Competing with Network Externalities and Price Discrimination*

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Abstract

The paper examines a competitive game between a dominant network and a challenging network with third-degree or perfect price-discrimination. Domination is defined in terms of reputation, and captured through an adequate resolution of the coordination game played by consumers. Allowing for any configuration of network externalities, the paper analyses the equilibria of the Stackelberg game and the simultaneous pricing Bertrand game. The general conclusion is that price-discrimination drastically reduces average prices and may generate substantial cross-subsidization. Moreover a network cannot extract any surplus generated by the network effects between groups. In the extreme case of perfect price-discrimination, the only pure strategy equilibrium outcome is that the most efficient network covers the market. We show also that this may generate excess momentum, inefficient market sharing, and that both networks benefits from becoming compatible.

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1 Introduction

While there is already a substantive literature on competition with network effects (see Katz and Shapiro (1994) and Economides (1996)), one feature of modern networks that has not received considerable attention is that network effects are often not isotropic: members may join for different reasons and value both the service and the participation of others in very different ways. In conjunction with that, and partly because of that, most suppliers of network goods practice some form of price-discrimination. For instance, telecommunication operators or software suppliers discriminate between residential and professional users, as well as geographic areas. As another example, consider the case of a provider of intermediation services in a vertical industry. Then downstream firms care about the upstream firms in the network while the reverse hold for upstream firms. This is sometime referred to as the "chicken and eggs" problem, the issue being to build a market strategy that convince both sides to join. Moreover the intermediaries propose very different deals to buyers and sellers. At the conceptual level one may view these situations as involving differentiated users of the network and price-discrimination. This general set-up covers a wide range of activities, involving either indirect network effects, or direct network effects with different types of participants. To give a few, it includes intermediation activities, telecommunication services, postal services, multi-media services, advertising, Internet services, operating systems and software applications, research centers,....

It is well known that in the presence of network externalities, consumers face a coordination problem in their purchasing decision that may generate multiple equilibria (Katz and Shapiro (1985)). A key determinant of the outcome of competition is the consumers confidence on the ability of a network to grow. Successful entry then requires to persuade the consumers of potential of the new network to attract a large clientele.[1] When consumers tend to perceive established firms as focal and coordinate on it, this effect, that can be interpreted as a reputation effect, creates barriers to entry. This is indeed the main motive for fast entry and aggressive strategies in building a customer base in infant stages of network industries.

What seems less acknowledged is that the combination of network effects and price-discrimination creates a very particular environment as it allows for aggressive strategies that subsidize some consumers to join and exploit the network externalities to recover the subsidy on other consumers

[1] See Arthur (1989), David (1985), Farell and Saloner (1986), Katz and Shapiro (1986,1992).

(a divide-and-conquer strategy)[2]. Such a strategy allows a network to overcome the coordination problem by passing part of the surplus to targeted customers and creating a bandwagon effect. This is the static version of the dynamic mechanism that leads a network to subsidize earlier customers in an attempt to build the customer base.[3] Because these strategies only make sense with network effects, the implications for market conduct are still to be investigated and may lead to revisit the conventional wisdom on both price-discrimination and network competition. One object of this paper is to show that price-discrimination has a strong pro-competitive effect, and that, as a result, markets for network goods may be subject to a higher competitive pressure than markets for standard goods.

The paper studies the interaction between the reputation effects at work with network externalities and the new complex marketing strategies that emerge when a differential price treatment of consumers is possible. It examines a competition game between an established network and a challenging network, where the former benefits from a strong reputational advantage. The population is decomposed into groups and third degree price-discrimination is possible. Network effects may occur within groups (intra-group) and between groups (inter-group). Different groups may have different valuations for the goods, as well as for the participations of other groups to the network. This allows for a wide range of differentiation (both horizontal and vertical) and of network effects.

The reputation of the established network is modeled by selecting a particular equilibrium of the subgame played by consumers when they choose where to buy, which can be interpreted as a particular choice of consumer's equilibrium beliefs. With positive network externalities, there is a well defined selection criterion that captures the idea of a reputation advantage for the established network, which amounts to impose that a consumer will buy from the established network whenever this is the outcome of at least one equilibrium of the allocation game. This is referred to as Domination in Beliefs and it maximizes the dominant firm market share for any price configuration.

The paper characterizes the equilibria of the Stackelberg game and of the simultaneous pricing game with a dominant network (the consequences for the case where is no dominant network are also derived). It then discusses the implications for technological choices such as compatibility or quality, and for efficiency.

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[2] See Innes and Sexton (1993) for similar ideas exploiting increasing returns to scale.

[3] See Bensaid and Lesne (1996) and Cabral, Salant and Woroch (1999) applications to dynamic monopoly pricing.

To overcome its reputation disadvantage, a newcomer relies on divide-and-conquer strategies as described above. This means that the nature of competition is changed in a non-trivial manner compared to uniform pricing, due to the possibility of cross-subsidizing. Firms will target some consumers by offering advantageous conditions, and the outcome of the market game may exhibit a substantive amount of cross-subsidization if there are asymmetries in the network effects.

This forces the established firm to set on average prices at a much lower level than it would do with uniform prices. It turns out that it is impossible for a network to capture in equilibrium the surplus generated by the inter-group network externalities.

The intuition behind the result is the following. Consider two firms I and C (with zero cost) competing for two consumers A and B. Suppose that A and B have identical valuations for the two network goods, and that they receive an extra value v if they both join the same firm. With uniform prices, there is an equilibrium where one firm (say I) sells to both consumers at price v. The reason is that for any positive price $p^C > 0$ set by C, the value of joining I together at price v is higher than the value of buying alone from C at price p^C. Now suppose that price-discrimination is allowed and that I charges a uniform price p^I to both consumers. Then the competitor C can react as follows: it charges a price below $p^I - v$ to consumer B (divide) and a price $p^I + v$ to consumer A (conquer). With such a price structure, B joins firm C irrespective of what A does because in relative terms, C offers him a value larger than the value attached to the participation of A (it is a dominant strategy for B to join C). A must then choose between buying alone from I and joining B in the network of C. He is thus willing to pay the premium v to join firm C. Notice that C achieves to serve the two consumers irrespective of the nature of the coordination process that determinates the allocation of consumers, and that it obtains the profit that would accrue to I if the two consumers joined I. The result is that equilibrium profits vanish in equilibrium.

In particular this reasoning will imply that with an homogeneous population and perfect price discrimination, any pure strategy equilibrium involves an efficient allocation of consumers, which stands at odds with inefficiency results that emerge with uniform pricing.

Competition is intense because firms attach a positive value to each consumer equal to the extra profit that his participation allows to generate on other consumers. In a sense, with network goods, each individual is not only a consumer but also an input. It is an input because this is the mere fact of consuming that creates value for other consumers. As an input, the consumer is a scarce resource. Firms then compete to sell to consumers but they also compete to buy the inputs, i.e. the participation of targeted individuals. Price-discrimination exacerbates this dual nature of competition and as a consequence leads to very low equilibrium profits.

The effect is even stronger when inter-group network effects are asymmetric. Consumers who value less the participation of others then becomes the object of an intense competition as they are more value-enhancing relatively to others. To see that suppose that B doesn't care for the participation of A. Then the firm that succeeds to sell to B gains a competitive advantage on A; equal to A's willingness to pay to join B. The competition for B then dissipates the profit, and may even prevent the existence of a pure strategy equilibrium.[4]

One consequence of this intense competitive process is that, when intra-group network effects are small and in particular when perfect price discrimination is feasible, both firms are better off by making their networks compatible, even a firm that benefits from a strong reputational advantage. With (almost) perfect price-discrimination, a competing network can build a very aggressive strategy that at the same time solves the coordination problem between consumers and transfers to them the value of the network externalities. This leads incompatible networks to be extremely aggressive in building market shares, while being enable to capture the extra surplus generated by network effects. Compatibility, by suppressing the interaction between the relative size of each network and the willingness to pay of consumers, eliminates the motive for cross-subsidization. Provided that the network goods are differentiated, even slightly, once the needs to enlarge a firm's market share disappear, the benefits of differentiation are restored, with a weakening of the competitive pressure and market segmentation. Unless one firm can exploit large intra-group network effects, a peaceful exploitation of differentiation by compatible networks is then more profitable than a head-to-head confrontation to conquer the whole market by incompatible networks.

The second consequence is that price-discrimination may be the source of excess momentum, with consumers switching to the challenger's network, while this is inefficient. This is at odds with standard results because the model is solved under assumptions that generate excess inertia when prices are uniform since the incumbent benefits from the most favorable treatment by consumers. Even when the incumbent offers a uniformly higher quality, it may fail to cover the market with probability one when network effects are asymmetric. The reason is that the challenger can capture a positive share of the value of network effects with a divide-and-conquer strategies. If the

[4] This is one of the reason for which the paper studies both the sequential pricing game where existence is granted, and the simultaneous pricing game.

quality differential in favor of the incumbent is smaller than this share, it will fail to cover the market.

A second type of excess momentum occurs when the incumbent is a Stackelberg leader. If the two networks are horizontally differentiated, it may be profitable for the incumbent to accommodate some challenger's sales by not competing for some group. By giving away consumers with a high valuation of the network effects, the leader reduces the aggressiveness of the challenger in the competition for the other types of users, as otherwise the challenger would fight for the whole population as a way to attract these particular valuable consumers. There is an excess incentive to segment the market: the market may be shared between the two firms while it would be more efficient to regroup all consumers in the dominant network.

A final point that emerges is that networks' quality choices may not be efficient. A network may in particular degrade the quality it offers to some targeted groups (and not for others), as a mean to increase the degree of horizontal differentiation and to induce market sharing. This can be viewed as a commitment device for the network, which doing so, commits not to compete for this group.

The paper is organized as follows. Section 2 presents the general model along with the notion of reputational advantage. Section 3 presents some preliminary key results. Section 4 presents a full-fledged analysis of the case with two groups. Section 5 discusses the results, in particular compatibility choices, quality choices and efficiency. Section 6 then analyses the situation where the population is homogeneous.

2 The model

2.1 A monopoly network

Consider first an incumbent sole supplier of a network good with a production cost normalized to 0; denoted I: The good is consumed by J different types of users, each represented by a group composed of a continuum of homogeneous consumers with mass m_j: The set of groups is denoted J: The perceived quality of the good varies across types of users, denoted u_j for type j users (intrinsic value). One may view the situation alternatively as one where various users uses a network for different purpose, or one in which the network can discriminate between types of users with different valuation for the same service. Network effects are allowed to be asymmetric, meaning that the consumers of a given group may value differently the participation of every other group to their network. Denoting by n^I_j the mass of consumers of type

j buying from i; the valuation of a consumer j for the participation of n_l^i members of group l is $\beta_{jl} n_l^i$. The coefficients β_{jl} are nonnegative and capture the inter-group network effects. There is also a network effect within groups (intra-group), captured by a nonnegative coefficient α_j. The case $\alpha_j = 0$ can be interpreted as a situation where j is a single individual, and thus one of perfect price discrimination.[5] Overall the gross utility that a type j consumer derives from consumption of the network good is

$$U_j^i = u_j + \alpha_j n_j^i + \sum_{l \neq j} \beta_{jl} n_l^i .$$

Firm i is able to charge a different price p_j^i for each group, and thus choose a vector of J prices $P^i = \{p_1^i; \ldots; p_J^i\}$. Given these prices, each consumer decides whether to buy or not. We assume that they coordinate on a rational expectation equilibrium of this allocation game.[6] Due to network effects, there may be a multiplicity of such equilibria. Indeed, with network effects, one consumer's behavior depends on his expectation about the other consumers' behavior. Thus there may be situations where both all consumers buying and no consumer buying are equilibria.

The maximal price that can be set by a monopoly firm is

$$\bar{p}_j^i = u_j + \alpha_j m_j + \sum_{l \neq j} \beta_{jl} m_l .$$

At these prices, each consumer is willing to buy provided that it anticipates that all others do. $\bar{p}_j^i$ is the maximal value that a consumer can expect. Since any price above this level is just equivalent to it, I shall from now on impose that prices are below the maximal monopoly price.

If beliefs are extremely unfavorable, the monopolists must set at least one price p_j^i below u_j. This is because when all prices are above the intrinsic value of the service, there is an equilibrium market allocation of customers where none of them buy. Suppose that $p_1^i = u_1$; then a member group 1 buys irrespective of what the others are doing. Given that, i can set a price $p_2^i = u_2 + \beta_{21} m_1$ and make sure that group 2 buys as well. Indeed for this price, since group 1 joins, a member of group 2 joins irrespective of what the others are doing. Using this reasoning recursively, we see that i can conquer

[5] The model thus also addresses the situation where there are some large users along with a population of small users. However, large users do not act strategically in the sense that they don't take into account the impact of their decision on the choice of other consumers. This could be addressed by using a sequential process of choice for consumers.

[6] Each individual's consumption decision maximizes his utility given the prices and the equilibrium allocation of other consumers, inculding others members of its group.

the market by setting prices $p^I_j = u_j + \sum_{l<j} \beta_{jl}\mu_l$, even faced with the least favorable market conditions. Obviously the reasoning doesn't depend on the order of the group so that, if $\sigma(.)$ denotes an order (a permutation) on the set of the groups, prices $p^I_j = u_j + \sum_{\sigma(l)<\sigma(j)} \beta_{jl}\mu_l$ allow I to conquer the market as well.

Total monopoly profit is thus bounded above by $\sum_j p^I_j \mu_j$, and below by $\sum_j u_j\mu_j + \max_{\sigma(.)} \left\{\sum_{\sigma(l)<\sigma(j)} \beta_{jl}\mu_l\mu_j\right\}$. Notice that even in the worst case, the monopoly can extract part of the value of network externalities. As an example consider a monopolist facing J identical customers, which corresponds to $\mu_j = 1$, $u_j = u$, $\alpha_j = 0$ and $\beta_{jl} = \beta$. With a uniform price, the minimal monopoly profit reduces to Ju. But if the monopolist is able to discriminate between consumers, it can charge prices u, $u+\beta$, $u+2\beta$, ... to different customers and raise its profit by $\frac{J(J-1)\beta}{2}$, half of the total value of the network effects.

The general conclusion from this section is that price-discrimination may serve two purposes for a monopoly network. First it may serve the standard purpose of increasing the rent extracted from high value users, and of extracting some rent from low value users who would not buy under uniform pricing. An indirect benefit is that by inducing the participation of low value users, the network enhances the value for other users. The second benefits from price-discrimination is that it may helps the network to overcome problems due to coordination failure, while capturing part of the surplus.

2.2 A challenged network

Suppose now that the incumbent I competes with a challenger C for the provision of network goods. The goods are differentiated by levels of perceived quality. Denoting by n^C_j the mass of consumers of type j buying from C, the valuation of a consumer j for good C is then

$$U^C_j = u^C_j + \alpha_j n^C_j + \sum_{l\neq j} \beta_{jl} n^C_l.$$

With such a formulation, the model is compatible with both vertical differentiation ($u^k_j > u^{-k}_j$ for all j) and horizontal differentiation ($u_j - u^C_j > 0 > u_l - u^C_l$).

This paper focuses on the case where the network externalities at stake are not too large compared to the intrinsic value of the service. More specifically it is assumed that:

Assumption 1: For all j, $\min\{u_j, u^C_j\} \geq \sum_{l\neq j} \beta_{jl}\mu_l$.

To interpret the condition, one should keep in mind that the network effects in the model are measured at the firm level not at the industry level. To give an example, internet services like Email or webpage hosting involves some network externalities at the firm level (there are some benefits of being both members of Yahoo) and others at a more general level (the World Wide Web). Here the intrinsic value should be viewed as including the network effects that are fixed or global. Another example is the case of mobile telecommunication operators or internet access providers competing on a local market. The intrinsic value then incorporates the value of the connectivity to the global market, while the coefficients β_{jl} capture the interactions at the local level. The situation may also be interpreted as one with partially compatible networks and imperfect interconnection: β_{jl} then captures the relative benefits from being in the same network and depends on the quality of interconnection.[7]

This assumption ensures that firm I can't expect to sell to group j at a price above $u_j + \alpha_j m_j$: Indeed if I sets $p^I_j > u_j + \alpha_j m_j$, the maximal utility that a member of group j can obtain with I is $U^I_j < \sum_{l \neq j} \beta_{jl} m_l \leq u^C_j$: But then C could sell to group j with a small positive price and would do so. This will unable us to restrict attention to prices below $u_j + \alpha_j m_j$, the "stand-alone" value that group j obtains with I when it is the only group to join. Technically, the advantage is that at these prices a group buys from I if not from C, which means that when dealing with out of equilibrium pricing strategies we will not have to keep track of the decision of consumers to leave the market or buy, but only have to consider the choice between the two firms. This simplifies greatly the exposition.

I must however emphasize that this assumption is not only a technical one. Although the logic behind aggressive strategies exploiting cross-subsidization is the same, the equilibrium strategies, profits and the economic consequences are different when the intrinsic value is small compared to the network effects. The case is analyzed in Caillaud-Jullien (2000b), and discussed in Section 5.

Before going to the pricing game, let us discuss the allocation of consumers between the firms. Faced to prices $P^I = \{p^I_1, ..., p^I_J\}$ and $P^C = \{p^C_1, ..., p^C_J\}$, each consumer has to decide whether to buy or not, and which good.[8] These

[7] With two partially compatible networks, consumers derives external benefits from the competing network as well. The utility then writes as $u^k_j + \alpha_j n^k_j + \sum_{l \neq j} \beta_{jl} n^k_l + \sum_l \gamma_{jl} (n^I_l + n^C_l)$ and β_{jl} captures the extra value attached to being in the same network. Because we will assume that the market is covered ($n^I_l + n^C_l = m_l$), this is equivalent to having an intrinsic value $u^k_j + \sum_l \gamma_{jl} m_l$. Obviously for this case, the monopoly analysis must be adapted.

[8] In this model, consumption is exclusive. See Caillaud and Jullien (2000b) for an

decisions are assumed to be taken simultaneously by all consumers, and as above, consumers coordinate on a rational expectation equilibrium allocation (REA). When the market is fully covered ($n^I_j + n^C_j = m_j$) this amounts to the fact that

$$n^k_j = m_j \text{ if } u^k_j + \alpha_j n^k_j + \sum_{l \neq j} \beta_{jl} n^k_l - p^k_j > u^{-k}_j + \alpha_j n^{-k}_j + \sum_{l \neq j} \beta_{jl} n^{-k}_l - p^{-k}_j.$$

Denote by $A(P^I; P^C)$ an allocation rule that assigns to every price vectors a REA. Denote by $\pi^k(P^I; P^C; A)$ the profit of firm k when the prices are P^I and P^C; and consumers are allocated according to A. An equilibrium consists in an allocation rule A; and equilibrium prices for the respective Stakelberg games and simultaneous pricing game with profit functions $\pi^k(P^I; P^C; A)$ The choice of the allocation rule is then key to the determination of equilibrium prices. We can interpret this choice as a reflecting the consumers beliefs (their conjectures about the composition of the two networks), and thus as a reputation effect.

The general idea of the paper is to capture the fact that I is dominant through an adequate choice of these beliefs. One should think about I as a well known established firm, and about C as a newcomer that has no reputation in the particular domain of activity concerned. In such a context, beliefs should be biased in favor of the established firm. In a somewhat tautological way, the incumbent is dominant when all consumers believe it is. This generates self-fulfilling beliefs that resolve the coordination problem of consumers in favor of I. While the notion of favorable beliefs may be ambiguous in a general context,[9] the following result shows that it takes the form of a simple selection criteria when there are positive network externalities.

Lemma 1 Fix the prices P^I and P^C; and consider all the equilibria of the allocation game played by consumers choosing where to buy. Denote K^I the set of groups buying from I in at least one equilibrium and K^C the set of groups buying from C in all equilibria. Then there exists a maximal equilibrium $D^I(P^I; P^E)$ such that all groups within K^I buy from I, all groups within K^C buy from C, others don't buy.

analysis of non-exclusive consumption.

[9] With negative externalities, consumers will tend to separate from each other. Then beliefs favourable to I would correspond to beliefs that others join C. Given that expectations must be rational in equilibrium, they may not be a simple and non ambiguous way to define beliefs that favor I over C.

Proof. See appendix. ■

The key feature is that the value of a network uniformly increases when new consumers are added to its customer base. According to a bandwagon effect, displacing consumers from C to I both raises the incentives to join I for all individuals and reduces the incentives to join C. Thus there exists a unique equilibrium that at the same time maximizes the market share of the incumbent and minimizes the market share of the challenger. In technical terms, there are strategic complementarities in the consumers allocation game (see Topkis (1979), Vives(1990), Milgrom-Roberts (1990)).

The incumbent "dominates" if for all prices P^I and P^C, the equilibrium allocation of consumers corresponds to the maximal equilibrium $D^I(P^I, P^E)$. However, as usual with price competition, fixing the allocation in a too rigid way may create inexistence problems because the best-response of C may not exist due to discontinuity at indifference points of consumers. For example with one group, no network effects and $u_1^C > u_1^I$, D^I implies that C doesn't sell at $p_1^C = u_1^I$ and $p_1^I = 0$, while it sells for any price $p_1^C < u_1^I$. So we assume that firm I dominates the coordination process in the following way, referred to as domination is beliefs:[10]

Assumption 2 (Domination in Beliefs):

For almost all (P^I, P^C), $A(P^I, P^E) = D^I(P^I, P^E)$;

For all (P^I, P^C), $\Pi^C(P^I, P^C, A) \le \lim_{\varepsilon \to 0} \Pi^C(P^I, P^C, D^I(P^I, P^C - \varepsilon))$.

Notice that the domination in beliefs implies that consumers will not always coordinate on a Pareto efficient allocation (for the consumers only). This contrasts with equilibrium concepts used in other works on competing network as for example the recent work by Fudenberg and Tirole (1998) on limit pricing. This well known non-efficiency is a key property of networks and will occur precisely when one network good is perceived as dominating the market although it is less efficient than an alternative network good.

The paper starts by analyzing the Stackelberg games with price competition. Combining the two Stackelberg games will then provide us with equilibria of the simultaneous pricing game. The reason for doing so is that the derivation of the strategies in the Stackelberg games helps to better understand the nature of the strategic interactions.

Moreover, the Stackelberg game with I being the leader provides us with the maximal profit that the incumbent can obtain as long as the challenger is allowed to react to the incumbent pricing. One may thus view the model alternatively as a way to assess the maximal extent of market power that a

[10] The alternative is to restrict attention to a discrete grid of prices.

firm may derive from being perceived as "focal", or as a representation of dynamics short-run competition.[11]

3 Preliminary results

For this first part of the paper, I shall focus on the following question: Assuming that firm k sells to all groups, within a subset K of groups, what are its maximal profit and pricing strategy? Results will then be applied to various cases.

3.1 The incumbent's maximal profit

Suppose that I is a Stackelberg leader, and wishes to cover the market. Since I dominates in beliefs, it will cover the market whenever $u_j + \alpha_j n_j + \sum_{l \neq j} \beta_{jl} n_l - p^I_j \geq u^C_j - p^C_j$ for all groups. Define the quality differential including intra-group network effects as

$$\delta_j = u_j - u^C_j + \alpha_j n_j :$$

The condition writes as:

$$p^I_j - p^C_j \leq \delta_j + \sum_{l \neq j} \beta_j n_l \text{ for all } j:$$

It is thus immediate that I must necessarily set prices such that

$$p^I_j \leq \delta_j + \sum_{l \neq j} \beta_j n_l = \bar{p}^I_j - u^C_j ; \qquad (1)$$

which ensures that the maximal utility that a type j consumer can derive from I is larger than the minimal utility obtained when buying from C at a zero price. If it were not the case, the challenger could attract a type j with a positive price. As mentioned above this implies that $p^I_j \leq u_j + \alpha_j n_j$; and, given our selection criterion for the REA, ensures that group j buys from I; if it doesn't buy from C:

[11] The game captures the market position of an established firm faced to potential entry when prices take some time to adjust to entry. In this context there will be room for short-run profitable entry strategies. Given that with network effects, the main cost of entry is the loss incurred while building a customer base, the ability of an established firm to increase this cost is crucial to maintain its market position. Although, the present model is not intended to capture the full dynamics of entry in network industries, it provides key insights on how cross-subsidization can be used by entrants, and how established firms should fight entry.

If all prices verify condition (1), C can't sell unless it charges a negative price for at least one group.[12] But one has to take into account the possibility that C favors one group over the other. C can charge a low or even negative price for one group if selling to this group allows to sell to another group at a high price. To follow insights from Innes and Sexton (1993), one may view the situation as one in which C faces consumers who have the possibility to form a coalition (to join I). To prevent the formation of the coalition, C needs to "bribe" some groups. However it needs not bribe all groups but only enough of them to ensure that the value of the sub-coalition composed of the remaining groups is reduced to a point where it becomes unattractive. The challenger's optimal strategy is then be to subsidize some groups and to charge high prices on other groups.

The key issue is then to understand how the incumbent should set prices when it accounts for such "divide-and-conquer" strategies.

To start with, suppose that there are only two groups. Under condition (1); I covers the market unless C finds a profitable strategy that attracts all consumers. For this, C needs to set one price below $p^I_j - \left[\delta_j + \beta_{jl} m_l\right]$. Assume that it sets the price p_1^C just below $p_1^I - \delta_1 - \beta_{12} m_2$: Then group 1 buys from the challenger irrespective of what the other group buys. This means that to attract group 2, the challenger needs only to price p_2^C such that $u_2^C + \beta_{21} m_1 - p_2^C \geq u_2^I - p_2^I$: The maximal price that C can set on group 2 is thus $p_2^I - \delta_2 + \beta_{21} m_1$: By attracting group 1, the challenger has reduced the attractiveness of the incumbent by $\beta_{21} m_1$ for group 2, and increased its own attractiveness by the same amount. The overall profit from this strategy is $\left[p_1^I - \delta_1 - \beta_{12} m_2\right] m_1 + \left[p_2^I - \delta_2 + \beta_{21} m_1\right] m_2$: If the incumbent wants to sell to both groups this must be non-positive, and it must set prices such that:

$$p_1^I m_1 + p_2^I m_2 \leq \sum_j \delta_j m_j - (\beta_{21} - \beta_{12}) m_1 m_2 \qquad (2)$$

The same hold true for the symmetric treatment of group 1 and group 2:

Proposition 1 *With two groups, the maximal profit that I can obtain by selling to all consumers is* $\Pi^I = \sum_{j=1}^{2} \delta_j m_j - \left|\beta_{21} - \beta_{12}\right| m_1 m_2$:

[12] Given that cost are normalised to zero, a negative price should be interpreted as a price below marginal cost. If the effect is large, the price can be not only below marginal cost but indeed negative. The best interpretation in this case is that the customer receives free access to the network, and is subsidized through additional free services (gifts, lotteries, ...). This can also be interpreted as the result of costly advertising campaigns.

Proof. Assume that $\beta_{21} \geq \beta_{12}$. We need only to show that I can obtain this profit. Consider prices $p^I_j \leq \delta_j + \beta_{jl} m_l$, satisfying (2). From above, C can't make positive profits by selling to one group only, or to both groups with a price $p^C_1 \leq p^I_1 - \delta_1 - \beta_{12} m_1$. The only alternative for C is to set $p^C_2 \leq p^I_2 - \delta_2 - \beta_{21} m_1$ and $p^C_1 \leq p^I_1 - \delta_1 + \beta_{12} m_2$, which yields less than $-2(\beta_{21} - \beta_{12}) m_1 m_2 \leq 0$. ∎

We see that the presence of inter-group network effects reduces the profit of the incumbent, while intra-group network effects enhance this profit (as δ_j includes $\alpha_j m_j$).

Notice that any pair of prices that verifies (1) and binds (2) are optimal prices for I. Thus the market may, but need not, exhibit cross-subsidization, with one group being charged a low price (even negative) while the other receives a zero surplus.

Let us now generalize the analysis to the case of J groups. Suppose that I tries to serves the whole market at prices p^I_j verifying (1). Suppose that C tries to sell to a subset K of groups, and for conciseness take $K = \{1, 2, ..., K\}$. The challenger then sets high prices to groups outside K, and designs a pricing scheme for the targeted groups.[13] It is somewhat in the situation that was described for a monopoly facing unfavorable market conditions. In order to sell, it has to set at least one price small enough so that $u^I_j + \sum_l \beta_{jl} m_l - p^I_j \leq u^C_j - p^C_j$, which ensures the willingness of a member of group j to buy alone. Say that this price is p^I_1. The resulting price is negative, but then, as above, this allows to induce group 2 to join C with a price such that $u^I_2 + \sum_{l>2} \beta_{2l} m_l - p^I_2 \leq u^C_2 + \beta_{21} m_1 - p^C_2$. The process can then continue, as group 3 joins for a price that makes it more attractive for its members to join groups 1 and 2, as opposed to staying with groups 3 and above. Thus group j is charged the largest price that ensures that its members join given that groups $l < j$ join.

For the order $1, 2, ...,$ we obtain the conditions (here equality generates the maximal profit but one should think of C as setting prices slightly below equality):

$$p^C_1 = p^I_1 - \delta_1 - \sum_{l>1} \beta_{1l} m_l$$

$$p^C_j = p^I_j - \delta_j + \sum_{l<j} \beta_{jl} m_l - \sum_{l>j} \beta_{jl} m_l$$

[13] It may seem strange to exclude a priori groups outside K with a high price as some may be willing to join, but this is just an analytic tool. The challenger will optimize on K, so that if attracting K implies that some other group wants to join as well, the targeted set will be extended.

Summing up over the groups we obtain the challengers' profit:

$$\sum_{j=1}^{K} p_j^C n_j = \sum_{j=1}^{K} p_j^I n_j - \sum_{j=1}^{K} \delta_j n_j + \sum_{j=1}^{K} \sum_{l=j+1}^{K} \left[\beta_{lj} - \beta_{jl} \right] n_l n_j - \sum_{j=1}^{K} \sum_{l=K+1}^{J} \beta_{jl} n_l n_j$$

More generally, the challenger has the choice of the set of groups targeted, and of the order in which groups are subsidized. Indeed C would benefit from targeting first the groups valuing less the network effects. To compute equilibrium conditions, we must thus account for any possible order between groups. Denote $\sigma(.)$ a permutation on the set of groups, where $\sigma(l) > \sigma(j)$ means that j is ranked before l. The interpretation is that C sets a price $p_j^C = p_j^I - \delta_j + \sum_{\sigma(l)<\sigma(j)} \beta_{jl} n_l - \sum_{\sigma(l)>\sigma(j)} \beta_{jl} n_l$ for group j such that a member of this group is willing to join provided that it is sure that all groups ranked below join as well. The maximal profit that C can get on a subset K of groups is now obtained by summing the prices over the groups and optimizing on the order. Define

$$\Pi_K^I = \sum_{j \in K} \delta_j n_j - \max_{\sigma(.)} \left\{ \sum_{j \in K} \sum_{\substack{l \in K \\ \sigma(l) > \sigma(j)}} (\beta_{lj} - \beta_{jl}) n_l n_j \right\} \tag{3}$$

Π_K^I depends only on the characteristics of groups within K and includes two components. The first component captures the absolute advantage of I on the groups targeted (δ_j), which includes the quality differential along with the intra-group network effect. The second term captures the impact of the ability to discriminate between groups, the maximization coming from the optimal choice by C of the order of targeting. This term has a very simple interpretation: when group j is attracted "before" group l, C must leave a subsidy $\beta_{jl} n_l$ to the members of group j to compensate then for the opportunity cost of leaving l; but C can charge an extra $\beta_{lj} n_j$ to the members of group l corresponding to the value of joining group j. The net effect is then $(\beta_{lj} - \beta_{jl}) n_l n_j$.

By selling to the subset K, the challenger can obtain at most $\sum_{j \in K} p_j^I n_j - \Pi_K^I - \sum_{j \in K} \sum_{l \notin K} \beta_{jl} n_l n_j$. The term $-\Pi_K^I$ captures the maximal surplus that it can extract on group K alone, to which is substracted the additional value accruing to I due to the fact that groups outside K stay with I.

The incumbent I covers the market whenever there is no subset K on which C can generate a positive profit. This amounts to the fact that

$$\text{for all } K, \quad \sum_{j \in K} p_j^I n_j \le \Pi_K^I + \sum_{j \in K} \sum_{l \notin K} \beta_{jl} n_l n_j. \tag{4}$$

This provides us with a set of necessary and sufficient conditions for I to serve the whole market, that put bounds on the total profit derived from any particular subset of groups.

Clearly an upper bound on the profit that I can obtain by selling to all groups is the bound Π_J obtained for the set of all groups J. Whether this upperbound can be reached or not depends on the ability to design prices that verify all the constraints. The next results shows that this is indeed the case.

Proposition 2 *The maximal profit that I can obtain by selling to all groups when it is leader is equal to Π^I_J.*

Proof. See appendix. ■

The prices that are designed to reach the maximal profit are

$$p^I_j = \delta_j - \sum_{\sigma(l)<\sigma(j)} \beta_{jl} m_l + \sum_{\sigma(l)>\sigma(j)} \beta_{jl} m_l;$$

where $\sigma(.)$ is the order that yields Π_J. The last ranked type receives a discount on δ_j equal to its valuation of the total externality. Then the discount is reduced as we move down along the order and a mark-up over δ_j is introduced, the first ranked type being charged the full value of the externality. These prices are designed in such a way that if it were the case that all groups ranked before j bought from C, the challenger could not sell to group j with a positive margin. One should be aware that these prices are not unique (see the case of two groups above) so that there is no clear prediction on which group receives a subsidize. The theory so far just points to the fact that some prices will be set below δ_j while others may be above.

Looking at this bound we can extend the results obtained with two groups. In the case where the valuation of network effects are symmetric ($\beta_{jl} = \beta_{lj}$), the effects of intergroup network externalities cancel out so that the maximal profit is $\Pi^I_J = \sum_{j=1}^{J} \delta_j m_j$. Then introducing some asymmetry can only reduce I's profit.

Lemma 2 *$\Pi^I_J \leq \sum_{j=1}^{J} \delta_j m_j$, with equality if and only if for all j, l, $\beta_{jl} = \beta_{lj}$.*

Proof. See appendix. ■

Thus, the presence of inter-group network effects hurts the dominant firm I, when firms are allowed to discriminate between groups. Divide-and-conquer strategies by the challenger eliminate the reputation advantage of the incumbent. The reason is that the challenger can choose which group to subsidy. It then targets the low externality groups with negative prices and more than recoups its losses on the high externality groups.

As a final remark, notice that the same reasoning generates a bound on the incumbent's profit when it sells only to groups within a subset K: Indeed suppose that I set prices in such a way that a subset K of groups stays with the incumbent and the remaining groups buy from the challenger in equilibrium. The challenger can fix the prices for groups $j \notin K$ at their equilibrium values minus an arbitrarily small amount, and changes the pricing for groups $j \in K$ so as to attract some of them. In doing so it can't loose its initial groups ($j \notin K$) because our criterion for the allocation of consumers ensures that bringing new groups to the challenger (which reduces the attractiveness of the incumbent) can't induce them to change their behavior and buy from the incumbent (otherwise they would have done so in the equilibrium configuration). The resulting game is thus the same game as above played on the groups $j \in K$ only, but where the intrinsic value of the good proposed by the challenger u_j^C is replaced by $u_j^C + \beta \sum_{l \notin K} n_l$.

An upper bound on the profit of that I can derive by selling to groups within K is thus the maximal profit of I when its covers a market consisting of only groups in K; with the new "stand-alone" value differential defined as $\delta_j - \beta \sum_{l \notin K} n_l$ instead of δ_j. It follows that the profit is bounded above by

$$\Pi^I \leq \Pi_K^I - \beta \sum_{j \in K} \sum_{l \notin K} n_l n_j, \tag{5}$$

However, we will see that this bound may not always be attainable, because the challenger can use more complex strategies involving a reduction of some prices for the groups its serves in equilibrium.

3.2 The challenger's maximal profit

To contrast the results, let us examine what occurs with the reverse order of moves, so that C set prices before I. Now the challenger is in a very weak position, since I benefits not only from its reputation advantage but also from the second mover advantage. The formal derivation of prices and profits is done in Appendix, the intuition behind the results is provided here. Let us consider the situation where C tries to sell to all the groups within K and let I sells to the groups outside K. The key difference with before is that I can capture all the surplus from network effects. In equilibrium, I generates a surplus from inter-group network effects equal to $\sum_{j \notin K, l \notin K} \beta_{jl} n_l n_j$. By selling to all groups, I can capture the full value of the inter-group network effects $\sum_{j\in J, l\in J} \beta_{jl} m_l m_j$; plus δ_j on each groups j belonging to K. Define

$$\Pi_K^C = -\sum_{j\in K} \delta_j m_j - \sum_{j\in J, l\in J} \beta_{jl} m_l m_j + \sum_{j\notin K, l\notin K} \beta_{jl} m_l m_j \qquad (6)$$

Then I can raises its profit when covering the market by an amount at least equal to $\sum_{j\in K} p_j^C m_j - \Pi_K^C$. This must be nonpositive if C sells to groups outside K. It is shown in Appendix that this is the only binding condition:

Proposition 3 *The maximal profit that C can obtain by selling to all groups within K when it is leader is equal to Π_K^C.*

Proof. See Appendix. ■

The profit is the same as if buying from C generates no network externality. The challenger's profit is the difference between the total intrinsic utility it offers to its customers and the total surplus that the incumbent would generate by attracting all of them, which includes not only the network effects for C's customers but also the increase in the utility of I's customers. In particular C can't cover the market with positive profit unless the total intrinsic utility it can offer is larger than the total surplus that would be generated if all groups joined I.

4 Heterogeneous population

Let us now focus on the case where the population is composed of two groups. For the sake of presentation, the groups are ranked:

Assumption 3: $J = 2$; $\beta_{12} \le \beta_{21}$.

Thus group 1 is the group that values less the externality. Assumption 3 is maintained throughout this section.

4.1 Sequential pricing

Let us start with the situation where I is leader of a Stackelberg game. From above, I can obtain Π_J^I by selling to both groups. The other possibility for I is to design prices in such a way that it sells to one group and C sells to the other group. We need thus to assess the profit of I when it sells only to one group.

When I gives up on group 2 and serves only group 1, it must set a price high enough so that it is profitable for the challenger to attract only group 2.

Intuition suggests that I should leave C the opportunity to gain a high profit on group 2 alone and thus set a very high price p_2^I. This is true here, but we need to consider the necessity to put the challenger in the worst position if it tries to cover the market. Indeed if $p_2^I > \hat{p}_2^I$, group 2 never buys from I, and C can convince group 1 to join with a relatively high price $p_1^C = p_1^I - \delta_1$. On the other hand if $p_2^I \leq \hat{p}_2^I$, a deviation of C from equilibrium prices induces group 2 to join group 1 in I's network. C must thus undercut p_1^I to $p_1^C \leq p_1^I - \delta_1 - \beta_{12} m_2$ to secure the whole market. Notice also that setting a price $\hat{p}_2^I$ just leaves a zero utility to group 2 consumers, so that C can obtain the maximal profit on group 2, namely $\Pi^C = u_2^C m_2$. It follows that $\hat{p}_2^I$ is the optimal price for group 2. The basic intuition is thus that I should set a high price for group 2 but not above the monopoly price, maintaining a potential participation of this group to its network. There remains only to determine the price for group 1.

Proposition 4 *The maximal profit that I can obtain by selling only to group j when it is leader is $\Pi_j^I = \delta_j m_j - \max\{\beta_{jl}, \beta_{lj} - \beta_{jl}\} m_j m_l$.*

Proof. Suppose that I sells only to group 1. Imposing $p_2^I \leq \hat{p}_2^I$, the equilibrium price of C is $p_2^C = p_2^I - \delta_2 - \beta_{21} m_1$.

C will conform to selling to group 2 only if

$$\begin{aligned} p_2^C m_2 &\geq p_1^I m_1 + p_2^I m_2 - \delta_1 m_1 - \delta_2 m_2 + (\beta_{12} - \beta_{21}) m_1 m_2 \\ p_2^C m_2 &\geq p_1^I m_1 - \delta_1 m_1 + u_2^C m_2 - \max\{u_2^I - p_2^I, 0\} m_2 + (\beta_{21} - \beta_{12}) m_1 m_2 \end{aligned}$$

which yields

$$\begin{aligned} \delta_1 m_1 - \beta_{12} m_1 m_2 &\geq p_1^I m_1 \\ \delta_1 m_1 + \max\{u_2^I - p_2^I, 0\} m_2 - u_2^I m_2 + p_2^I m_2 - \beta_{21} m_1 - (\beta_{21} - \beta_{12}) m_1 m_2 &\geq p_1^I m_1 \end{aligned}$$

The LHS of the second condition is maximal at the monopoly price $p_2^I = \hat{p}_2^I$, and yields:

$$\begin{aligned} \delta_1 m_1 - (\beta_{21} - \beta_{12}) m_1 m_2 &\geq p_1^I m_1 \\ \Pi^C &= u_2^C m_2. \end{aligned}$$

■

In a market sharing equilibrium, C sells to group l at a price $p_l^C = u_l^C$. An interpretation is that the incumbent induces cooperation by the challenger with some type of "stick and carrot" strategy which can be stated as: "you can serve one group with high profit, but don't try to be aggressive on the other group, or I will take it back".

If $\beta_{21} > 2\beta_{12}$; I must set a price strictly below $\delta_1 - \beta_{12} m_2$ to sell to group 1 only. The reason is that at this price, C will still find profitable to subsidize heavily group 1 and to raise the price on group 2: In this case, irrespective of whether I covers the whole market or sell to 1 only, the main limitation on I's strategy is the indirect profit that C can obtain on group 2 by attracting group 1.

The equilibrium allocation for the Stackelberg game is obtained by comparing the three levels of profit:

$$
\begin{aligned}
\Pi^I_J &= \delta_1 m_1 + \delta_2 m_2 - (\beta_{21} - \beta_{12}) m_1 m_2 \\
\Pi^I_1 &= \delta_1 m_1 - \max\{\beta_{12}; \beta_{21} - \beta_{12}\} m_1 m_2 \\
\Pi^I_2 &= \delta_2 m_2 - \beta_{21} m_1 m_2 :
\end{aligned}
$$

The first point is that I may be unable to sell, even in cases where both δ_1 and δ_2 are positive. Even a firm that dominates both in terms of quality ($u_j > u^C_j$) and reputation may not be able to generate a positive profit on the market. This is due to the ability of C to exploit the inter-group network effects.

Suppose that I sells to at least one group. A key point is that the profit loss due to inter-group network effects is always higher when I serves only one group than when it serves to two groups. As a consequence, I will not give up on a group that it would serve if they were no inter-group effect. Indeed it is immediate to see that

$$
\Pi^I_J \geq \Pi^I_j \text{ if } \delta_l \geq 0; \ l \neq j:
$$

The motive for abandoning one group is thus grounded into a quality advantage of the challenger (see the discussion in the next section). In particular market sharing can only occur when there is a sufficient degree of horizontal differentiation.

More precisely, I prefers not to sell to group 1 if $\delta_1 < -\beta_{12} m_2$: It prefers not to sell to group 2 if $\delta_2 < \{\beta_{21} - 2\beta_{12}; 0\} m_1$:

The market share of the incumbent are represented in the space $(\delta_1; \delta_2)$; for the case $\beta_{12} < \beta_{21} < 2\beta_{12}$ in Figure 1, and the case $\beta_{21} > 2\beta_{12}$ in Figure 2 (grey areas and dotted lines will be explained below).

Comparing with the case where there is no inter-group network effect (where I serves group j if $\delta_j \geq 0$); we see that the range of parameters for which the market is shared between the two firms is reduced. But this reduction of market sharing can be the result of an increase or a decrease in the market share of I: When I has a large quality differential on some group

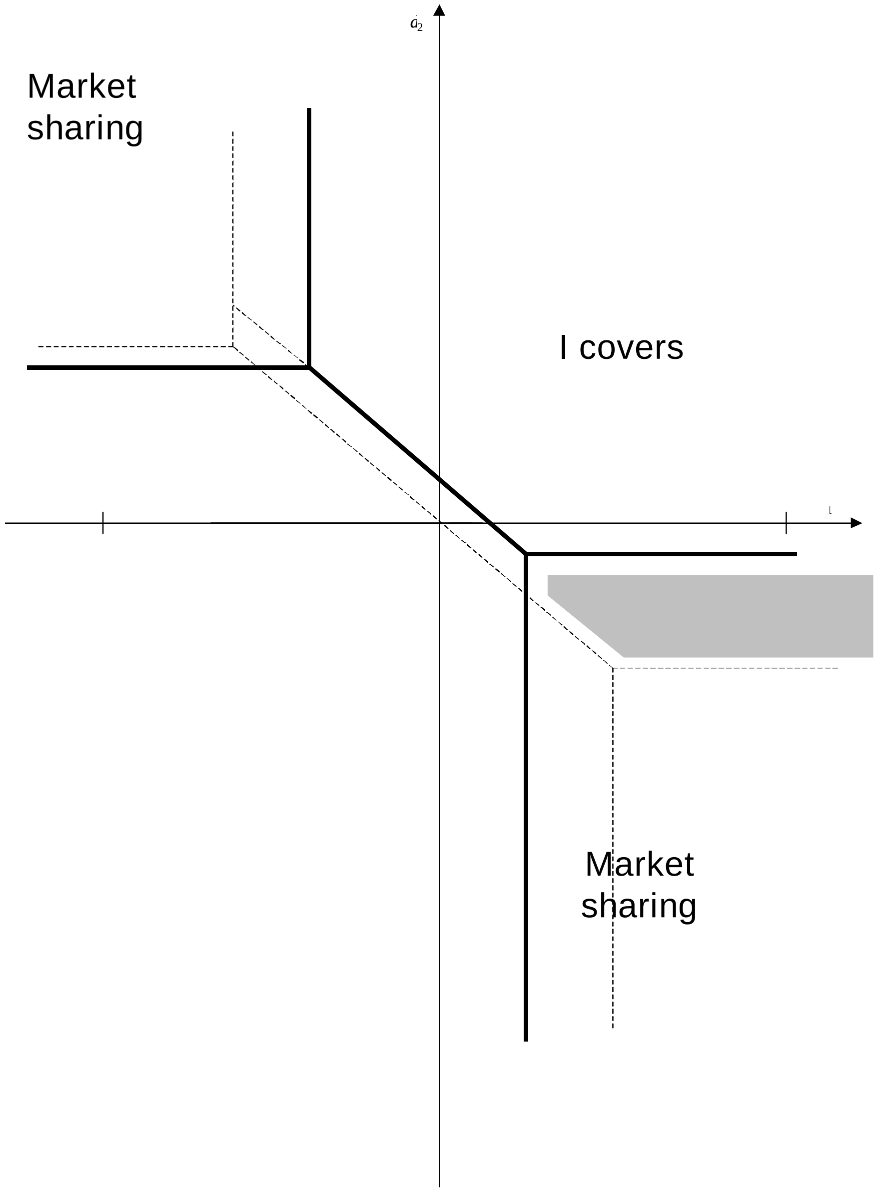


Figure 1:

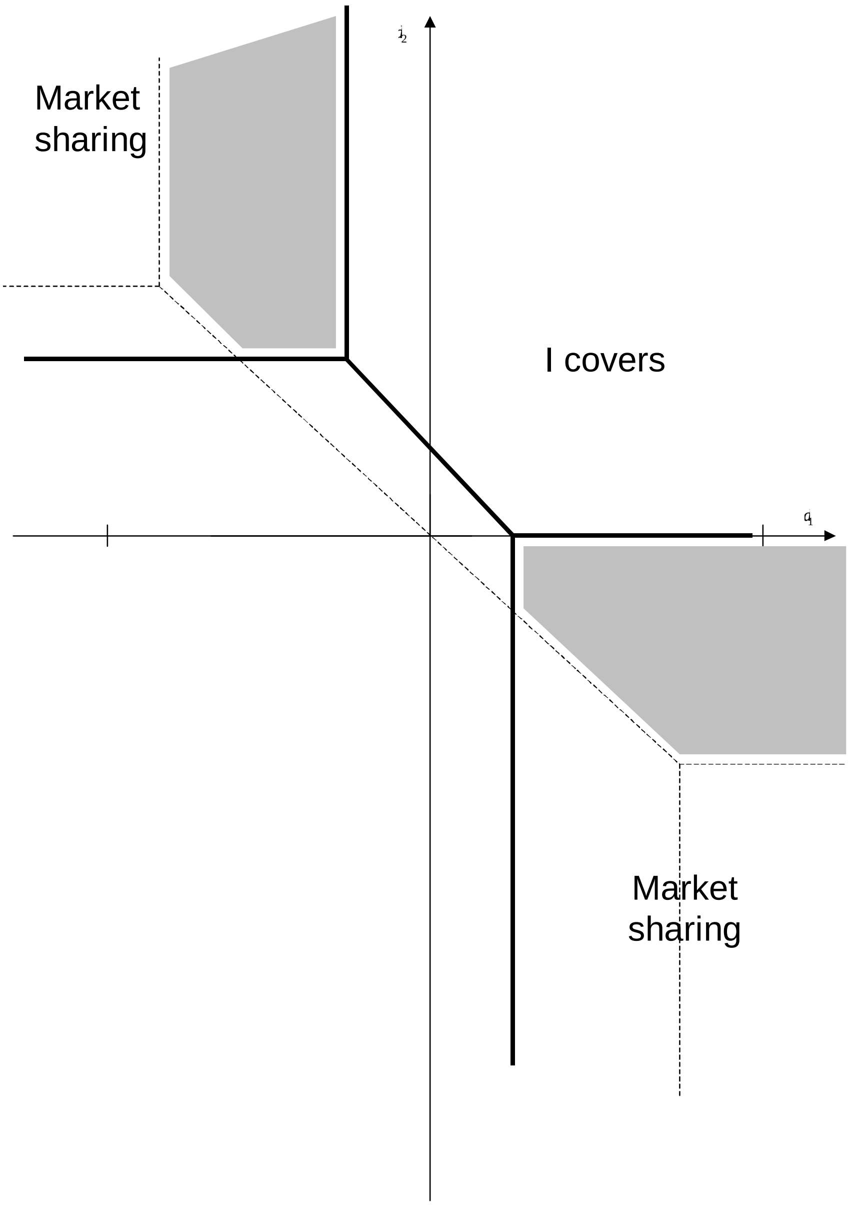


Figure 2:

(δ_j high) and a rather small disadvantage on the other (δ_l negative but small in absolute value), I will cover the market, in order to secure group j. On the other hand, if the quality levels are similar, I will not be able to obtain a positive profit.

When β_{12} increases from 0 to β_{21}, Π^I_J increases, while Π^I_2 is unchanged. The effects on Π^I_1 is twofold: it increases up to $\beta_{12} = \frac{\beta_{21}}{2}$, then decreases. However $\Pi^I_J - \Pi^I_1$ evolves as $\max\{2\beta_{12} - \beta_{21}, 0\}$ and is thus non-decreasing. Overall, increasing the value of the smallest network effect favors the situation where I serves both groups and raises I's profit.

Reduce β_{21} clearly raises all levels of profits. At the same time it doesn't affect the comparison between Π^I_J and Π^I_2, while Π^I_J raises faster than Π^I_1. As a general conclusion we obtain:

When the value attached to the inter-group network effects by one group moves closer to the value attached by the other group, I's profit raises (weakly) and I is more likely to serve both groups.

4.2 Simultaneous pricing

Let us now turn to the case where prices are set simultaneously. Before stating the results, we need to examine quickly the reverse timing where C is a Stackelberg leader. Then by selling to group j only C would obtain

$$\Pi^C_j = -\delta_j m_j - (\beta_{12} + \beta_{21}) m_1 m_2$$

which is larger than Π^C_J if $\delta_l > 0$. This implies that C prefers to cover the market whenever $\delta_j \leq 0$ for both groups.

For situations where one firm would cover the market in any of the Stackelberg games, an equilibrium is easy to obtain as there is some freedom in the choice of the prices set by the other firm.

Proposition 5 Suppose that prices are set simultaneously. There exists an equilibrium where firm k covers the market (with profit $\Pi^k \in [0, \Pi^k_J]$) if and only if $\Pi^k_J \geq 0$ and:

If $k = I$, $\delta_j \geq -\left(\beta_{jl} + 2\beta_{lj}\right) m_l$ for both groups;

If $k = C$, $\delta_1 \leq (\beta_{12} + 2\beta_{21}) m_2$ and $\delta_2 \leq \beta_{21} m_1$.

The conditions hold if firm k chooses to cover the market when it is a Stackelberg leader.

Proof. First a firm can't cover the market in equilibrium unless $\Pi^k_J \geq 0$.

Suppose that $\Pi^I_J \geq 0$. Set the prices $p^I_j \leq \theta^I_j - u^C_j$ such that $\sum_j p^I_j m_j = \Pi^I \leq \Pi^I_J$. Then C's best response is not to sell at all. The smallest possible prices for C are $p^C_j = p^I_j + u^C_j - u^I_j$: at these prices, each consumer is indifferent between C and I. Moreover, it is impossible for I to sell at a group above p^I_j. Thus this is an equilibrium if I prefers to cover the market than to sell to one group. To sell to group j alone, I must set a price $\hat{p}^I_j$ such that

$$u^I_j + \alpha_j m_j - \hat{p}^I_j \geq u^C_j + \beta_{jl} m_l - p^C_j$$

which amounts to $\hat{p}^I_j = p^I_j - 2\beta_{jl} m_l$. This is not profitable if $-2\beta_{jl} m_1 m_2 \leq p^I_l m_l$. Thus an equilibrium must verify $p^I_j m_j + p^I_l m_l \geq p^I_j m_j - 2\beta_{jl} m_l m_j$ or: $p^I_l \geq -2\beta_{jl} m_j$. Equilibrium prices then exist if $u^I_j - u^C_j \geq -2\beta_{lj} m_l$ for both groups or $\delta_j \geq -\beta_{jl} m_l - 2\beta_{lj} m_l$.

When $\Pi^I_J \geq \max\{\Pi^I_j; 0\}$ we know from above that $u^I_1 - u^C_1 \geq 0$ and $u^I_2 - u^C_2 > 0$.

Suppose now that $\Pi^C_J \geq 0$. Choose prices $p^C_j \leq -\beta_{jl} m_l - \delta_j (\leq u^C_j)$ such that $\Pi^C = \sum p^C_j m_j \leq \Pi^C_J$. I can't obtain a positive profit because its profits is the maximum between $\sum p^C_j m_j - \Pi^C_J$; $\left(p^C_1 + \delta_1 - \beta_{12} m_2\right) m_1$ and $\left(p^C_2 + \delta_2 - \beta_{21} m_1\right) m_2$; and they are all nonpositive. C can't obtain more than Π^C by covering the market if

$$\sum p^I_j m_j - \Pi^I_J \leq \Pi^C.$$

The unique prices that are compatible with the fact that C covers the market and obtains Π^C are $p^I_1 = p^C_1 + \delta_1 + \beta_{12} m_2$; $p^I_2 = p^C_2 + \delta_2 - \beta_{21} m_1$. Notice that at these prices, group 1 buys alone from C. So C would cover the market only if $p^C_2 \geq 0$. On the other hand, C could reduce its price by $2\beta_{21} m_1 m_2$ and sell only to group 2. So C would cover the market only if $p^C_1 m_1 \geq -2\beta_{21} m_1 m_2$.

We can find such prices whenever $\delta_1 \leq -\beta_{12} m_2 + 2\beta_{21} m_2$ and $\delta_2 \leq -\beta_{21} m_1$. ■

The conditions in the proposition are necessary and sufficient. Their purpose is to ensure that the prices set by the firm covering the market are not too small, so that this firm has no incentive to exclude some consumers.

The analysis of market sharing is more complex as there is less flexibility in the design of prices. Moreover it must be possible for each the two firms to obtain a positive profit with market sharing when it is leader. It is thus clearly more difficult to sustain market sharing with simultaneous pricing then with sequential pricing.

Proposition 6 When prices are set simultaneously, there exists an equilibrium where firm I sell to group j with profit Π^I and firm C sells to group l with profit Π^C if $\Pi^C_l - \beta_{jl} m_1 m_2 \geq \Pi^C \geq 0$ and $\Pi^I_j - X_j \geq \Pi^I \geq 0$, where $X_2 = 0$; $X_1 = \inf\{\beta_{21}; 2(\beta_{21} - \beta_{12})\} > 0$.

Proof. See appendix. ■

It is not sufficient that the Stackelberg profit be positive for both firms to obtain a market sharing equilibrium. Profits are reduced because firms should not be tempted to cover the market which limits the sustainable levels of prices.

The global equilibrium configurations is always the same and is depicted on Figure 3 for the case $\beta_{21} = 3; \beta_{12} = 1$.

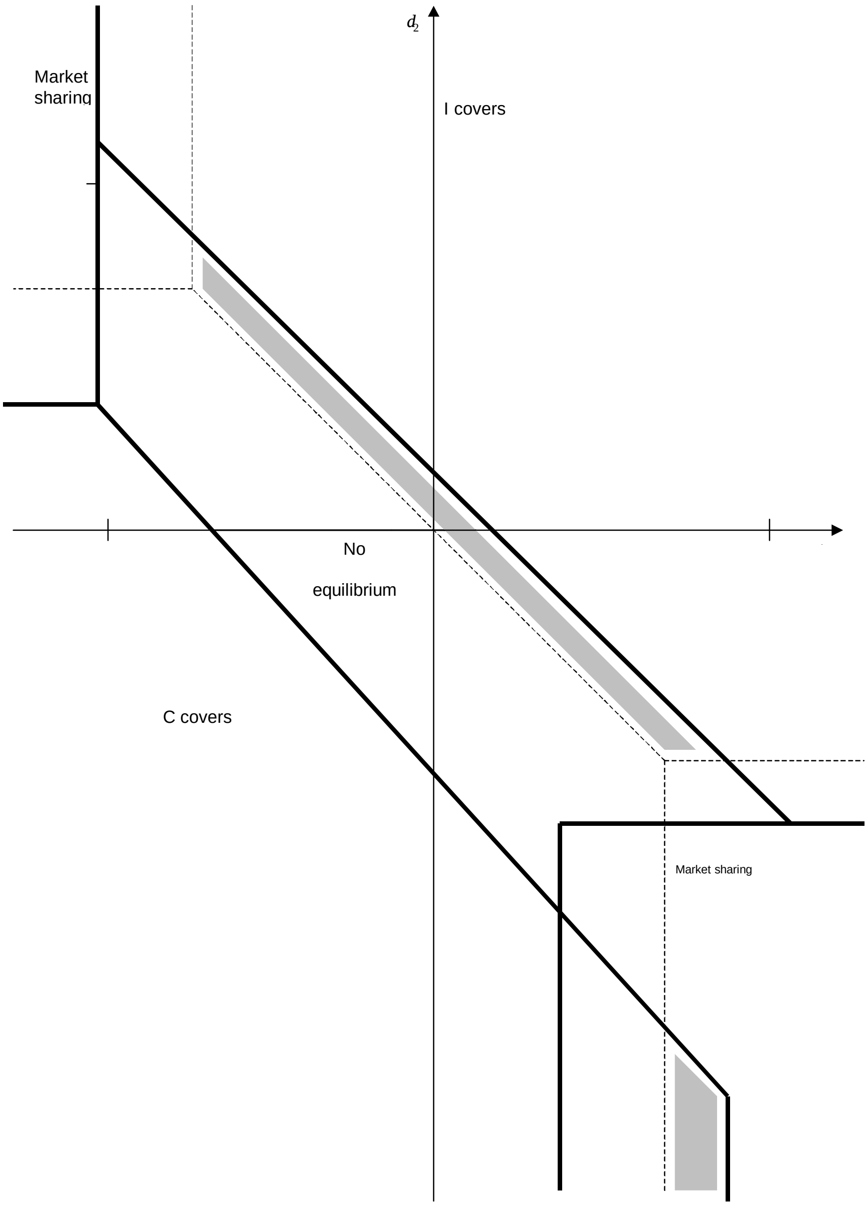
In the middle range, a pure strategy equilibrium fail to exists, and in particular no firm can cover the market with probability one. We can also see that market sharing equilibria and market covering equilibria hardly coexist. The only case where this can occur is when there are two equilibria: one in which C covers the market, and one in which I sells only to the low externality group 1.

5 Discussion

5.1 Compatibility choice

In the analysis, the two networks are assumed to be incompatible. A natural question is whether firms have an incentive to become compatible. With full compatibility, the model amounts to a Bertrand price game with J distinct markets as consumers would ignore the network effects when choosing their supplier. The profit of firm I is then $\sum_j \max(u_j^I - u_j^C; 0) m_j$, while C receives $\sum_j \max(u_j^C - u_j^I; 0) m_j$. Whether firms prefers to be compatible or incompatible turns out to be a complex issue. The reason is that when networks are incompatible, the ability to exploit the inter-group network effects, with divide-and-conquer strategies, strengthens competition and is extremely harmful for profits. While the incumbent benefits from incompatibility because this allows it to capture the value of the intra-group network effects, it has to bear the cost of this exacerbation of competition. Whether the incumbent prefers to be compatible or not then depends on the comparison between the intra-group network effects and the inter-group network effects. When the latter effects dominate, firms will prefer to be compatible. To illustrate this we focus on the case where there is no intra-group network effects ($\alpha_j = 0$). As mentioned above, this case corresponds to perfect price discrimination as each group can be interpreted as a single individual.

Corollary 1 *Assume perfect price discrimination ($\alpha_j = 0$); an arbitrary size of the population ($J \geq 2$) and asymmetric network effects ($\beta_{jl} \neq \beta_{lj}$). Suppose that prices are set simultaneously, and that a pure strategy equilibrium*



d_2
Market sharing
I covers
No
equilibrium
C covers
Market sharing

Figure 3:

exists when network are incompatible. Then both firms prefer to be compatible.

Proof. The reasoning used to derive the bounds on firm k's profit when it is a Stackelberg leader shows that if there is a simultaneous pricing equilibrium where k sells to K with profit π^k, then there exists a strategy for its opponents that allows it to cover the market with an extra profit larger then $\pi^k - \pi^k_K$. Thus $\pi^k \leq \max_K \left\{\pi^k_K\right\}$. But $\max_K \left\{\pi^k_K\right\} < \max_K \left(\sum_{j \in K} (u_j^k - u_j^{i^k}) m_j\right) \leq \sum_{j \in J} \max(u_j^k - u_j^{i^k}, 0) m_j$. ■

A natural conjecture is that this holds also with mixed strategy equilibrium as there should involve zero profit.

Notice that the same reasoning shows that a Stackelberg leader also prefers to be compatible. On the other hand, a follower in the Stackelberg game may prefer to be incompatible. This holds obviously for I as it benefits both from a second mover advantage and from a reputation effect, but also for C. Indeed when I acts as a leader and there is enough horizontal differentiation, the incumbent will choose to share the market. In this case, the challenger charges the full intrinsic value u_1^C to one group, while it would charge only $u_1^C - u_1^I$ under compatibility. Thus the challenger prefers to be incompatible.

5.2 Welfare: excess inertia or excess momentum?

Let us compare the equilibrium configuration with the efficient allocation. Given that there is enough flexibility in designing prices to transfer the surplus from one group to another, the welfare criterion used here is total surplus. In what follows, excess inertia refers to the cases where it would be optimal to reduce the market share of I from a welfare point of view. Our domination assumption favoring the incumbent, it tends to generate excess inertia, as it is well known with uniform pricing. In our model, the extent of excess inertia increases with the size of intra-group network effects. As this is a standard result, we now focus on the case where there is no intra-group network effects:

$$\alpha_1 = \alpha_2 = 0.$$

When this is the case, total welfare is maximal with market sharing whenever $(\beta_{12} + \beta_{21}) m_1 m_2$ is smaller $\min\{\delta_1, -\delta_2\}$ or $\min\{-\delta_1, \delta_2\}$, which corresponds to a strong pattern of horizontal differentiation and small network effects. Otherwise firm I should cover the market if $\delta_1 + \delta_2 > 0$, while firm 2 should cover the market if $\delta_1 + \delta_2 < 0$. Dotted lines in Figures 1, 2 and 3 show the delimitations of the various range of quality differentials.

Let us consider the game with simultaneous pricing. Then Figure 3 clearly shows that there can be excess inertia which can either take the form of the inexistence of a pure strategy equilibrium implying that C can't cover the market with probability one while it should, or the form of inefficient market sharing.

However, divide-and-conquer strategies allow the challenger to capture part of the value of inter-group network effects. There is then the possibility of excess momentum, which occurs when it would be optimal to increase the market share of firm I. The situations with excess momentum correspond the grey areas in Figure 3. Excess momentum can take two forms. First, I may fail to cover the market with probability 1 while it should, because there is no pure strategy equilibrium. Second, C may cover the market in equilibrium while it would be more efficient to share the market.

If we examine the game where I is a Stackelberg leader we obtain a new form of excess momentum (grey areas in Figures 1 and 2). When I is a leader and there is enough horizontal differentiation, it may choose to accommodate by inducing an allocation with market sharing, although total welfare maximization requires that both groups buy from I.

To illustrate this effect, suppose that there are two consumers ($\alpha_j = 0$; $m_j = 1$) with values: $u_1^C = u_2^I = \beta_{12} = 0$; $u_1^I > \beta_{21} > 0$; $u_2^C > 0$. The optimal strategy for I is to give up on group 2 by setting a price β_{21} on group 2. This allows to charge $u_1^I - \beta_{21}$ to 1, while C charges u_2^C to 2. Indeed, the only alternative strategy for the challenger would be to secure 1 with a price $p_1^I - u_1^I$ and to raise the price charged to 2 by β_{21}, which is not profitable when

$$p_1^I \leq u_1^I - \beta_{21}. \tag{7}$$

This conclusion holds even if $u_2^C << \beta_{21}$, despite the fact 2 would be willing to pay a larger amount to join I than to join C. To see why this occurs, consider what would happen if the incumbent attempted to sell to 2 by lowering its price p_2^I below $\beta_{21} - u_2^C$. C would have no other option if it wants to sell than to attract both individuals. The same strategy as above would allow the challenger to generate a positive revenue unless $p_1^I - u_1^I + p_2^I + u_2^C + \beta_{21} \leq 0$ or

$$p_1^I + p_2^I \leq u_1^I - \beta_{21} - u_2^C. \tag{8}$$

Comparing (8) with (7), we see that I's profit is reduced by u_2^C when I tries to sell to both individuals. The reason is that in the former case, selling to 1 only allows C to capture the network effect of 2, while in the latter case it allows it to capture both the network effect and the intrinsic value.

Thus when 2 buys from the incumbent, the nature of competition on 1 is changed because the challenger can't sell to 2 unless it sells to 1. Individual 1 becomes like a bottleneck whose access has value u_2^C for the challenger. This value adds to the network effects that limits I's prices in any case. By giving away 2, the incumbent allows the challenger to extract the full intrinsic value u_2^C without having to fight for the other individual. Thus the incumbent weakens the competition.

5.3 Strategic degradation of quality for targeted customers

One of the general principle that emerges is that head-to-head competition to conquer the whole market is extremely costly when price-discrimination is possible. As usual, one way to escape from such situation of intense competition is to achieve enough horizontal differentiation. In the present model, this means shifting to a more peaceful market sharing situation. When firms can control the quality of the good at the individual level, one way to reach this objective is to degrade the quality for some customers and to provide the best possible quality for the others. The consumers targeted for quality degradation will be those for which the network can't propose a higher quality than its opponent.

To illustrate this phenomenon, consider the simultaneous pricing game and suppose that I covers the market in equilibrium with maximal profit $\Pi^I_J = (\delta_1 + \beta_{12} m_2) m_1 + (\delta_2 - \beta_{21} m_1) m_2 > 0$. Suppose in addition that $\delta_1 + \beta_{12} m_2 < 0$ (which is compatible with equilibrium conditions). Consider what happens if I succeeds in degrading quality for group 1 (holding u_2^I constant) up to a point where the new quality differential $\hat{\delta}_1 < -(\beta_{12} + 2\beta_{21}) m_2$. Then the new equilibrium involves market sharing: I sells to group 2 only with profit $(\delta_2 - \beta_{21} m_1) m_2 > \Pi^I_J$. It is thus profitable to do so. The point here is that I would like to commit not to compete on group 1; because the lack of commitment creates a situation where I is 'forced' to include group 1 in its network despite a competitive hedge in favor of C on this group. A targeted degradation of quality is one way to achieve such a commitment.

In a similar vein, if we start from an equilibrium where C covers the market and $\delta_1 > \beta_{12} m_2$; by degrading quality for group 1 up to a point where $\hat{\delta}_1 > (\beta_{12} + 2\beta_{21}) m_2$; C induces a market sharing equilibrium where its profits is $-\delta_2 m_2 - (\beta_{12} + \beta_{21}) m_1 m_2 - \beta_{12} m_1 m_2 > \Pi^C_J = -\delta_2 m_2 - (\beta_{12} + \beta_{21}) m_1 m_2 - \delta_1 m_1$.

More generally when a network can choose the technology and affect perceived qualities, and when it can't gain a large quality advantage on both groups, it will have incentives to shift its technological choices toward the preferred technology of one group and the least preferred technology for the other group. This may result in inefficiencies in technological choices and even in the choice of a dominated technology. For example suppose that the network has access to two technologies: one technology generates $\underline{u}_1$ and $\underline{u}_2$; the other $\bar{u}_1$ and $\bar{u}_2$: Suppose that $\underline{u}_j < \bar{u}_j$ for both groups, but the network would serve only one group with the least efficient technology and both groups with the most efficient. Then if the difference between the two technologies is small, the network may choose the least efficient one.

5.4 Large network effects.

In the paper, network effects are assumed to be small compared to intrinsic values, or at least networks are assumed to be partially compatible so that the network effects have a relatively small impact (Assumption 1). For pure network goods that are totally incompatible, the only source of value is the presence of other users. Then the assumption will not hold. The situation were $u^C_j < \sum_{l\neq j} \beta_{jl} m_l$ is dealt with in Caillaud-Jullien (2000a and 2000b), along with other issues that emerges in the context of intermediation. Nothing is changed when the market is shared, but the analysis of market covering by I differs. The key point is that I may set a price $p^I_j > u^I_j + \alpha_j m_j$ and still cover the market, provided that $p^I_j < \bar{p}^I_j - u^C_j$; while in the present model $\bar{p}^I_j - u^C_j < u^I_j + \alpha_j m_j$. This means that if C convinces group l to join with a low price, group j stops to buy from I: The price set by C for group j must then be $u^C_j + \beta_{jl} m_l < p^I_j - \delta_j + \beta_{jl} m_l$: The challenger's ability to compete is thus reduced. This effect works in favor of the dominant firm as it allows I to extract part of the value of the inter-group externality that C can't capture through cross-subsidization (the value vanishes as soon as group l joins C): It is then shown that firm I maximal profit is higher and exceeds $\sum_j \delta_j m_j$ when intrinsic values are low. The dominant firm may thus be able to capture part of the surplus generated by inter-group network effects, and there will be more situations where it covers the whole market. Moreover the equilibria exhibit a much stronger pattern of cross-subsidization.

5.5 Equilibrium and domination

One of the problem encountered is that an equilibrium may fail to exist. The question is then whether this is due to the assumption of domination in beliefs (Assumption 2) that gives too much weight to firm I: It is clear that if we drop Assumption 2; there is much more flexibility in assigning the consumers between the networks for out-of-equilibrium prices.

Indeed, if we define $D^C(P^I; P^C)$ as the allocation rule that maximizes the market share of C; we can now assume that if any of the two firms deviates from the equilibrium prices, consumers allocate on the least favorable REA. Thus if I deviates, they coordinate on D^C; while if C deviates they coordinate on D^I. While this helps to extend the set of equilibria this will not completely resolve the existence problem. Inexistence still occurs if there is not enough differentiation between the networks. In particular if $\left|\sum_j \left(u_j^I - u_j^C\right) m_j + \alpha_j m_j^2\right| < \min\{\beta_{12}; \beta_{21} - \beta_{12}\} m_1 m_2$ a pure strategy equilibrium doesn't exist in the simultaneous pricing game. Inexistence occurs when networks are of similar quality, with small intra-group network effects and strong asymmetry in inter-group network effects. This points to the high instability of situations of strong competition between similar networks.

The second point concerning domination is that while being dominant is clearly the most favorable case when one covers the market, this is not so when the market is shared between the two networks. Indeed a firm may gain some form of commitment if deviations from equilibrium strategies are costly, which occurs if it looses its domination out of the equilibrium path. In fact this remark is true even if the dominant firm I is a Stackelberg leader. The reason here is that domination on the equilibrium path prevents the opponent from capturing the value of the intra-group network effect on its clients. It thus limits the profit that the challenger can obtain in equilibrium and raises its incentives to deviate. The dominant firm may then benefit from leaving more profit to the challenger than domination implies. When the challenger is allowed to capture the value of the intra-group effect on its clients, but consumers coordinate on D^I when it deviates, the threat of loosing this additional value provides additional incentives for the challenger to conform to the equilibrium strategy. In turn this allows to raise the incumbent's equilibrium prices. A formal analysis of this point is provided in the last Appendix, although the type of equilibrium involved is not very compelling because it implies a strong discontinuity in C's payoff (it payoff drops down by $\alpha_l m_l^2$ as soon as it changes its prices, in any direction).

6 Homogeneous population

The most straightforward application of the results of Section 3 is the case where the population is homogeneous. Consider a population of identical individuals of total size M, divided into J subgroups of size m_j. Suppose also that individuals care only about the mass of consumers buying the same good as them. For this situation, parameters are

Assumption 4 $\alpha_j = \beta_{jl} = \beta;\ u_j^I = u^I;\ u_j^C = u^C.$

The first intuitive point is that a firm should sell to all individuals or to none. This is immediate to show for the case where the challenger is leader. Indeed the profit of C when it sells to K is

$$\Pi^C_K = \left(u_C - u_I - \beta M - \beta\left(M - \sum_{j\in K} m_j\right)\right)\sum_{j\in K} m_j;$$

which if positive is maximal when C covers the market and takes value

$$\Pi^C_J = \left[u_C - u_I - \beta M\right] M:$$

Thus a Stackelberg leader C would cover the market if $u_C - u_I \geq \beta M$; and not at all otherwise.

This intuition is less obvious for firm I because when the quality of C is higher than the quality of I; there may be groups for which $u_I + \beta m_j > u_C$ and groups for which this is not the case: But, it is shown in Appendix that if I can profitably sell to some groups, then I is better-off selling to the whole population.

Then when firm k is a Stackelberg leader, it sells to the whole population with profit

$$\Pi^k_J = (u_I - u_C)M - \beta\sum_j m_j^2$$

if it is nonnegative, and it doesn't sell at all otherwise.

The main effect of price-discrimination in this context is to prevent the incumbent from appropriating all the surplus generated by the inter-group network effects. The very same logic applies with simultaneous pricing, and the various timing are easily related to each others.

Proposition 7 *Assume 4 and that prices are set simultaneously. Define $h = \sum_j \left[\frac{m_j}{M}\right]^2$: Then:*
If $u_C - u_I \leq \beta h M$; I covers the market with profit $\Pi^I \in [0; \Pi^I_J]$;
If $\beta h M < u_C - u_I < \beta M$; there is no pure strategy equilibrium;
If $\beta M \leq u_C - u_I$; firm C covers the market with profit $\Pi^C \in [0; \Pi^C_J]$:

Proof. See appendix. ■

When the challenger has a higher quality but not too high, there is no equilibrium. The reason is that I can exploit its reputation advantage while C can always react to the prices set by I by exploiting the possibility to price-discriminate. The two effects go in opposite directions and prevent any stability of the competitive process. For very high levels of quality, C covers the market.

h is a measure of the concentration of the groups, similar to an Herfindhal index. Thus whenever $u^C > u^I$; the most efficient network may fail to cover the market if the concentration is high enough. As the number of groups increases, or more generally as the concentration decreases, it is less and less likely that the least efficient firm serves the market. For a given total population, the effect is maximal when the sub-populations are of equal size. In particular the ability to separate a small group from the rest of the population has a low impact. The reason is that it is extremely costly for the challenger to persuade this small group to leave the large group. And the gain from doing so is small as the propensity of members of the large group to join the small group is proportional to the size of this group.

Notice that the inexistence result is due to the assumption that assigns I dominates in beliefs: If we drop the assumption of domination, then we are free to assign domination at any prices to any firm, and an equilibrium always exists.

Corollary 2 *Assume that the population is homogeneous, prices are set simultaneously and Assumption 2 is not imposed. Then an equilibrium in pure strategy exists. If $u^C - u^I \leq \beta h M$ any of the two firms can cover the market in equilibrium. If $u^C - u^I > \beta h M$, only the highest quality firm can cover the market in equilibrium.*

Proof. Immediate, as a firm can cover the market in equilibrium when it can do so with the allocation rule $D^k(P^I; P^C)$ that favors it. ■

Thus, when $u^I \leq u^C$; there always exists an equilibrium of the simultaneous pricing game with no specific dominant firm, where C covers the market. There is also an equilibrium where I covers the market if the difference in utility is smaller than the concentration index.

At the limit when the number of groups increases with a fixed population, the model converges to the situation of perfect price-discrimination. Alternatively, perfect price-discrimination is obtained by setting $\alpha_j = 0$ instead of β and $m_j = 1$: Then the model is equivalent to a model with J individuals. In both cases, the firm offering the highest quality covers the market.

Corollary 3 *Assume that the population is homogeneous, prices are set simultaneously and Assumption 2 is not imposed. With perfect price-discrimination, the set of equilibria generates the same allocation and profit levels as if there were no network effects at all. In particular the equilibrium allocation is efficient.*

The result contrasts heavily with the case where price-discrimination is not allowed. Indeed with uniform pricing, I may cover the market when $u^E - u^I < \beta M$; which is the key to the fact that the market may fail to coordinate on the most efficient standard. But with perfect price discrimination, a firm is able to completely overcome the coordination problem and to pass to its customers the resulting efficiency gain. The unique equilibrium is then efficient.

7 Conclusion

It is commonly admitted that network externalities are sources of market failures that favor incumbents and generate barriers to entry. This paper points to the fact that when competitive strategies exploit the potential for price-discrimination, the nature of competition is more complex and the conventional wisdom needs to be revisited. In particular we found that when the scope for price discrimination is high, the competitive pressure is stronger in the presence of network externalities than without. This can be seen as a source of instability for network industries. We emphasized mostly two consequences of this: the possibility of excess momentum and the incentives for compatibility.

On the positive side, it appears that networks are vulnerable to strategies that build a customer base with a subsidy and exploit the bandwagon effect (see also Fudenberg and Tirole (2000) for a related idea). This can explain very aggressive strategies of existing networks toward small competitors, even when these competitors do not stand as an immediate threat and specialize on a small segment of the market. In this case, what is at stake is the future possibility to leverage existing market shares to conquer new markets linked by network externalities.

Obviously our model of competition is extreme and its conclusions need to tempered for practical purposes. Whether barriers to entry are low or not may depend heavily on specific market conditions. The model suggests that it may be easier than expected for a superior technology to enter, provided that the quality improvement is large enough. On the other hand, for small quality improvements, the fact that competition is intense may act as a stronger barrier to entry than reputation effects. Most of network industries involve large sunk costs. While a firm may expect to succeed in building its reputation and conquering the market in a reasonable delay under uniform pricing, thus recovering its sunk costs, the intensity of ex-post competition with price-discrimination may reduce its profitability. The issue will then be similar to the one involved in the innovation process where ex-post competition may slow down innovation, and there may then be an optimal balance between reputation effects and discrimination possibilities.

The results of the paper suggest several lines of extension.

One caveat on the analysis is that, because of the static nature of the model, it is not able to addressed financial issues. Viewed in a more dynamic context, divide-and-conquer strategies require high financial possibilities as they imply negative cash-flows for some period of time. Firms may have difficulties in finding the required external funding for such a strategy because the analysis also suggests that the prospect of conquering the market are uncertain. A more dynamic perspective is certainly called for. Even in our static context, the strategies may be very risky if demand characteristics are imperfectly known, because a firm faces the risk of attracting only unprofitable users.

While the paper focuses on third-degree and perfect discrimination, it would be worth investigating the implications of second-degree price discrimination. This includes the use of non-linear tariffs that proliferate in network industries, but also more specific pricing schemes that succeed in linking the price to the size of the network. Caillaud and Jullien (2000b) shows that allowing for such complex pricing schemes increases the competitiveness of the industry.

This includes also the choices of bundles of services and of technologies in multi-attribute networks. Indeed most networks offers various services that may be bundled in complex ways. The present model can be interpreted as one with multiple services provided that each individual buys only one service. It would be worth extending the analysis to situations where consumers can choose various combinations of services.

It is also worth investigating further the implications of competition for technological choices and compatibility decisions. Basically, networks face two possibilities. They can try to differentiate their services under compatible technologies, or they can try to conquer the market by providing high quality with incompatible technology. In this respect, compatibility or standardization agreements resemble cross-licensing agreements in R&D.

As mentioned in the paper, the analysis of pure network goods (no intrinsic value) that are incompatible doesn't fit into the paper's framework. The analysis in Caillaud and Jullien (2000a and b) shows that reputation effects are stronger and that the competitive pressure is weaker in this case. It also shows that multi-homing possibilities (the possibility to join several networks at a time) modify the nature of the pricing strategies and strengthens competition.

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A Appendix to Section 2

Proof of lemma 1. Consider two continuation equilibria $i = 1;2$, and the respective K_i^I and K_i^C. Then

$$j \notin K_i^C \Rightarrow \max\{u_j^I + \alpha_j m_j + \sum_{l \in K_i^I} \beta_{jl} m_l - p_j^I; 0\} \geq u_j^C + \sum_{l \in K_i^C} \beta_{jl} m_l - p_j^C$$

$$j \in K_i^I \Rightarrow u_j^I + \alpha_j m_j + \sum_{l \in K_i^I} \beta_{jl} m_l - p_j^I \geq 0$$

Now suppose that all groups in $J \backslash K_1^C \cap K_2^C$ don't buy from C, and that all groups in $K_1^I \cup K_2^I$ buy from I. Since for the firms in $J \backslash K_1^C \cap K_2^C$

$$\max\{u_j^I + \alpha_j m_j + \sum_{l \in K_1^I \cup K_2^I} \beta_{jl} m_l - p_j^I; 0\} \geq u_j^C + \sum_{l \in K_1^C \cap K_2^C} \beta_{jl} m_l - p_j^C;$$

the minimal benefit that a member of a group in $J \backslash K_1^C \cap K_2^C$ can obtain when not buying from C is larger than the maximal benefit it can gain when buying from C. Therefore the optimal strategy is either to buy from I or not to buy at all.

Moreover for a firm in $K_1^I \cup K_2^I$:

$$u_j^I + \alpha_j m_j + \sum_{l \in K_1^I \cup K_2^I} \beta_{jl} m_l - p_j^I \geq 0:$$

Thus the optimal strategy is indeed to buy from I. There is thus an equilibrium with $K_1^I \cup K_2^I \subset K^I$ and $K^C \subset K_1^C \cap K_2^C$: Taking a maximal element complete the proof. ■

B Appendix to Section 3

Lemma 3 Given prices $\{p_1^I; \ldots; p_J^I\}$ and Π; the minimal profit that firm C can obtain in a continuation equilibrium is smaller or equal to Π if and only if

$$\sum_{j \in K} p_j^I m_j \leq \Pi_K^I + \sum_{j \in K} \sum_{l \notin K} \beta_{jl} m_l + \Pi \quad \text{for all } \sigma \text{ and } K \subset J: \tag{9}$$

Proof. It is true if $J = 1$: Suppose the proposition holds up to $\# J = J - 1$ and consider $\# J = J$: Suppose first that C decides to sell to only $J - 1$

group, leaving outside one group (say $j = J$) by setting p_J^I high. Given that group J stay with I, the problem of attract $K < J$ groups within J groups is the same as with only $J - 1$ groups but with a utility for group j:

$$u_j + \alpha_j n_j^I + \beta_{jJ} m_J + \sum_{l=1}^{J-1} \beta_{jl} n_l^I$$

when buying from I. Condition 9 for $K < J$ then ensures that attracting less than J group can't yield more than Π.

Consider now attracting all the groups. For at least one of the group the price p_j^C must be smaller than $p_j^I - (\delta_1 + \sum_{l \neq j} \beta_{jl} m_l)$. Let us say it is group 1. We then set $p_1^C = p_1^I - \left(\delta_1 + \sum_{l>1} \beta_{1l} m_l\right)$ and group 1 buys from C for sure. Here the problem is reduced to $J - 1$ groups, but now with utility

$$u_j + \alpha_j n_j^I + \sum_{l=2}^{J} \beta_{jl} n_l^I$$

at firm I, and utility

$$u_j^C + \alpha_j n_j^C + \sum_{l=2}^{J} \beta_{jl} n_l^C + \beta_{j1} m_1$$

at C. The total profit on the $J-1$ groups is less than $\Pi - (p_1^I - \left[\delta_1 + \sum \beta_{jl} m_l\right]) m_1$, which is precisely condition 9 for $\sigma(1) = 1$. Therefore C can't get more than Π in the worst case faced to the pricing strategy of firm 1. ■

Proof of proposition 2. As a consequence, the incumbent serves all the market if and only if

$$\text{for all } K: \quad \sum_{j \in K} p_j^I m_j \leq \Pi_K^I + \sum_{j \in K} \sum_{l \notin K} \beta_{jl} m_l m_j.$$

W.l.o.g. let us assume that Π_J^I is obtained for the order $1, 2, \ldots J$. Suppose that I sets prices

$$p_j^I = \delta_j - \sum_{l<j} \beta_{jl} m_l + \sum_{l>j} \beta_{jl} m_l.$$

Then

$$\sum_{j \in J} p_j^I m_j = \Pi_J^I$$

Now take any subset K and let σ be such that

$$\Pi^l_K = \sum_{j\in K}\delta_j\mu_j - \sum_{j\in K}\sum_{\substack{l\in K\\ \sigma(l)>\sigma(j)}}(\beta_{lj}-\beta_{jl})\mu_l\mu_j$$

Suppose that

$$\sum_{j\in K} p^l_j\mu_j > \Pi^l_K + \sum_{j\in K}\sum_{l\notin K}\beta_{jl}\mu_l\mu_j$$

and let us show that it leads to contradiction. The condition writes

$$-\sum_{j\in K}\left(\sum_{l<j}\beta_{jl}\mu_l - \sum_{l>j}\beta_{jl}\mu_l\right)\mu_j > -\sum_{j\in K}\sum_{\substack{l\in K\\ \sigma(l)>\sigma(j)}}(\beta_{lj}-\beta_{jl})\mu_l\mu_j + \sum_{j\in K}\sum_{l\notin K}\beta_{jl}\mu_l\mu_j$$

Define $\hat\sigma$ as the order K is ranked below $J\backslash K$; σ on K and the increasing order $J\backslash K$. Then

$$\begin{aligned}
-\sum_{j\in J}\sum_{l>j}(\beta_{lj}-\beta_{jl})\mu_l\mu_j &> -\sum_{j\notin K}\left(\sum_{l<j}\beta_{jl}\mu_l - \sum_{l>j}\beta_{jl}\mu_l\right)\mu_j \\
&\quad - \sum_{j\in K}\sum_{\substack{l\in K\\ \sigma(l)>\sigma(j)}}(\beta_{lj}-\beta_{jl})\mu_l\mu_j + \sum_{j\in K}\sum_{l\notin K}\beta_{jl}\mu_l\mu_j \\
&> -\sum_{j\notin K}\sum_{\substack{l\notin K\\ l>j}}(\beta_{lj}-\beta_{jl})\mu_l\mu_j - \sum_{j\in K}\sum_{\substack{l\in K\\ \sigma(l)>\sigma(j)}}(\beta_{lj}-\beta_{jl})\mu_l\mu_j \\
&\quad + \sum_{j\notin K}\sum_{\substack{l\in K\\ l>j}}\beta_{jl}\mu_l\mu_j - \sum_{j\notin K}\sum_{\substack{l\in K\\ l<j}}\beta_{jl}\mu_l\mu_j + \sum_{j\in K}\sum_{l\notin K}\beta_{jl}\mu_l\mu_j \\
&> -\sum_{j\in J}\sum_{\hat\sigma(l)>\hat\sigma(j)}\big(\beta_{lj}-\beta_{jl}\big)\mu_l\mu_j + \sum_{j\in K}\sum_{l\notin K}(\beta_{lj}-\beta_{jl})\mu_j\mu_l \\
&\quad + \sum_{j\notin K}\sum_{\substack{l\in K\\ l>j}}\beta_{jl}\mu_l\mu_j - \sum_{j\notin K}\sum_{\substack{l\in K\\ l<j}}\beta_{jl}\mu_l\mu_j + \sum_{j\in K}\sum_{l\notin K}\beta_{jl}\mu_l\mu_j \\
&> -\sum_{j\in J}\sum_{\hat\sigma(l)>\hat\sigma(j)}\big(\beta_{lj}-\beta_{jl}\big)\mu_l\mu_j + 2\sum_{j\in K}\sum_{\substack{l\notin K\\ l<j}}\beta_{lj}\mu_j\mu_l
\end{aligned}$$

But this implies that

$$\Pi^l_J > \sum_{j\in J}\delta_j\mu_j - \sum_{j\in J}\sum_{\hat\sigma(l)>\hat\sigma(j)}\big(\beta_{lj}-\beta_{jl}\big)\mu_l\mu_j;$$

a contradiction. Therefore, it must be the case that

$$\sum_{j\in K} p^I_j \mu_j \le \Pi^I_K + \sum_{j\in K}\sum_{l\notin K} \beta_{jl}\mu_l\mu_j \text{ for all } K.$$

■

Proof of lemma 2. For any order σ on J there is an exact reverse ordering $\hat\sigma$. But then

$$\sum_{j\in J}\sum_{\hat\sigma(l)>\hat\sigma(j)}(\beta_{lj}-\beta_{jl})\mu_l\mu_j = -\sum_{j\in J}\left(\sum_{\sigma(l)>\sigma(j)}(\beta_{lj}-\beta_{jl})\mu_l\mu_j\right).$$

Hence the maximum over all permutations is non-negative. Now, suppose that the maximum is zero. This means that the summation term is 0 for any permutation. Consider a permutation σ such that $\sigma(J-1) = J-1$ and $\sigma(J) = J$; and a permutation $\tilde\sigma$ such that $\tilde\sigma(j) = \sigma(j)$ if $j < J-1$; $\tilde\sigma(J-1) = J$; $\tilde\sigma(J) = J-1$. We have just reversed the order of the last two in the order. Then

$$0 = \sum_{j<J-1}\left(\sum_{\sigma(l)>\sigma(j)}(\beta_{lj}-\beta_{jl})\mu_l\mu_j\right) + (\beta_{J,J-1}-\beta_{J-1,J})\mu_{J-1}\mu_J;$$

$$0 = \sum_{j<J-1}\left(\sum_{\sigma(l)>\sigma(j)}(\beta_{lj}-\beta_{jl})\mu_l\mu_j\right) + (\beta_{J-1,J}-\beta_{J,J-1})\mu_{J-1}\mu_J.$$

Taking the difference yields $\beta_{J-1,J} = \beta_{J,J-1}$. Extending the reasoning to any pair of groups, we obtain $\beta_{jl} = \beta_{lj}$. ■

Proof of proposition (3). In equilibrium I sets prices for $j \in K$ at

$$p^I_j = u^I_j + \alpha_j\mu_j - \max\{u^C_j + \sum_{l\in K}\beta_{jl}\mu_l - p^C_j; 0\} + \sum_{l\notin K}\beta_{jl}\mu_l$$

which is the maximal price that ensures that a member of group j buys from I when groups $l \in K$ do. If I decides to attract groups in $L \subset K = \{1; \ldots K\}$, it can do so by setting prices for all groups outside $K \setminus L$; including those groups it already serves:

$$\hat p^I_j = u^I_j + \alpha_j\mu_j - \max\{u^C_j + \sum_{l\in K\setminus L}\beta_{jl}\mu_l - p^C_j; 0\} + \sum_{l\notin K\setminus L}\beta_{jl}\mu_l$$

The gain in profit is then

$$\sum_{j\in L}\hat{p}_j\mu_j + \sum_{j\notin K}(\hat{p}_j - p^I_j)\mu_j$$

Notice first that for $j \notin K$; the price differential $\hat{p}_j - p^I_j$ is larger than $\sum_{l\in L}\beta_{jl}\mu_l\mu_j$; with equality when p^C_j is high enough so that I can capture the full surplus of the transaction. It is thus optimal for C to set very high prices for these groups. The gain in profit is then

$$\sum_{j\in L}\hat{p}_j\mu_j + \sum_{j\notin K}\sum_{l\in L}\beta_{jl}\mu_l\mu_j$$

If C want to sell to all groups in K; it must then set prices such that for all subset L:

$$\sum_{j\in L}p^C_j\mu_j \le -\sum_{j\in L}\delta_j\mu_j + \sum_{j\in L}\sum_{l\in K\setminus L}\beta_{jl}\mu_l\mu_j - \sum_{j\in L}\sum_{l\notin K\setminus L}\beta_{jl}\mu_l\mu_j - \sum_{j\notin K}\sum_{l\in L}\beta_{jl}\mu_l\mu_j$$

An upper bound on the profit is then obtained for $L = K$;

$$\Pi^C \le -\sum_{j\in K}\delta_j\mu_j - \sum_{j\in K;l\in J}\beta_{jl}\mu_l\mu_j - \sum_{j\notin K;l\in K}\beta_{jl}\mu_l\mu_j = \Pi^C_K$$

It is easily seen that it can be obtained by setting prices $p^C_j = -\delta_j - \sum_{l\in J}\beta_{jl}\mu_l - \sum_{l\notin K}\beta_{lj}\mu_l\mu_j$; so that this is indeed the maximal profit. ∎

C Appendix to Section 4

Proof of proposition 6. Assume that I sells to group 1. Each consumers should prefer to stay with its group rather than to join the other group. Using $\Pi^I = p^I_1\mu_1$ and $\Pi^C = p^C_2\mu_2$ this yields

$$u^I_1\mu_1 + \alpha_1\mu_1^2 - \max\left\{u^C_1\mu_1 + \beta_{12}\mu_1\mu_2 - p^C_1\mu_1; 0\right\} \ge \Pi^I$$
$$u^C_2\mu_2 - \max\left\{u^I_2\mu_2 + \alpha_2\mu_2^2 + \beta_{21}\mu_1\mu_2 - p^I_2\mu_2; 0\right\} \ge \Pi^C$$

Clearly to sustain the equilibrium, the best is to put prices p^C_1 and p^I_2 at their minimal value given profits; which yields:

$$\Pi^I = p^C_1\mu_1 + \delta_1\mu_1 - \beta_{12}\mu_1\mu_2$$
$$\Pi^C = p^I_2\mu_2 - \delta_2\mu_2 - \beta_{21}\mu_1\mu_2$$

Equilibrium conditions then write for C:

$$\Pi^C \geq \Pi^I - \delta_1\mu_1 - \beta_{12}\mu_1\mu_2 + u^C_2\mu_2 - \max\left\{u^I_2\mu_2 + \alpha_2\mu_2^2 - p^I_2\mu_2; 0\right\} + \beta_{21}\mu_1\mu_2$$

$$\Pi^C \geq \Pi^I + p^I_2\mu_2 - \delta_1\mu_1 - \delta_2\mu_2 + \beta_{12}\mu_1\mu_2 - \beta_{21}\mu_1\mu_2$$

or

$$\max\left\{-\beta_{21}\mu_1\mu_2; \Pi^C - u^C_2\mu_2\right\} + \delta_1\mu_1 + (\beta_{12} - \beta_{21})\mu_1\mu_2 \geq \Pi^I$$
$$\delta_1\mu_1 - \beta_{12}\mu_1\mu_2 \geq \Pi^I$$

For I we then get

$$\Pi^I \geq \Pi^C + p^C_1\mu_1 + \delta_1\mu_1 + \delta_2\mu_2 + (\beta_{12} + \beta_{21})\mu_1\mu_2$$

or

$$-\delta_2\mu_2 - (2\beta_{12} + \beta_{21})\mu_1\mu_2 \geq \Pi^C$$

which implies $-\beta_{21}\mu_1\mu_2 > \Pi^C - u^C_2\mu_2$. Overall this yields

$$\delta_1\mu_1 + (\beta_{12} - 2\beta_{21})\mu_1\mu_2 \geq \Pi^I$$
$$\delta_1\mu_1 - \beta_{12}\mu_1\mu_2 \geq \Pi^I$$
$$-\delta_2\mu_2 - (2\beta_{12} + \beta_{21})\mu_1\mu_2 \geq \Pi^C$$

This can be written as

$$\Pi^I_1 - X_1 \geq \Pi^I$$
$$\Pi^C_2 - \beta_{12}\mu_1\mu_2 \geq \Pi^C$$

If I sells to group 2 the conditions are the same and yields $X_2 = 0$.

Now suppose that there exists an equilibrium with market covering. Then if I covers the market $\delta_j \geq -\beta_{jl}\mu_l - 2\beta_{lj}\mu_l$ for both groups which implies that $\Pi^C_j - \beta_{lj}\mu_l \leq 0$. If C covers the market, $\delta_1 \leq \beta_{12}\mu_2 + 2\beta_{21}\mu_2$ and $\delta_2 \leq \beta_{21}\mu_1$, which implies $\Pi^I_2 \leq 0$. The only possibility is then a market sharing equilibrium where I sells to group 1. ■

D Appendix to section 6

Let us first show that I must cover the market if it sells as a Stackelberg leader. The maximal profit it can gain but selling to a subset K of groups is

$$(u^I - u^C)\sum_{j\in K}\mu_j + \alpha\sum_{j\in K}\mu_j^2 - \beta\left(\sum_{j\in K}\mu_j\right)\left(\sum_{l\notin K}\mu_l\right)$$

while serving the whole population yields $\pi^I_J = \left(u^I - u^C\right)M + \beta\sum_{j\in J} m_j^2$. The difference between the profit with the whole population and the profit with K is thus

$$(u^I - u^C)\sum_{j\notin K} m_j + \beta\sum_{j\notin K} m_j^2 + \beta\left(\sum_{j\in K} m_j\right)\left(\sum_{j\notin K} m_j\right)$$

Suppose that this is nonpositive so that I prefers to sell to K only. Then

$$u^I - u^C \le -\beta\frac{\sum_{j\notin K} m_j^2}{\sum_{j\notin K} m_j} - \beta\sum_{j\in K} m_j.$$

This implies that

$$\begin{aligned}\pi^I_K &\le -\beta\left(\frac{\sum_{j\notin K} m_j^2}{\sum_{j\notin K} m_j} + \sum_{j\in K} m_j\right)\sum_{j\in K} m_j + \beta\sum_{j\in K} m_j^2\\ &< -\beta\left(\sum_{j\in K} m_j\right)^2 + \beta\sum_{j\in K} m_j^2 \le 0.\end{aligned}$$

Thus I doesn't sell at all. As a consequence I covers the market or doesn't sell.

Proof of proposition 7. Suppose that $u^C - u^I \le \beta\frac{\sum_j \alpha_j m_j^2}{M}$. Consider equilibrium prices $0 \le p^I_j \le u^I - u^C + \beta m_j$; $p^C_j = p^I_j - \beta M$. C doesn't sell and we know that it is playing a best response. The maximal deviation profit that I can obtain is

$$\begin{aligned}&\sum_{j\in K}\left(p^I_j + \beta\sum_{j\notin K} m_j - \beta M\right)m_j\\ =&\sum_{j\in K} p^I_j m_j + \beta\left(\sum_{j\notin K} m_j - M\right)\sum_{j\in K} m_j\\ \le&\sum_{j\in K} p^I_j m_j.\end{aligned}$$

So this is an equilibrium as long as I obtains a positive profit.

Suppose that $u^C - u^I \ge \beta M$. Let $\hat{p}_j$ be the prices exhibited in the proof of proposition 2 that allows I to obtain π^I_J: $\hat{p}_j = u^I - u^C + \beta\sum_{l\le j} m_l - \beta\sum_{l>j} m_l$. Let $p^I_j = \hat{p}_j + \frac{\epsilon}{M}$; $p^C_j = \frac{\epsilon}{M}$; $0 \le \epsilon \le \pi^C_J$. At these prices there is a REA where C covers the market (and another one where I covers the market). C can't obtain more than ϵ because its maximal profit is $\sum p^I_j m_j -$

$\Pi^I_J = \Pi$. I can't obtain a positive profit because its profits is bounded by $p^C_j m_j - \Pi^C_J \le 0$. Therefore this an equilibrium. It is also clear that these are the unique one as I can't sell at a nonnegative price (C would attract such a group for sure with a zero price).

Suppose now $\beta M > u^C - u^I > \beta \frac{\sum_j \alpha_j m_j^2}{M}$. Suppose that C sell to groups within K with prices p^C_j. Then I sells to the others (since $p^C_j \ge u^C_j$ for them,). I can leave all its price unchanged, except that it charges $p^C_j + u^I - u^C + \beta M - \varepsilon$ to groups in K. Because D^I applies for almost all prices, I would then sell to all groups but then it would increase its profit by $\Pi^C + \left(u^I - u^C + \beta M\right)\sum_{j\in K} m_j > \Pi^C$. Therefore C can't sell. Since C can always sell to a customer j not willing to buy from I at a price u^C_j, I must cover the market. But I can't cover the market with positive profit since $\Pi^I_J < 0$. Thus there is no equilibrium. ■

E Dropping Assumption 2 with an heterogeneous population

This part discusses the consequences of dropping Assumption 2 while maintaining Assumption 3. First it is immediate that the set of equilibria is enlarged. In particular there are more equilibria with market covering by one firm in the simultaneous pricing game, although as discussed in Section 5, a pure strategy equilibrium may fail to exists. We focus here on market sharing.

The most profitable market sharing equilibrium

Suppose that I is leader of the Stackelberg game. Clearly it is optimal from the viewpoint of I that I dominates in beliefs if C deviates from equilibrium prices. Thus we can impose $A(P^I; P^C) = D^I(P^I; P^C)$ if C deviates. But suppose it is not the case on the equilibrium path. Let I sell to j and C sell to l.

The prices for l must verify:

$$u^C_l + \alpha_l m_l - \max\left\{u^I_l + \beta_{lj} m_j - p^I_l, 0\right\} \ge p^C_l.$$

The analysis of deviations is unchanged, so that incentive compatibility conditions are (imposing $p^I_l \le \tilde{p}^I_l$):

$$p^C_l m_l \ge u^C_l m_l - \max\left\{u^I_l m_l + \alpha_l m_l^2 - p^I_l m_l, 0\right\} + \beta_{lj} m_1 m_2 + p^I_j m_j - \delta_j m_j - \beta_{jl} m_1 m_2$$

$$p^C_l m_l \ge p^I_j m_j + p^I_l m_l - \delta_j m_j - \delta_l m_l + \left(\beta_{jl} - \beta_{lj}\right) m_1 m_2$$

which yields

$$\max\left\{u_l m_l + \alpha_l m_l^2 - p_l^l m_l; 0\right\} + \alpha_l m_l^2 - \max\left\{u_l m_l + \beta_{lj} m_1 m_2 - p_l^l m_l; 0\right\}$$
$$\geq p_j^l m_j - \delta_j m_j + (\beta_{lj} - \beta_{jl}) m_1 m_2$$

and

$$u_l m_l - p_l^l m_l + 2\alpha_l m_l^2 - \max\left\{u_l m_l + \beta_{lj} m_1 m_2 - p_l^l m_l; 0\right\}$$
$$\geq p_j^l m_j - \delta_j m_j + \left(\beta_{jl} - \beta_{lj}\right) m_1 m_2$$

For both conditions, the LHS is maximal at $u_l m_l + \beta_{lj} m_1 m_2 = p_l^l m_l$; where they become

$$\delta_j m_j - (\beta_{lj} - \beta_{jl}) m_1 m_2 - \min\left\{\beta_{lj} m_1 m_2; \alpha_l m_l^2\right\} + 2\alpha_l m_l^2 \geq \Pi^l$$
$$\delta_j m_j - \beta_{jl} m_1 m_2 + 2\alpha_l m_l^2 \geq \Pi^l$$

which yields a maximal profit for l

$$\Pi^{1l}_j = \delta_j m_j - \max\left\{(\beta_{lj} - \beta_{jl}) m_1 m_2 + \min\left\{\beta_{lj} m_1 m_2; \alpha_l m_l^2\right\}; \beta_{jl} m_1 m_2\right\} + 2\alpha_l m_l^2:$$

Notice that it may well be the case that the bound obtained is higher than $u_j m_j + \alpha_j m_j^2$ if $\alpha_l m_l^2$ is high, in which case l can only charge $u_j + \alpha_j m_j$: the monopoly profit in a market sharing equilibrium: The maximal profit is thus this case set

$$\hat{\Pi}^l_j = \max\{u_j m_j + \alpha_j m_j^2; \Pi^{1l}_j\}:$$

It is immediate to verify that $\hat{\Pi}^l_j > \Pi^l_j$:

Lemma 4 Assume 3; simultaneous pricing but not Assumption 2. Let $\hat{\Pi}^k_j$ be the maximal profit that firm k can obtain in an equilibrium where it is leader but not necessarily dominant. Then there exists $X^k > 0$; such that an equilibrium where k serves group 1 and its opponent serves group 2 exists if and only if $\hat{\Pi}^k_1 \geq X^k$ and $\hat{\Pi}^{-k}_2 \geq 0$:

Proof. Assume that l sells to group 1.

$$\begin{aligned}
\hat{\Pi}^l_1 &= \min\left\{u_1 m_1 + \alpha_1 m_1^2; \delta_1 m_1 + 2\alpha_2 m_2^2 - \max\{A_1; \beta_{12} m_1 m_2\}\right\} \\
\text{where } A_1 &= (\beta_{21} - \beta_{12}) m_1 m_2 + \min\left\{\beta_{21} m_1 m_2; \alpha_2 m_2^2\right\} \\
\hat{\Pi}^c_2 &= \min\{u_2^c m_2 + \alpha_2 m_2^2; -\delta_2 m_2 + 2\alpha_1 m_1^2 + 2\alpha_2 m_2^2 - \beta_{21} m_1 m_2\}
\end{aligned}$$

The REA of consumers implies that each consumers should prefer to stay with its group rather than to join the other group and benefit from the inter-group network effect.

Using $\Pi^I = p_1^I m_1$ and $\Pi^C = p_2^C m_2$

$$
\begin{aligned}
u_1^I m_1 + \alpha_1 m_1^2 - \max\left\{u_1^C m_1 + \beta_{12} m_1 m_2 - p_1^C m_1; 0\right\} &\geq \Pi^I \\
u_2^C m_2 + \alpha_2 m_2^2 - \max\left\{u_2^I m_2 + \beta_{21} m_1 m_2 - p_2^I m_2; 0\right\} &\geq \Pi^C
\end{aligned}
$$

As before, the best is to put prices p_1^C and p_2^I at their minimal value given profits:

$$
\begin{aligned}
p_1^C m_1 &= \Pi^I - \delta_1 m_1 + \beta_{12} m_1 m_2 \\
p_2^I m_2 &= \Pi^C + \delta_2 m_2 - 2\alpha_2 m_2^2 + \beta_{21} m_1 m_2
\end{aligned}
$$

with

$$
\begin{aligned}
u_1^I m_1 + \alpha_1 m_1^2 &\geq \Pi^I \\
u_2^C m_2 + \alpha_2 m_2^2 &\geq \Pi^C
\end{aligned}
$$

Equilibrium conditions then write:

$$
\begin{aligned}
\Pi^C \geq\ & \Pi^I - \delta_1 m_1 - \beta_{12} m_1 m_2 \\
& + u_2^C m_2 - \max\left\{u_2^I m_2 + \alpha_2 m_2^2 - p_2^I m_2; 0\right\} + \beta_{21} m_1 m_2 \\
\Pi^C \geq\ & \Pi^I - \delta_1 m_1 + \beta_{12} m_1 m_2 \\
& + p_2^I m_2 - \delta_2 m_2 - \beta_{21} m_1 m_2
\end{aligned}
$$

$$
\begin{aligned}
\Pi^I \geq\ & \Pi^C + \delta_1 m_1 - 2\alpha_2 m_2^2 - \beta_{21} m_1 m_2 \\
& + u_1^I m_1 - \max\left\{u_1^C m_2 + \alpha_1 m_1^2 - p_1^C m_2; 0\right\} + \beta_{12} m_1 m_2 \\
\Pi^I \geq\ & \Pi^C + \delta_2 m_2 - 2\alpha_2 m_2^2 + \beta_{21} m_1 m_2 \\
& p_1^C m_2 + \delta_1 m_1 - 2\alpha_1 m_1^2 - \beta_{12} m_1 m_2
\end{aligned}
$$

(Notice that the first constrain for I is implied by the second.)

Reporting the value of the prices into the conditions we find:

$$
\begin{aligned}
\max\left\{2\alpha_2 m_2^2 - \beta_{21} m_1 m_2; \Pi^C - u_2^C m_2\right\} &\geq \Pi^I - \delta_1 m_1 + (\beta_{21} - \beta_{12}) m_1 m_2 \\
\delta_1 m_1 - \beta_{12} m_1 m_2 + 2\alpha_2 m_2^2 &\geq \Pi^I
\end{aligned}
$$

$$
-\delta_2 m_2 + 2\alpha_2 m_2^2 + 2\alpha_1 m_1^2 - \beta_{21} m_1 m_2 \geq \Pi^C
$$

Define

$$X^I = \min\left\{\beta_{21} m_1 m_2; \alpha_2 m_2^2\right\} + \min\left\{\beta_{21} m_1 m_2 - \alpha_2 m_2^2; \gamma m_2 + 2\alpha_2 m_2^2 - \pi_2^C\right\}$$

Notice that $X^I > 0$. We can find nonnegative profits Π^k when

$$\begin{aligned} \delta_1 m_1 + 2\alpha_2 m_2^2 - A_1 &\geq X^I \\ \delta_1 m_1 + 2\alpha_2 m_2^2 - \beta_{12} m_1 m_2 &\geq 0 \\ \pi_2^C &\geq 0 \end{aligned}$$

Notice that this implies that $\pi_1^I > 0$. The condition for I holds if

$$\pi_1^I \geq X^I.$$

On the other hand if $\pi_1^I < X^I$, it is impossible to fulfill the first condition.
■