RECENT DEVELOPMENTS IN (GENERAL) EQUILIBRIUM MODELING

1. Mixed Complementarity Programming

An economic equilibrium can be cast as a mixed complementarity problem. For illustration, we consider a standard Arrow-Debreu economy with n commodities (incl. factors), m sectors and h households (incl. government).[1]

The endogenous variables of the Arrow-Debreu economy can be classified into 3 categories (Mathiesen 1985):

- $\boldsymbol{p}$:= a non-negative n-vector in prices for all goods and factors (I ={1,...,n})
- $\boldsymbol{y}$:= a non-negative m-vector for activity levels of CRTS-production sectors (J={1,...,M}), and
- $\boldsymbol{M}$:= a non-negative k-vector in incomes (H ={1,...,k})

In equilibrium the variables must fulfill three classes of conditions:

- zero profit of CRTS-producers (exhaustion-of-product constraint)

  $$-\Pi_j(p) = C_j(p) - R_j(p) \geq 0 \quad \forall j$$

  where (using Hotelling's Lemma):

  $\Pi_j(p)$ the unit profit function,

  $$C_j(p) \equiv \min\left\{\sum_i p_i \frac{\partial \Pi_j}{\partial p_i} \middle| f_j(\cdot) = 1\right\}$$ the unit costs function, and

  $$R_j(p) \equiv \max\left\{\sum_i p_i \frac{\partial \Pi_j}{\partial p_i} \middle| g_j(\cdot) = 1\right\}$$ the unit revenue function.

  The functions f_j and g_j characterize feasible input- and output-combinations of production in sector j.

- market clearance for all goods and factors:

[1] The competitive setup can be easily extended by price restrictions and quantity constraints that represent market imperfections..

$$\sum_j y_j \frac{\partial \Pi_j(p)}{\partial p_i} + \sum_h b_{ih} \geq \sum_h d_{ih} \quad \forall i$$

where:

b_{ih} the initial endowment of household h with commodity i, and

$d_{ih}(p, M_h) \equiv \arg\max \left\{ U_h(x) \middle| \sum_i p_i x_i = M_h \right\}$ the demand for good i by household h maximizing utility.[2]

- budget constraints for households:

$$\sum_h p_i b_{ih} = M_h \geq \sum_h p_i d_{ih} \quad \forall h$$

For common utility functions (non-satiation), households are always on their budget line, i.e.. $\sum_h p_i b_{ih} = M_h = \sum_h p_i d_{ih}$, and Walras' law holds.[3]

Using Walras' law, aggregation of market clearance conditions and zero profit conditions yields:

$$\sum_j y_j \Pi_j(p) = 0\,; \quad y_j \Pi_j(p) = 0 \quad \forall j$$

and

$$p_i \left(\sum_j y_j \frac{\partial \Pi_j(p)}{\partial p_i} + \sum_h b_{ih} - \sum_h d_{ih} \right) = p_{i\xi} \xi_i = 0 \quad \forall i$$

as well as:

$$M_h \left(\sum_h p_i b_{ih} - \sum_h p_i d_{ih} \right) = 0 \quad \forall h\,.$$

Thus, economic equilibrium features complementarity between equilibrium variables and equilibrium conditions: positive market prices imply market clearance – otherwise commodities are in excess supply and the respective prices fall to zero. Activities will be operated as long as they break even - negative revenues imply the shutdown of the respective production activities.

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[2] U_h denotes the utility function of household h.

[3] Due to linear-homogeneity of profit functions and homogeneity of demand functions of degree zero in prices, the economic equilibrium only determines relative prices.

The MCP (Cottle and Pang 1992, Rutherford 1995):

$$\begin{aligned} Given: f: R^n \to R^n \\ Find: z \in R^n \\ s.t.: f(z) \geq 0,\ z \geq 0,\ z^T f(z) = 0 \end{aligned} \qquad \textbf{(MCP)}$$

corresponds to the problem of finding an economic equilibrium for *z = [y, p, M]* and *f(z)* = $[\,\Pi_j(p), \xi_i, (\sum_h p_i b_{ih} - \sum_h p_i d_{ih})\,]$, hereby stating complementarity between variables and equilibrium conditions..

In this context the term „*mixed complementarity problem*" (MCP) is straightforward: „*mixed*" indicates that the mathematical program includes equalities as well as inequalities; „*complementarity*" refers to complementary slackness between system variables and system conditions. The above formulation of an economic equilibrium is very compact: Variables such as consumption and factor demand or commodity supplies are treated implicitly which reduces computation time significantly for higher-dimensional models.

2. Functional Forms in Calibrated Share Form

Numerical calculation of an economic equilibrium, requires the choice of concrete functional forms for production possibilities and preferences. In applied modeling, combinations of Leontief-, Cobb-Douglas-, or constant-elasticity-of-substitution (CES) functions are most common.

Tables 1 and 2 provide an overview of standard functional forms in conventional coefficient form and (less common) calibrated share form.

The following notations are adopted with respect to functional forms characterizing production:

α_i	distribution parameter for input *i*,
γ	scale parameter,
ρ	substitution parameter,
σ	substitution elasticity ($\rho := \frac{\sigma - 1}{\sigma}$),

$\overline{x}_i$	benchmark demand for input *i*,
$\overline{w}_i$	benchmark price for input *i*,
$\overline{y}$	benchmark output level $\left[\overline{y}=f\left(\overline{x}\right)=1\right]$,
$\overline{p}$	benchmark output price,
$\overline{C}$	benchmark cost,
$\overline{c}$	benchmark unit cost, and
$\theta_i=\frac{\overline{x}_i\cdot\overline{w}_i}{\overline{y}\cdot\overline{p}}$	benchmark value share of input *i*.

Equivalent notations for functional forms characterizing final demand include:

$\overline{d}_i$	benchmark consumption demand for good *i*,
$\overline{p}_i$	benchmark consumer price for good *i*,
$\overline{U}$	benchmark utility level $\left[\overline{U}=g\left(\overline{d}\right)=1\right]$,
$\overline{E}$	benchmark expenditure,
$\overline{e}$	benchmark unit expenditure,
$\overline{M}$	benchmark income, and
$\theta_i=\frac{\overline{d}_i\cdot\overline{p}_i}{\overline{M}}$	benchmark value share of consumption demand *i*.

The so-called calibrated share form eases the calculation of free parameters of functional forms, because there is no need (as opposed to the coefficient form approach) to invert demand functions. Equivalence of the coefficient form and the calibrated share form is straightforward. For example, let us consider CES production functions. Using the inverted factor demand functions in coefficient form

$$\alpha_i=\gamma^{(1-\sigma)/\sigma}\cdot\left(\frac{x_i}{y}\right)^{1/\sigma}\frac{w_i}{p}$$

we obtain:

$$\alpha_i=\gamma^{(1-\sigma)/\sigma}\cdot\theta_i\cdot\overline{x}_i^{(1-\sigma)/\sigma}\quad .$$

Substituting this expression within the coefficient forms for production, cost and demand functions we can derive the equivalent calibrated share forms:

- for the CES production function:

$$y=\gamma\left(\sum_i \alpha_i x_i^{\rho}\right)^{1/\rho} = \bar{y}\cdot\gamma\left(\sum_i\left(\gamma^{-\rho}\cdot\theta_i\cdot\bar{x}_i^{-\rho}\right)x_i^{\rho}\right)^{1/\rho} = \bar{y}\cdot\left[\sum_i\left(\theta_i\cdot\left(\frac{x_i}{\bar{x}_i}\right)^{\rho}\right)\right]^{1/\rho},$$

- for the CES cost function:

$$C=\gamma^{-1/\sigma}\left[\sum_i \alpha_i^{\sigma}\gamma^{(\sigma-1)\cdot\rho}w_i^{1-\sigma}\right]^{1/(1-\sigma)}\cdot y = \gamma^{-1/\sigma}\left[\sum_i\left(\theta_i^{\sigma}\cdot\bar{x}_i^{1-\sigma}\gamma^{1-\sigma}\right)\gamma^{(\sigma-1)\cdot\rho}w_i^{1-\sigma}\right]^{1/(1-\sigma)}\cdot\frac{y}{\bar{y}}$$

$$=\left[\sum_i\theta_i\cdot\left(\bar{y}\cdot\bar{p}\right)^{1-\sigma}\cdot\left(\frac{w_i}{\bar{w}_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}\cdot\frac{y}{\bar{y}} = \bar{C}\cdot\left[\sum_i\theta_i\cdot\left(\frac{w_i}{\bar{w}_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}\cdot\frac{y}{\bar{y}}$$

and

- for the CES demand function:

$$x_i=\bar{C}\cdot\left[\sum_i\theta_i\cdot\left(\frac{w_i}{\bar{w}_i}\right)^{1-\sigma}\right]^{\sigma/(1-\sigma)}\cdot\frac{\theta_i}{\bar{w}_i}\cdot\left(\frac{w_i}{\bar{w}_i}\right)^{-\sigma}\cdot\frac{y}{\bar{y}} = C^{\sigma}\cdot\bar{C}^{1-\sigma}\cdot\frac{\bar{x}_i}{\bar{y}\cdot\bar{p}}\cdot\left(\frac{\bar{w}_i}{w_i}\right)^{\sigma}\cdot\left(\frac{y}{\bar{y}}\right)^{1-\sigma}$$

$$=\bar{x}_i\cdot\left(\frac{y}{\bar{y}}\right)^{1-\sigma}\cdot\left(\frac{C}{\bar{C}}\cdot\frac{\bar{w}_i}{w_i}\right)^{\sigma} = \bar{x}_i\cdot\frac{y}{\bar{y}}\cdot\left(\frac{p}{\bar{p}}\cdot\frac{\bar{w}_i}{w_i}\right)^{\sigma}$$

References

Cottle, R.W and J.S. Pang (1992), The Linear Complementarity Problem, Academic Press.

Mathiesen, L. (1985), Computation of Economic Equilibrium by a Sequence of Linear Complementarity Problems, in: A. Manne (ed.): Economic Equilibrium - Model Formulation and Solution, Mathematical Programming Study 23, S. 144-162.

Rutherford, T.F. (1995), Extensions of GAMS for complementarity problems arising in applied economics, *Journal of Economic Dynamics and Control* 19, 1299-1324.

Table 1: Functional forms in production

		Coefficient Form	**Calibrated Share Form**
Production function	Leontief	$y=\min_i\left(\frac{x_i}{\alpha_i}\right)$	$y=\bar{y}\cdot\min_i\left(\frac{x_i}{\bar{x}_i}\right)$
	CD	$y=\gamma\prod_i x_i^{\alpha_i}$	$y=\bar{y}\cdot\prod_i\left(\frac{x_i}{\bar{x}_i}\right)^{\theta_i}$
	CES	$y=\gamma\left(\sum_i\alpha_i x_i^{\rho}\right)^{1/\rho}$	$y=\bar{y}\cdot\left[\sum_i\left(\theta_i\cdot\left(\frac{x_i}{\bar{x}_i}\right)^{\rho}\right)\right]^{1/\rho}$
Cost function	Leontief	$C=\left(\sum_i\alpha_i\cdot w_i\right)\cdot y$	$C=\bar{C}\cdot\left[\sum_i\theta_i\cdot\left(\frac{w_i}{\bar{w}_i}\right)\right]\cdot\frac{y}{\bar{y}}$
	CD	$C=\frac{1}{\gamma}\prod_i\left(\frac{w_i}{\alpha_i}\right)^{\alpha_i}\cdot y$	$C=\bar{C}\cdot\left[\prod_i\left(\frac{w_i}{\bar{w}_i}\right)^{\theta_i}\right]\cdot\frac{y}{\bar{y}}$
	CES	$C=\gamma^{-1/\sigma}\left[\sum_i\alpha_i^{\sigma}\cdot\gamma^{(\sigma-1)\cdot\rho}\cdot w_i^{1-\sigma}\right]^{1/(1-\sigma)}\cdot y$	$C=\bar{C}\cdot\left[\sum_i\theta_i\cdot\left(\frac{w_i}{\bar{w}_i}\right)^{1-\sigma}\right]^{1/(1-\sigma)}\cdot\frac{y}{\bar{y}}$
Demand function	Leontief	$x_i=\alpha_i\cdot y$	$x_i=\bar{x}_i\cdot\frac{y}{\bar{y}}$
	CD	$x_i=\frac{\alpha_i p}{w_i}\cdot y$	$x_i=\bar{x}_i\cdot\frac{y}{\bar{y}}\cdot\frac{c}{\bar{c}}\cdot\frac{\bar{w}_i}{w_i}$
	CES	$x_i=\gamma^{\sigma-1}\cdot\left(\frac{\alpha_i p}{w_i}\right)^{\sigma}\cdot y$	$x_i=\bar{x}_i\cdot\frac{y}{\bar{y}}\cdot\left(\frac{c}{\bar{c}}\cdot\frac{\bar{w}_i}{w_i}\right)^{\sigma}$

Table 2: Functional forms in consumption

		Coefficient Form	**Calibrated Share Form**
Utility function	Leontief	$U = \min_i \left[\frac{d_i}{\alpha_i} \right]$	$U = \bar{U} \cdot \min_i \left(\frac{d_i}{\bar{d}_i} \right)$
	CD	$U = \prod_i d_i^{\alpha_i}$	$U = \bar{U} \cdot \prod_i \left(\frac{d_i}{\bar{d}_i} \right)^{\theta_i}$
	CES	$U = \left(\sum_i \alpha_i^{1/\sigma} d_i^{\rho} \right)^{1/\rho}$	$U = \bar{U} \cdot \left[\sum_i \left(\theta_i \cdot \left(\frac{d_i}{\bar{d}_i} \right)^{\rho} \right) \right]^{1/\rho}$
Expenditure function	Leontief	$E = \left(\sum_i \alpha_i \cdot p_i \right) \cdot U$	$E = \bar{E} \cdot \left[\sum_i \theta_i \cdot \left(\frac{p_i}{\bar{p}_i} \right) \right] \cdot \frac{U}{\bar{U}}$
	CD	$E = \prod_i \left(\frac{p_i}{\alpha_i} \right)^{\alpha_i} \cdot U$	$E = \bar{E} \cdot \left[\prod_i \left(\frac{p_i}{\bar{p}_i} \right)^{\theta_i} \right] \cdot \frac{U}{\bar{U}}$
	CES	$E = \left[\sum_i \alpha_i \cdot p_i^{1-\sigma} \right]^{1/(1-\sigma)} \cdot U$	$E = \bar{E} \cdot \left[\sum_i \theta_i \cdot \left(\frac{p_i}{\bar{p}_i} \right)^{1-\sigma} \right]^{1/(1-\sigma)} \cdot \frac{U}{\bar{U}}$
Demand function	Leontief	$d_i = \alpha_i \frac{M}{\left(\sum_i \alpha_i p_i \right)}$	$d_i = \bar{d}_i \left(\frac{\sum_j \theta_j \bar{p}_j}{\sum_j \theta_j p_j} \right) \left(\frac{M}{\bar{M}} \right)$
	CD	$d_i = \frac{\alpha_i M}{p_i}$	$d_i = \bar{d}_i \frac{M}{\bar{M}} \frac{\bar{p}_i}{p_i}$
	CES	$d_i = \frac{\alpha_i M}{e} \left(\frac{e}{p_i} \right)^{\sigma}$	$d_i = \bar{d}_i \frac{M}{\bar{M}} \left(\frac{e}{\bar{e}} \right)^{\sigma-1} \left(\frac{\bar{p}_i}{p_i} \right)^{\sigma}$