is indeed saddle-path stable. The eigenvector associated with the latter, stable, eigenvalue points upwards to the right, therefore the saddlepath approaches the steady state from the left.

The curve of $\dot{\mu} = 0$ is increasing and divides the space (μ, S) into two halves. In the lower half $\dot{\mu}$ is positive, and in the upper half it is negative. This means that as the saddle path approaches the steady state it must eventually be below this curve. For this very reason, and since S is increasing on every path, it can never have been above it. Therefore on the saddle path $\dot{\mu} > 0$.

As concerns market share β, from (21) we can calculate

$$\dot{\beta} = \frac{1}{54}\frac{\alpha}{S^2}\left(1 - S\right)\frac{9\delta\mu z S^2 - \left(S\left(z + \Delta\right) + 2z\left(1 + L\right) + \alpha\left(1 - S\right)\mu S\right)^2}{z^2},$$

which is certainly negative while $\mu < 0$. On the other hand it is also negative at the steady state (μ^*, S^*), therefore $\dot{\beta}$ remains negative for positive μ.