Compensated Own- and Cross-Price Demand Elasticities for Separable Nested CES Technologies

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December 19, 2000

Dear CGE modellers

I got a nested CES function and want to extract the demand elasticities from this function. One nest is no problem, but 2 nests or even more seems to get tedious.

Any hints?

1 Calibrated Demand Functions

Suppose that we have a nested cost function of arbitrary depth and complexity. The prices for goods i and j, p_i and p_j are arguments of $C(p)$. Assume that the cost function $C(p)$ is *nested*. In the simplest two level case, we would have:

$$C(p) = \left(\sum_k \theta_k c_k(p)^{1-\sigma_0} \right)^{1/1-\sigma_0}$$

in which:

$$c_k(p) = \left(\sum_{i \in I_k} \alpha_{ik} p_i^{1-\sigma_k} \right)^{1/1-\sigma_k}$$

where I_k indicates the set of commodities entering nest k.

In a more general case, we could have cost aggregates as arguments to other cost aggregates. Figure 1 indicates a graph of the nested CES cost function in which we number the nested cost functions which lead from the top level to nest containing good i as C_0 (top level), C_1, ..., C_L.

If we construct the cost function from a calibrated benchmark in which input prices and total cost are unity , we can scale the benchmark values of the subaggregate cost functions as unity and express the demand for good i as:

$$x_i = \bar{x}_i \left(\frac{C_L}{p_i} \right)^{\sigma_L} \left(\frac{C_{L-1}}{C_L} \right)^{\sigma_{L-1}} \cdots \left(\frac{C_0}{C_1} \right)^{\sigma_0} \quad = \quad \bar{x}_i \; p_i^{-\sigma_L} \; C_0^{\sigma_0} \; \prod_{\ell=1}^{L} C_\ell^{\sigma_\ell - \sigma_{\ell-1}}$$

2 The Own-Price Elasticity of Demand

By Shephard's lemma the derivative of C_ℓ with respect to p_i equals the demand for good i per unit of aggregate ℓ. Recalling that all prices are scaled to unity, the benchmark "quantity" of aggregate ℓ equals the sum of the inputs which enter directly or indirectly into that cost function:

$$\bar{X}_\ell = \sum_{j \in I_\ell} \bar{x}_j$$

and

$$\left.\frac{\partial C_\ell}{\partial p_i}\right|_{p=1} = \begin{cases} 0 & i \notin I_\ell \\ \frac{\bar{x}_i}{\bar{X}_\ell} & i \in I_\ell \end{cases}$$

We then can compute the compensated own-price elasticity of demand for good i:

$$\eta_i \equiv \left.\frac{\partial x_i}{\partial p_i}\right|_{p=1} = -\sigma_L + \bar{x}_i \left(\sigma_0 + \sum_{\ell=1}^{L} \frac{\sigma_\ell - \sigma_{\ell-1}}{\bar{X}_\ell} \right)$$

3 The Cross-Price Elasticity of Demand

When we evaluate the elasticity of demand for i with respect to a change in the price of good j, we can let k denote the deepest price aggregate which contains both p_i and p_j. (See Figure 1).

The cross derivative can then be computed using the demand function for x_i, taking into account the impact of p_j on C_k, C_{k-1}, ..., C_0:

$$\left.\frac{\partial x_i}{\partial p_j}\right|_{p=1} = \bar{x}_i \left[\sigma_0 \frac{\partial C_0}{\partial p_j} + \sum_{\ell=1}^{k} (\sigma_\ell - \sigma_{\ell-1}) \frac{\partial C_\ell}{\partial p_j} \right]$$

One means of representing the dependence of x_i on p_j is with the Allen-Uzawa elasticity-of-substitution which:

$$\sigma_{ij} \equiv \frac{\partial x_i}{\partial p_j} \frac{C_0}{x_i x_j} = \sigma_0 + \sum_{\ell=1}^{k} \frac{\sigma_\ell - \sigma_{\ell-1}}{X_\ell}$$

As a logical check on this elasticity, consider two special cases:

$\sigma_\ell = \sigma_0 \quad \forall \ell$	Single level CES implies a constant cross elasticity of substitution between all input pairs, $\sigma_{ij} = \sigma_0$
$\sigma_\ell = 0 \quad \forall \ell < k$	Leontief demand for aggregate k, implies that the elasticity of substitution between i and j is defined as $\sigma_{ij} = \sigma_k$ (N.B. The cross elasticity between i and j is independent of $\sigma_\ell \quad \forall \ell > k$).

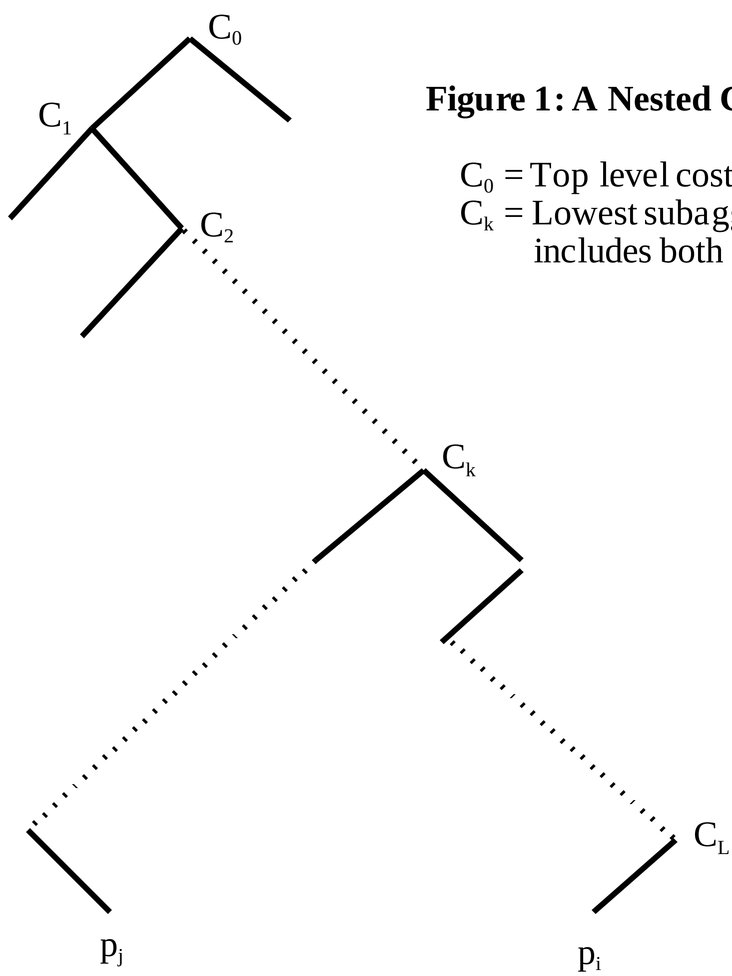

Figure 1: A Nested CES Cost Function

C_0 = Top level cost function
C_k = Lowest subaggregate which includes both p_i and p_j