1. The "mixed-complementarity problem" (MCP):

Given: $$F : R^N \to R^N, \quad \ell, u \in R^N \qquad \text{(MCP)}$$

Find: $$z, w, v \in R^N$$

s.t. $$F(z) - w + v = 0$$

$$\ell \leq z \leq u, \quad w \geq 0, \quad v \geq 0$$

$$w^T(z - \ell) = 0, \quad v^T(u - z) = 0$$

in which $-\infty \leq \ell \leq u \leq +\infty$.

Special case i: *a linear system of equations*

$$\text{Given:} \quad A \in R^{n\times n}, \quad b \in R^n \quad \text{(LSYS)}$$

$$\text{Find:} \quad x \in R^n$$

$$\text{s.t.} \quad Ax = b$$

which is represented as an MCP by letting $\ell = -\infty$, $u = +\infty$, $z = x$, and $F(z) = Az - b$;

Special case ii: *a nonlinear system of equations*

$$\text{Given:} \qquad f : R^n \rightarrow R^n, \quad b \in R^n \qquad \text{(NLSYS)}$$

$$\text{Find:} \qquad x \in R^n$$

$$\text{s.t.} \qquad f(x) = 0$$

which is represented as an MCP by letting $\ell = -\infty$, $u = +\infty$, $z = x$, and $F(z) = f(z)$;

Special case iii: *a linear complementarity problem*

Given: $M \in R^{n \times n}, \quad q \in R^n$ (LCP)

Find: $z \in R^n$

s.t. $q + Mz \geq 0, \quad z \geq 0, \quad z^T(q + Mz) = 0$

which is represented as an MCP by letting $\ell = 0$ $u = +\infty$, and $F(z) = q + Mz$;

Special case iv: *a nonlinear complementarity problem*

Given: $f : R^n \to R^n$ (NCP)

Find: $z \in R^n$

s.t. $f(z) \geq 0, \quad z \geq 0, \quad z^T f(z) = 0$

which is represented as an MCP by setting $\ell = 0$, $u = +\infty$, and $F(z) = f(z)$;

Special case v: *a nonlinear program*

Given: $f : R^n \to R, \quad g : R^n \to R^m, \quad \widehat{\ell}, \widehat{u} \in R^n$ (NLP)

Find: $x \in R^n$ to

max $f(x)$

s.t. $g(x) = 0$

$$\widehat{\ell} \leq x \leq \widehat{u}$$

which (when $f()$ is concave and $g()$ is convex) may be represented as an MCP by setting $N = n + m$, and partitioning

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \ell = \begin{pmatrix} \widehat{\ell} \\ -\infty \end{pmatrix}, \quad u = \begin{pmatrix} \widehat{u} \\ +\infty \end{pmatrix}$$

$$F(z) = \begin{cases} \nabla f(x) - \nabla g(x)^T y \\ g(x) \end{cases}$$

Special case v: *a finite-dimensional system of variational inequalities*

$$\text{Given:} \qquad f : R^n \to R^n, \quad g : R^n \to R^m \qquad \text{(VIP)}$$

$$\text{Find:} \qquad x^* \in X \equiv \{\xi \in R^n | g(\xi) \geq 0\}$$

$$\max\ f(x^*)^T(x - x^*) \geq 0 \quad \forall x \in X$$

which (when $f()$ is convex and $g()$ is concave) may be represented as an MCP by setting $N = n + m$, and partitioning:

$$z = \begin{pmatrix} x \\ y \end{pmatrix}, \quad \ell = \begin{pmatrix} -\infty \\ 0 \end{pmatrix}, \quad u = (+\infty), \quad F(z) = \begin{cases} f(x) - \nabla g(x)^T y \\ g(x) \end{cases}$$

The Hitchcock-Koopmans Transportation Problem

A linear program seeks a transport schedule which minimizes the cost of meeting demands in markets (j) from suppliers (i):

$$\begin{aligned} \text{min} \quad & \textstyle\sum_{ij} c_{ij} x_{ij} \\ \text{s.t.} \quad & \textstyle\sum_j x_{ij} \le a_i \\ & \textstyle\sum_i x_{ij} \ge b_j \\ & x \ge 0 \end{aligned}$$

GAMS representation:

```
SETS
       I   canning plants    / SEATTLE, SAN-DIEGO /
       J   markets           / NEW-YORK, CHICAGO, TOPEKA / ;

PARAMETERS
        A(I)  capacity of plant i in cases
         /    SEATTLE     325
              SAN-DIEGO   575  /,
        B(J)  demand at market j in cases
         /    NEW-YORK    325
              CHICAGO     300
              TOPEKA      275  /,
        F  freight in dollars per case per thousand miles  /90/ ;

TABLE DIST(I,J)  distance in thousands of miles
                    NEW-YORK       CHICAGO      TOPEKA
      SEATTLE          2.5            1.7          1.8
      SAN-DIEGO        2.5            1.8          1.4   ;

PARAMETER C(I,J)  transport cost in thousands of dollars per case ;

  C(I,J) = F * DIST(I,J) / 1000 ;
```

```
VARIABLES       X(I,J)          SHIPMENT QUANTITY FROM I TO J
                COST            MINIMAND - TOTAL COST OF SHIPMENT

POSITIVE VARIABLE X;

EQUATIONS
                SUPPLY(I)       SUPPLY LIMIT
                DEMAND(J)       DEMAND CONSTRAINT (FIXED)
                OBJDEF          DEFINES COST;

SUPPLY(I)..     A(I) =G= SUM(J, X(I,J));

DEMAND(J)..     SUM(I, X(I,J))  =G=  B(J);

OBJDEF..        COST =E= SUM((I,J), C(I,J) * X(I,J));

MODEL MINCOST / SUPPLY, DEMAND, OBJDEF/;

SOLVE MINCOST USING LP MINIMIZING COST;
```

Interpretation as a Market Equilibrium Problem

w_i marginal cost in supply market i

p_j market price in demand market j

$$\textstyle\sum_j x_{ij} \le a_i, \quad w_i \ge 0, \quad w_i \left(a_i - \sum_j x_{ij}\right) = 0 \quad \forall i$$

$$\textstyle\sum_i x_{ij} \ge b_j, \quad p_j \ge 0, \quad p_j \left(b_j - \sum_i x_{ij}\right) = 0 \quad \forall i$$

$$w_i + c_{ij} \ge p_j, \quad x_{ij} \ge 0, \quad x_{ij} \left(w_i + c_{ij} - p_j\right) = 0 \quad \forall i, j$$

This is a *linear* complementarity problem.

GAMS representation:

```
POSITIVE VARIABLES
                W(I)            SHADOW PRICE AT SUPPLY NODE I,
                P(J)            SHADOW PRICE AT DEMAND NODE J,
                X(I,J)          SHIPMENT QUANTITIES IN CASES;

EQUATIONS
                SUPPLY(I)       SUPPLY LIMIT AT PLANT I,
                DEMAND(J)       FIXED DEMAND AT MARKET J,
                PROFIT(I,J)     ZERO PROFIT CONDITIONS;

SUPPLY(I)..     A(I) =G= SUM(J, X(I,J));

DEMAND(J)..     SUM(I, X(I,J))  =G=  B(J);

PROFIT(I,J)..   W(I) + C(I,J)   =G= P(J);

MODEL TRNSP / PROFIT.X, SUPPLY.W, DEMAND.P/ ;

SOLVE TRNSP USING MCP;
```

Price-Responsive Supply and Demand

Demand: $D_j(p_j) = \beta_j p_j^{-\sigma_j}$

Supply: $S_i(w_i) = \alpha_i w_i^{\eta_i}$

$$\textstyle\sum_j x_{ij} \le S_i(w_i), \quad w_i \ge 0, \quad w_i \left(S_i(w_i) - \sum_j x_{ij}\right) = 0 \quad \forall i$$

$$\textstyle\sum_i x_{ij} \ge D_j(p_j), \quad p_j \ge 0, \quad p_j \left(D_j(p_j) - \sum_i x_{ij}\right) = 0 \quad \forall i$$

$$w_i + c_{ij} \ge p_j, \quad x_{ij} \ge 0, \quad x_{ij} \left(w_i + c_{ij} - p_j\right) = 0 \quad \forall i, j$$

This is a *nonlinear complementarity problem*.

NLP Formulations

When the nonlinear complementarity problem is *integrable*, the solution corresponds to the first order conditions for one (or two) nonlinear programming problem. In this case, we could solve a *primal* NLP:

$$\max \; \sum_i \gamma_i y_i^{\epsilon_i^y} + \sum_j \kappa_j d_j^{\epsilon_j^d} - \sum_{ij} c_{ij} x_i j$$

$$\text{s.t.} \qquad \sum_j x_{ij} \leq y_i$$

$$\sum_i x_{ij} \geq d_j$$

$$x_{ij} \geq 0, \quad d_j \geq 0, \quad y_i \geq 0$$

where:

$$\gamma_i = \frac{\eta_i}{1+\eta_i}\left(\frac{1}{\alpha_i}\right)^{1/\eta_i},$$

$$\kappa_j = \frac{\sigma_j}{1+\sigma_j}\beta_j^{1/\sigma_j},$$

$$\epsilon_i^y = \frac{1+\eta_i}{\eta_i},$$

and

$$\epsilon_j^d = \frac{\sigma_j - 1}{\sigma_j};$$

or we could solve a *dual* NLP model:

$$\min \ \sum_i \frac{\alpha_i}{1+\eta_i} w_i^{1+\eta_i} - \sum_j \frac{\beta_j}{1-\sigma_j} p_j^{1-\sigma_j}$$

$$\text{s.t.} \qquad w_i + c_{ij} \geq p_j$$

N.B. Although this example is *integrable*, this is not always true: *NLP* $\subset$ *MCP*

Economic equilibrium problems addressing interesting policy questions are typically *not* integrable. Taxes, income effects, spillovers and other externalities interfere with the skew symmetry property which characterizes first order conditions for nonlinear programs.

2. General Equilibrium Models

p = a non-negative *n*-vector of commodity prices including all final goods, intermediate goods and primary factors of production;

y = a non-negative *m*-vector of activity levels for constant returns to scale production sectors in the economy; and

M = an *h*-vector of income levels, one for each "household" in the model, including any government entities.

$\Pi_j(p)$ is the unit profit function for sector j, the difference between unit revenue and unit cost, defined as:

$$C_j(p) \equiv \min \left\{ \sum_i p_i x_i | f_j(x) = 1 \right\}$$

and

$$R_j(p) \equiv \max \left\{ \sum_i p_i y_i | g_j(y) = 1 \right\}$$

$d_{ih}(p, M_h)$ is a demand function derived from budget-constrained utility maximization:

$$d_{ih}(p, M_h) = \text{argmax} \left\{ U_h(x) | \sum_i p_i x_i = M_h \right\}$$

in which U_h is the utility function for household h.

Equilibrium

Market Clearance:

$$\sum_j y_j \frac{\partial \Pi_j(p)}{\partial p_i} + \sum_h \omega_{ih} \geq \sum_h d_{ih}(p, M_h)$$

Zero Profit:

$$-\Pi_j(p) = C_j(p) - R_j(p) \geq 0 \qquad \forall\ j$$

Income Balance:

$$M_h = \sum_i p_i \omega_{ih}$$

Model Representation: MPSGE

Social accounting data:

	Sectors (S)	Consumers Households(H)	Government
Goods Markets (G):	A(G,S)-B(G,S)	-C(G,H)	
Factor Markets (F):	-FD(F,S)	E(F,H)-D(F,H)	
Capital taxes:	-T("K",S)		GREV
Transfers:		TRN(H)	-GREV

MPSGE model:

```
$SECTORS:
      AL(S)      ! Activity levels

$COMMODITIES:
      P(G)       ! Commodity prices
      W(F)       ! Factor return
      PT         ! Tax revenue transfer market

$CONSUMERS:
      RA(H)      ! Representative households
      GOVT       ! Government

$PROD:AL(S)  s:0  va:ELAS(S)
      O:P(S)      Q:A(S)
      I:P(G)      Q:B(G,S)
      I:W(F)      Q:FD(F,S)   P:PF(F,S)   A:GOVT   T:TF(F,S)  va:

$DEMAND:RA(H)  s:1  gds:ESUB(H)
      D:P(G)      Q:C(G,H)   gds:
      D:W(F)      Q:D(F,H)
      E:W(F)      Q:E(F,H)
      E:PT        Q:TRN(H)

$DEMAND:GOVT
      D:PT        Q:GREV
```

Definition of Mixed Complementarity Problem (MCP)

The MCP format:

$$\text{Given: } F: R^N \to R^N,\ \ell, u \in R^N$$
$$\text{Find: } z, w, v \in R^N$$
$$\text{s.t. } \quad F(z) - w + v = 0 \qquad \textbf{(MCP)}$$
$$\ell \le z \le u,\ w \ge 0,\ v \ge 0$$
$$w^T(z - \ell) = 0, \quad v^T(u - z) = 0$$
$$\infty \le l \le u \le \infty$$

(with *z*: decision variable; *v,w*: slack variables, *u*:upper bound, *l*:lower bound) encompasses a number of special cases, e.g.:

(i) *a nonlinear system of equations*:

$$\text{Given: } f: R^n \to R^n$$
$$\text{Find: } x \in R^n \quad \text{s.t. } f(x) = 0 \qquad \textbf{(NLSYS)}$$

with l=-∞, *u*=+∞, *z* = *x* and *F(z)* = *f(z)*;

(ii) *a nonlinear complementarity problem*:

$$\text{Given: } f: R^n \to R^n$$
$$\text{Find: } z \in R^n_+ \ \text{ s.t. } f(z) \ge 0,\ \ z^T f(z) = 0 \qquad \textbf{(NLCP)}$$

with l=0, *u*=+∞, and letting *F(z)* = *f(z)*;

(iii) *a nonlinear program:*

$$\text{Given: } f: R^n \to R,\ g: R^n \to R^m,\ \hat{\ell}, \hat{u} \in R^n$$
$$\text{Find: } x \in R^n_+ \ \text{ to}$$
$$\max\ f(x)$$
$$\text{s.t. } g(x) = 0 \qquad \textbf{(NLP)}$$
$$\hat{\ell} \le x \le \hat{u}$$

(when *f*() is concave and *g*() is convex) with *N* = *n*+*m*, and partitioning:

$$z = \begin{pmatrix} x \\ y \end{pmatrix},\ \ell = \begin{pmatrix} \hat{\ell} \\ -\infty \end{pmatrix},\ u = \begin{pmatrix} \hat{u} \\ +\infty \end{pmatrix},\ F(z) = \begin{cases} \Delta f(x) - \Delta g(x)^T y \\ \\ g(x) \end{cases};$$