The All-Pay Auction with Non-Monotonic Payoff

Subhasish Modak Chowdhury*

*School of Economics, ESRC Centre for Competition Policy, and Centre for Behavioural and Experimental Social Sciences, University of East Anglia

CCP Working Paper 10-6

Abstract: We analyse an endogenous prize all-pay auction under complete information where it is possible that none of the bidders wins and the winning payoff becomes non-monotonic in own bid. We derive the conditions for the existence of pure strategy equilibria and fully characterize the unique mixed strategy equilibrium when pure strategy equilibria do not exist. The high-valuation bidder places two distinct mass points in his equilibrium mixed strategy and the equilibrium support of the low-valuation bidder is not continuous. Under common valuation, in the equilibrium mixed strategy both bidders place mass points at the same point of the support. Possible applications are discussed.

JEL Classification: C62, C72, D74, D44

Keywords: Contests, All-pay auction, No-win, Non-monotonicity, Pure Strategy

Acknowledgements:
I appreciate useful comments from Dan Kovenock, Stephen Martin, Bill Novshek, Roman Sheremeta, Ron Siegel, Justin Tobias, Iryna Topolyan, participants at the 2009 International Conference in Game Theory, Spring 2009 Midwest Economic theory meetings, and the seminar participants at Purdue University, University of Cincinnati and the University of East Anglia. Any remaining errors are mine.

ISSN 1745-9648

Contact Details:
Subhasish Modak Chowdhury, ESRC Centre for Competition Policy, and Centre for Behavioural and Experimental Social Sciences, University of East Anglia, Norwich, NR4 7TJ, email: s.modak-chowdhury@uea.ac.uk, Phone: +44 1603 592 099, Fax: +44 1603 456 259

The All-pay Auction with Non-monotonic Payoff

Subhasish M. Chowdhury*

School of Economics, Centre for Competition Policy, and Centre for Behavioral and Experimental Social Sciences, University of East Anglia, Norwich NR4 7TJ, UK

Abstract

We analyze an endogenous prize all-pay auction under complete information where it is possible that none of the bidders wins and the winning payoff becomes non-monotonic in own bid. We derive the conditions for the existence of pure strategy equilibria and fully characterize the unique mixed strategy equilibrium when pure strategy equilibria do not exist. The high-valuation bidder places two distinct mass points in his equilibrium mixed strategy and the equilibrium support of the low-valuation bidder is not continuous. Under common valuation, in the equilibrium mixed strategy both bidders place mass points at the same point of the support. Possible applications are discussed.

JEL Classifications: C62, C72, D74, D44

Keywords: Contests, All-pay auction, No-win, Non-monotonicity, Pure Strategy

---

* E-mail: S.Modak-Chowdhury@uea.ac.uk, Phone: +44 1603 592 099, Fax: +44 1603 456 259

I appreciate useful comments from Dan Kovenock, Stephen Martin, Bill Novshek, Roman Sheremeta, Ron Siegel, Justin Tobias, Iryna Topolyan, participants at the 2009 International Conference in Game Theory, Spring 2009 Midwest Economic theory meetings, and the seminar participants at Purdue University, University of Cincinnati and the University of East Anglia. Any remaining errors are mine.

1. Introduction

In an all-pay auction, bidders simultaneously bid for prize(s) and pay their bids irrespective of the outcome. Because of its wide applications to patent races, innovation tournaments, electoral contests, rent-seeking activities and legal disputes, to name a few, the all-pay auction has become a popular research topic. The basic first-price all-pay auction (where a single prize is awarded with certainty to the highest bidder) equilibrium under complete information is fully characterized by Baye et al. (1996).[1] They show that if there are unique bidders with highest and second-highest valuations for the prize, then a symmetric mixed strategy Nash equilibrium exists. For more than two bidders with the second-highest valuation, a continuum of asymmetric mixed strategy Nash equilibria exist. Also, the two highest valuation bidders randomize their bids from zero to the second-highest valuation and the other bidders bid zero. Only the highest valuation bidder earns a positive expected payoff. However, in their structure the valuation of prize is not directly affected by the bid and hence the payoff is monotonically decreasing in own bid.

There are numerous practical situations when the valuation of the prize in an all-pay auction is affected by the bid. An example of this is the dependence of a patent's value on the corresponding R&D expenditures in a patent race. A firm may earn a patent on a particular product if it can innovate that product before its rivals; and at the same time the firm's expected payoff from the patent is bigger if the product is of higher quality due to a higher volume of R&D expenditures.[2] Another example is the relationship between the size of the gains by lobbyists and the corresponding lobbying expenses in a rent-seeking game. A lobby group might succeed in influencing government decision by making more lobbying

[1] See also Baye et al. (1993) and Hilman and Riley (1989)
[2] For example, Arora et al. (2008) show a positive relationship between patent value and R&D investment for US manufacturing industry.

expenditure than its rivals; at the same time, the degree of influence might well be affected by the amount of lobbying expenditure.[3]

To our knowledge, Amegashie (2001) is the first to propose this nature of problem in an all-pay auction setting. He analyzes a job interview contest under complete information where the candidate with the highest qualification gets the job, and the salary offered to the successful candidate depends on his level of qualification. In this model the payoff becomes non-monotonic in own bid. As a result there is a possibility of the existence of a pure strategy equilibrium, where one bidder places a high bid and the other bidder abstains from bidding.[4] However, the payoff structure in this model has a very restrictive parametric form that restricts bidders to simultaneously abstaining from bidding. Also, possible mixed strategies equilibria under this non-monotonic payoff are not analyzed. Kaplan et al. (2003) construct a complete information model where 'innovation time' is the choice variable. Here a higher reward as well as a higher cost is incurred with a choice of lower time. The authors characterize equilibria under both symmetric and asymmetric valuation cases. Che and Gale (2006) model lobbying as an all-pay auction and take into account a possible cap on bidding. This structure can also be used for solving the problem of bid-dependent valuation where the choice variable positively influences the cost as well as the prize value.

Bos and Ranger (2008) and Sacco and Schmutzler (2008) analyze all-pay auctions under complete information with a specific emphasis on bid-dependent prize valuations. These two independent studies are closely related to the current study. Bos and Ranger

[3] An example of this type in rent seeking (Tullock, 1980) literature is given by Amegashie (1999). Other examples include singing contests such as American Idol; and business plan competitions. The best singer wins the singing contest and at the same time, a better performance gives a higher valued contract to the winner. The best proposal in a business plan competition wins the contest and a higher quality of the proposal attracts more funds from venture capitalists. For further examples in sports and labor contests see Bos and Ranger (2008).

[4] Araujo et al. (2008), on the other hand, use an incomplete information structure to obtain a non-monotonic payoff. They also show the existence of a pure strategy Nash equilibrium. Kaplan et al. (2002) also construct an incomplete information model where the prize is separable in bidder type.

(2008) construct a two-bidder all-pay auction where the prize value is increasing in own bid in a non-decreasing returns-to-scale fashion. The authors, however, make a strong assumption that makes the winning payoff monotonically decreasing in own bid. They characterize the unique mixed strategy Nash equilibrium. Sacco and Schumtzler (2008) construct an n-bidder model. They assume that the winning prize value is an increasing concave function of own bid minus the second highest bid and that the cost is convex. They find conditions under which a pure strategy Nash equilibrium can be obtained. To solve the mixed strategy equilibria, they make a strong assumption of monotonically decreasing payoff in bids. Both studies find equilibrium mixed strategies similar to Baye et al. (1996).

Siegel (2009a) constructs a general family of games called 'all-pay contests'. This model provides a generic structure that incorporates the majority of the features of the previous analyses in the literature. Specifically, it is an n-bidder model under complete information where the bidders possess a degree of asymmetry in terms of their prize valuations and cost functions. In addition, the bidders choose a costly 'score' (similar to a bid) that monotonically affects the prize value. Siegel (2009a) gives a generic formula for the equilibrium payoffs of this type of auction. But, even in this generic structure, the highest bidder wins a prize with certainty and the winning payoff is assumed to be monotonically decreasing in own bid. Siegel (2009b) is an extension of Siegel (2009a) where the author characterizes the equilibrium strategies and participation rules under similar assumptions.

In summary, there are two apparent features of all the existing models of all-pay auction. First, the highest bidder wins a prize with certainty (there is no possibility that no bidder wins). Second, in most cases the winning payoff is monotonically decreasing in own bid. But the possibility that no one wins and non-monotonicity of the winning payoff are consistent with several real life phenomena that are modeled by all-pay auctions. For

example, in a patent race two firms can bear costly investments in order to innovate a new product. But there is always a chance that none of them is successful (Loury, 1979).[5] Another example of no-win is a successful technological innovation that is not marketable. For example, after making costly investments, jetpacks and teleportation process were invented long back in 1961 and 1993; but because of non-marketability issues none of them returned any profit to the inventors.[6] In another case, two firms can expend costly resources to create a product prototype and place the prototype for a procurement auction (Che and Gale, 2003). There is always a possibility that the demand-side governing body does not like either of the prototypes and rejects both. Interest groups may lobby a government agency to influence the details of regulations (Baye et al, 1993), yet regulations may be issued that do not favor any special interest. Finally, several men may try to courtship a particular woman. But, despite their effort, she might decide to choose none.[7]

Interestingly enough, under each of the cases the winning payoff may turn out to be non-monotonic in own bid. For example, the R&D expenditure or rent-seeking effort is known to have diminishing returns.[8] It is also very much plausible to observe diminishing returns on the investments to improve product-prototype quality or the investment on courtship. In such cases (and even for the cases of constant returns to effort), for a weakly convex cost function, the payoff function becomes non-monotonic in own bid.

[5] Nti (1997) incorporates the no-win possibility in a Tullock contest.
[6] See Wilson (2007) for a detailed discussion of successful but non-marketable innovations.
[7] See Sozou and Seymour (2005) for failure in courtship efforts (giving gifts). Even non-human animals expend a variety of costly irreversible effort in terms of courtship dance, songs, use of pheromone, and physical contest with other males etc with the intention of mating a particular female (Daly, 1978 and Bastock, 2007). However, in the post courtship period, it is very much possible that none from a set of competing males is able to mate with the particular female they intend to.
[8] Tullock (1997) discusses declining returns of rent-seeking investments. Griliches and Mairesse (1990), Mairesse and Sassenou (1991) and Wakelin (2001), among others, observe diminishing returns of R&D investments in firm level USA, Japan, and UK data.

In the current study we construct a 2-bidder single-prize all-pay auction model where there is a possibility that no bidder wins the prize and the prize value becomes increasing and concave in own bid. This results in a non-monotonic winning payoff in own bid. We find sufficient conditions for the existence of pure strategy Nash equilibria and fully characterize the unique mixed strategy Nash equilibrium when pure strategy equilibria do not exist.

2. Theoretical Model

2.1 Construction of the All-pay Auction with Non-monotonic Payoff

There are two bidders 1 and 2 with initial value for a prize V_1 and V_2 with $V_1 \geq V_2 > 0$. The bidders place costly bids to win the prize and lowest bidder never wins the prize. The bids are denoted by x_1 and x_2. There is a possibility that none of them wins the prize. We can explain this as a 'No success' case of innovation driven by nature or quality standard of the buying party in a procurement auction. We incorporate this by including a random threshold $\widetilde{R}$ with known, twice differentiable and continuous cumulative probability distribution $G(.)$ described by nature, where $G(0) = 0, G'(.)=g(.)>0$, and $G''(.)=g'(.)<0$.[9] The winner is determined by the highest bid that is higher than the random threshold $\widetilde{R}$. Irrespective of the result, the bidders bear cost according to the known, twice differentiable and continuous cost function $C(.)$. The cost function starts from origin, is increasing and weakly convex in own bid i.e., $C(0) = 0, C'(.)=c(.)>0$, and $C''(.)=c'(.)\geq 0$. Hence, the payoff function (neglecting a tie) is written as:

$$\pi_t(x_t, x_{-t}; \widetilde{R}) = \begin{cases} V_t\text{-}C(x_t) & \text{if } x_t > Max(x_{-t}, \widetilde{R}) \\ -C(x_t) & \text{otherwise} \end{cases} \qquad (2.1)$$

[9] Bertoletti (2006) introduces a fixed, common knowledge threshold (reserve price) in bids under complete information whereas Araujo et al. (2008) use the same under incomplete information.

The expected prize value for the winner becomes $G(x_t)V_t$-x_t, if $x_t > x_{-t}$, where –t is denoted as the bidder 'not t'. In case of a tie in asymmetric initial value ($V_1 > V_2$), if both bidders bid more than the random threshold ($\widetilde{R}$), then the highest initial value bidder, i.e., bidder 1 wins the prize. In the case of common initial value ($V_1 = V_2$), such a tie is resolved by a coin toss. Given the conditions, we can rewrite the payoff function as:

$$\pi_t(x_t, x_{-t}) = \begin{cases} G(x_t)V_t - C(x_t) & \text{if } (x_t > x_{-t}) \text{ or } (x_t = x_{-t}) \text{ and } (V_t > V_{-t}) \\ G(x_t)\frac{V_t}{2} - C(x_t) & \text{if } (x_t = x_{-t}) \text{ and } (V_t = V_{-t}) \\ -C(x_t) & \text{Otherwise} \end{cases} \quad (2.2)$$

$G(.)$ captures the diminishing returns nature. Let us call the payoff at the winning state the winning payoff and denote the same for bidder t as $W_t(x_t)$. The losing payoff is $L_t(x_t)$. Hence$W_t(x_t) = G(x_t)V_t$-$C(x_t)$ and $L_t(x_t) = -C(x_t)$. Denote the game as $\Gamma(1,2; \widetilde{R})$. Also, call the graph of the winning (losing) payoff in the bid-payoff space the winning (losing) curve.

2.2 Shape of the Winning Curves

Given the payoff function stated in (2.2), it is possible to characterize the shapes of the winning and losing curves. The winning curves for both bidders start from the origin and are strictly concave. If $\frac{C'(0)}{G'(0)} \geq V_t$ (where t = 1, 2) then the winning payoff of bidder t is non-positive; otherwise his winning curve is inverted U-shaped with an unique maximum. The curve shows the non-mnotonicity of the payoff in this all-pay auction. In case of initial asymmetric valuation ($V_1 > V_2$), the maximum winning payoff of bidder 1 is strictly higher than that of bidder 2. Also, if we denote the bid that maximizes the winning payoff of bidder t by x_t^{Wmax}, thn $x_1^{Wmax} > x_2^{Wmax}$. As the bid increases, eventually the winning payoff becomes negative. The maximum bid that can earn bidder t a non-negative winning payoff (the 'reach' as defined in Siegel, 2009a) is denoted by $\bar{x}_t$. It is easy to show that $\bar{x}_1 > \bar{x}_2$.

We prove the above mentioned complete characterization of the winning curves in Appendix 1(Claims 1 through 7). It is trivial to check the shape of the losing curve. The diagrammatic representations of the curves under the different cases are shown in Figures 1.1 to 1.3 and in Figure 2. Given the shapes of the curves, we characterize below the equilibria of the game. Subsections 2.3 and 2.4 deal with the initial asymmetric value case ($V_1 > V_2$) whereas Subsection 2.5 deals with the initial common value case ($V_1 = V_2$).

2.3 Characterization of Equilibria under Initial Asymmetric Values: Pure Strategy Cases

Lemma 1. An equilibrium in pure strategies for the game $\Gamma(1,2;\widetilde{R})$ with $(V_1 > V_2)$ exists under condition (i) $\left\{\frac{C'(0)}{G'(0)} \geq V_1\right\}$, or (ii) $\left\{\frac{C'(0)}{G'(0)} \in [V_2, V_1)\right\}$, or (iii) $\left\{\frac{C'(0)}{G'(0)} < V_2 \text{ and } (x_1^{Wmax} \geq \bar{x}_2)\right\}$. Under condition (i) there exist unique equilibrium strategies $(x_1^*, x_2^*) = (0,0)$, whereas under condition (ii) or (iii) the unique equilibrium strategies are $(x_1^*, x_2^*) = \left(x_1^{Wmax}, 0\right)$.

Proof: The cases are shown graphically in Figures 1.1, 1.2 and 1.3. For mathematical proof, see Appendix 2.

Figure 1.1 PSNE Case (i)

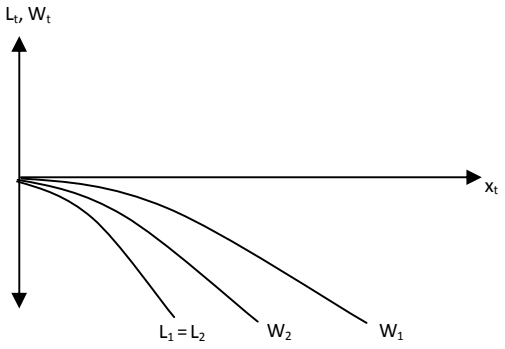


Figure 1.2 PSNE Case (ii)

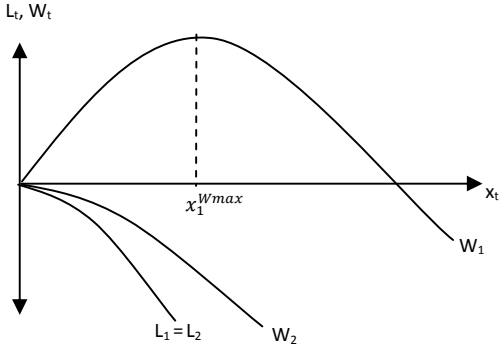
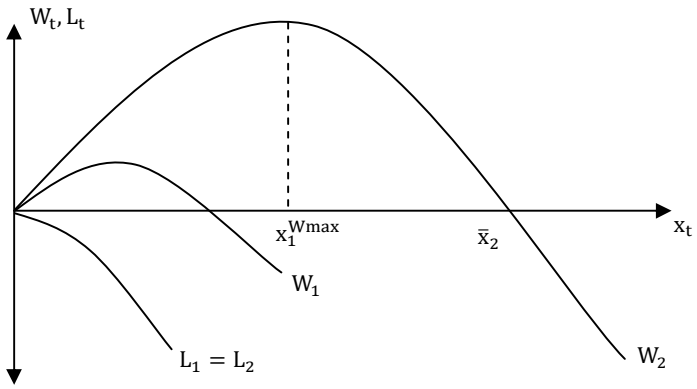


Figure 1.3 PSNE Case (iii)



Case (i) of Lemma 1 resembles the situation of a standard all-pay auction with reserve price (Bertoletti, 2006) if the reserve price is higher than the highest valuation and the bidders bid zero. Previous studies of all-pay auctions with non-monotonic payoff overlooked this case because of restrictive payoff function specifications. Cases (ii) and (iii) correspond to the results of previous studies. These cases are essentially the same, but we distinguish them as the intuitions are different for the two cases. In particular, a cap on bidding can improve competition in case (iii), but cannot in case (ii). These two cases, unlike the standard all-pay auction results, explain why in some situations only a single big company invests on some particular type of R&D project and small companies do not.

Lemma 2. If $\frac{C'(0)}{G'(0)} < V_2$ and $(x_1^{Wmax} < \bar{x}_2)$ then there exists no pure strategy Nash equilibrium for the game $\Gamma(1,2;\widetilde{R})$ with $(V_1 > V_2)$.

Proof: See Appendix 2.

Proposition 1. A pure strategy equilibrium for the game $\Gamma(1,2;\widetilde{R})$ with $V_1 > V_2$ exists if and only if any of the conditions (i) $\left\{\frac{C'(0)}{G'(0)} \geq V_1\right\}$, or (ii) $\left\{\frac{C'(0)}{G'(0)} \in [V_2, V_1)\right\}$, or (iii) $\left\{\frac{C'(0)}{G'(0)} < V_2 \text{ and } (x_1^{Wmax} \geq \bar{x}_2)\right\}$ holds. Moreover, under condition (i) there exist unique equilibrium

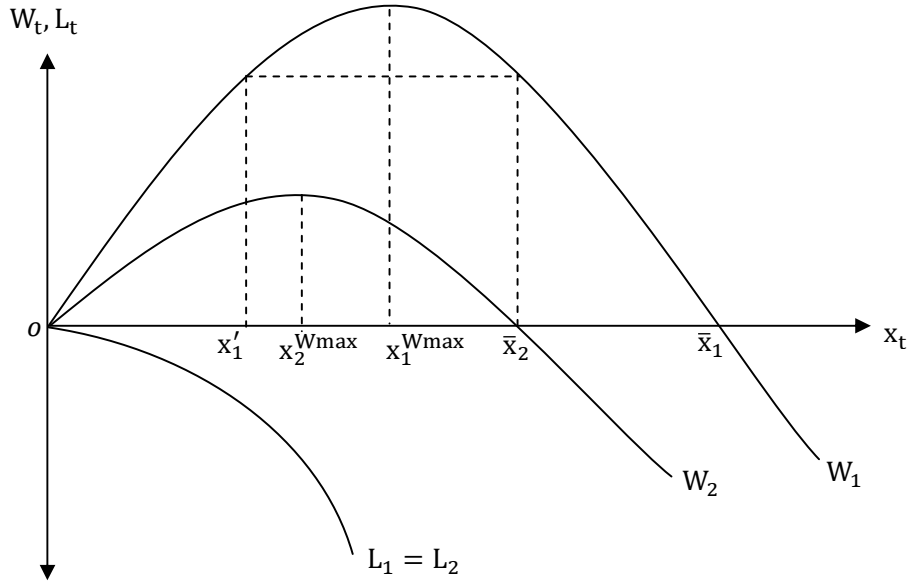
strategies $(x_1^*, x_2^*) = (0,0)$ whereas under condition (ii) or (iii) the unique equilibrium strategies are $(x_1^*, x_2^*) = \left(x_1^{Wmax}, 0\right)$.

Proof: Lemmas 1 and 2 imply Proposition 1. ■

Proposition 1 confirms that unlike the standard all-pay auction results as in Baye et al. (1996) or Siegel (2009a, b), under the non-monotonic payoff case we might end up attaining pure strategy Nash equilibria (as in Amegashie, 2001; Sacco and Schmutzler, 2008; and Araujo et al, 2008). The other studies, however, indicate only one possible pure strategy equilibrium, where the high value bidder places a positive bid and the low value bidder bids zero (cases (ii) and (iii) of Proposition 1). The current analysis shows that there can be another pure strategy equilibrium, where both bidders bid zero (case (i) of Proposition 1). More interestingly, for all the pure strategy equilibria under non-monotonic payoff, the payoff characterization results of Siegel (2009a) and strategy characterization results of Siegel (2009b) do not hold.

2.4 Characterization of Equilibria under Initial Asymmetric Values: Mixed Strategy Case

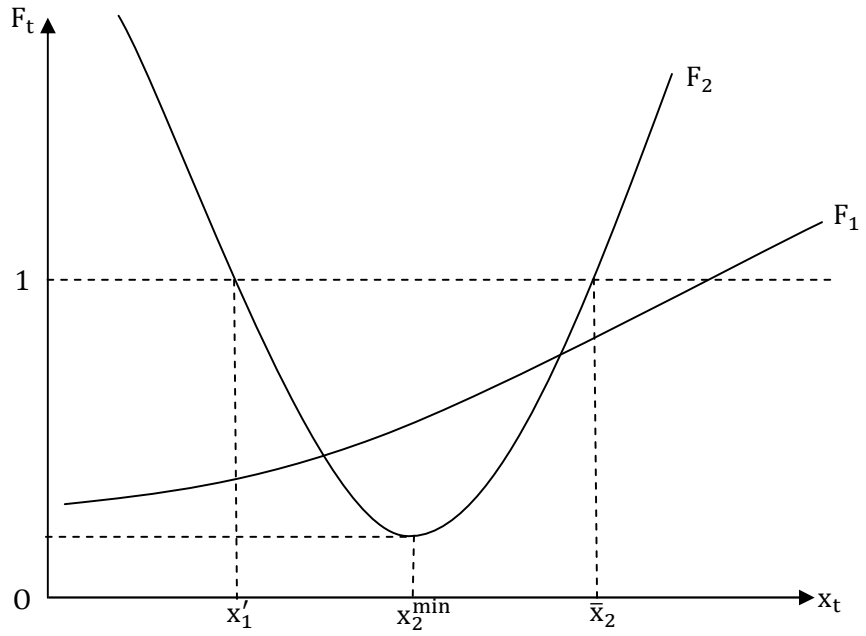
Figure 2. No Pure Strategy Equilibrium Case



In this section we discuss only the case of $\left\{\frac{C'(0)}{G'(0)} < V_2 < V_1 \text{and } (x_1^{Wmax} < \bar{x}_2)\right\}$, i.e., the case with no PSNE. We fully characterize the mixed strategy Nash equilibrium for the game $\Gamma(1,2;\widetilde{R})$ under this condition. This, in turn, proves the existence of equilibrium in mixed strategies that follows directly from theorem 5 of Dasgupta and Maskin (1986).

We define the No-arbitrage Bid Function (NBF) of bidder t to keep the other bidder indifferent as $F_t(x)$. In Lemmas 3 to 17 we derive the No-arbitrage Bid Functions as: $F_1(s) = \frac{C(s)}{V_2 G(s)}$, and $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1 G(s)}$. In Lemmas 18 to 20 we construct the shapes of the NBFs that are graphically represented in Figure 3. All the Lemmas (3 to 20) are stated and proved in Appendix 3.

Figure 3. No-arbitrage Bid Functions



It is clear that the No-arbitrage Bid Functions are not strategies. In particular, $F_2(s)$ is not nondecreasing. However, following Osborne and Pitchick (1986) and Deneckere and Kovenock (1996), the NBFs remain the basis for the construction of equilibrium. Let

$IF_2(s) = \underset{x \geq s}{\text{Inf}}\big(F_2(x)\big)$ be the nondecreasing floor of $F_2(s)$. $IF_2(s)$ equals $F_2(s)$ except the interval$[0, x_2^{\min})$. Then the strategy $Q_2(s) = \begin{cases} IF_2(s) & \text{for } (s < \bar{x}_2) \\ 1 & \text{for } (s \geq \bar{x}_2) \end{cases}$ is an equilibrium strategy for bidder 2. Note that $Q_2(s)$ is nondecreasing, non-negative, right continuous and is less than or equal to 1 for all s; hence, $Q_2(s)$ is a strategy. When bidder 2 does not bid according to $Q_2(s)$, it earns a strictly negative payoff . Given Q_2, if bidder 1 were indifferent between all bids in the interval $(0, \bar{x}_2]$, then F_1would be an equilibrium strategy, since it makes bidder 2 indifferent between all prices in the interval, and earns a strictly lower payoff otherwise. However, since Q_2 is strictly less than F_2 over the interval $[0, x_2^{\min})$, bidder 1 will attach zero probability to those set of bids. Since bidder 1 must set the strategy that keeps bidder 2 indifferent in the points of support, it will place a mass point at $x_2^{\min}$, the size of which equals $F_2\big(x_2^{\min}\big)$. Finally, bidder 1 will place zero probability to bid more than $\bar{x}_2$, hence it must place another mass point at $\bar{x}_2$ with a size$\big(1 - F_2(\bar{x}_2)\big)$. Hence the strategy of bidder 1 is: $Q_1(s) = \begin{cases} 0 & \text{for } (s < \bar{x}_2) \text{ and} \big(F_2(s) > Q_2(s)\big) \\ F_1(s) & \text{for } (s < \bar{x}_2) \text{ and} \big(F_2(s) = Q_2(s)\big) \\ 1 & \text{for } (s \geq \bar{x}_2) \end{cases}$. It is also easy to check that $Q_1(s)$ is nondecreasing, non-negative, right continuous, less than or equal to 1 for all s; and hence, is a strategy. In the following proposition we show that $Q_1(x)$ and $Q_2(x)$ constitute the unique mixed strategy Nash equilibrium of this game.

Proposition 2. The unique mixed strategy Nash equilibrium of the game $\Gamma(1,2;\widetilde{R})$ with restrictions $\left\{\frac{C'(0)}{G'(0)} < V_2 < V_1 \text{ and } (x_1^{Wmax} < \bar{x}_2)\right\}$ is characterized by the CDF pair $Q_1^*(s)$ and $Q_2^*(s)$. Where, the equilibrium CDF for bidder 1 is

$$Q_1^*(s) = \begin{cases} 0 & \text{for } s < x_2^{\min} \\ \frac{C(s)}{V_2 G(s)} & \text{for } s \in \big[x_2^{\min}, \bar{x}_2\big) \\ 1 & \text{for } s \geq \bar{x}_2 \end{cases}$$

i.e., there are two atoms: at x_2^{min} with a mass of size $\alpha_1(x_2^{min}) = \frac{C(x_2^{min})}{V_2 G(x_2^{min})}$ and at $\bar{x}_2$ with a mass of size $\alpha_1(\bar{x}_2) = \left(\frac{W_2(\bar{x}_2)}{V_2 G(\bar{x}_2)}\right)$. And the equilibrium CDF for bidder 2 is

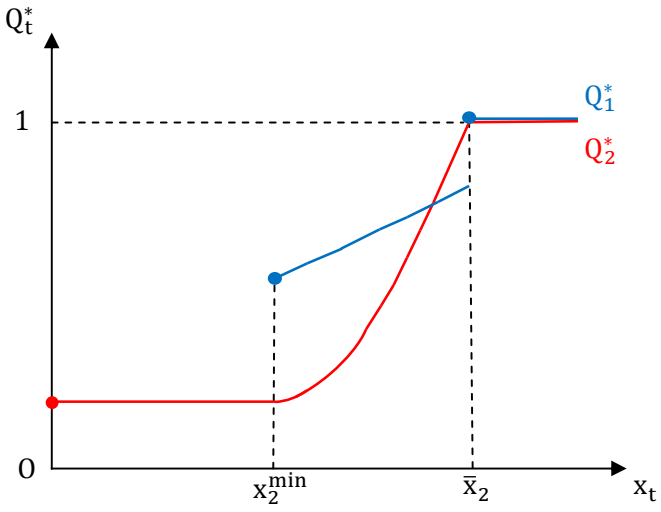
$$Q_2^*(s) = \begin{cases} \dfrac{C(x_2^{min}) + W_1(\bar{x}_2)}{V_1 G(x_2^{min})} & \text{for } s \le x_2^{min} \\ \dfrac{C(s) + W_1(\bar{x}_2)}{V_1 G(s)} & \text{for } s \in [x_2^{min}, \bar{x}_2] \\ 1 & \text{for } s \ge \bar{x}_2 \end{cases}$$

i.e., there is an atom at 0 with the size of the mass: $\alpha_2(0) = \frac{C(x_2^{min}) + W_2(\bar{x}_2)}{V_1 G(x_2^{min})}$.

Proof: We prove this proposition in two parts. First, we conclude that the pair $\{Q_1(s), Q_2(s)\}$ indeed characterizes an equilibrium. Then we show that the equilibrium is unique.

It is easy to show that $Q_1^*(s) = Q_1(s)$ and $Q_2^*(s) = Q_1(s)$. Therefore, from the previous discussion, $Q_1^*(s)$and $Q_2^*(s)$ are strategies and are also best response to each other. Hence, the strategy pair $\{Q_1^*(s)Q_2^*(s)\}$ characterize a mixed strategy Nash equilibrium for the game $\Gamma(1,2;\tilde{R})$. The diagrammatic representation of the equilibrium is described in Figure 4.

Figure 4. Equilibrium Distribution Functions



Now, if we show that the equilibrium support is unique, then the uniqueness of the equilibrium will also be proved. It is clear that the supports of equilibrium distribution coincide, and are equal to the interval $\left[x_2^{\min}, \bar{x}_2\right]$. In addition, bidder 2 has a mass point at 0. If $s \in \text{Support}(Q_t^*)$ then for any $F_t \neq Q_t^*$, $\pi_t(F_t(s, \pi_{-t}^*)) < \pi_t^*$. Also, if $\pi_t(F_t(s, \pi_{-t}^*)) > \pi_t^*$, then $s \notin \text{Support}(Q_t^*)$. Hence the respective supports of the bidders' mixed strategies unique and so the equilibrium mixed strategies. ■

There are two important features of the equilibrium. First, unlike the standard all-pay auction equilibrium, where the high-value bidder places no atom, here the high-value bidder places two atoms at the two extreme points in his support. Also, the low value bidder's support has a discontinuous point at zero. Although the equilibrium distributions are different from the standard all-pay auction, the equilibrium payoffs of the bidders are similar to the standard case and resemble the payoff characterization results of Siegel (2009a). However, because of the possibility of no-reward, the expected payoff is lower than that of the standard case.

2.5. Characterization of Equilibria under Initial Common Value Case

In the case of initial common value ($V_1 = V_1 = V$) all-pay auction with non-monotonic payoff, define $\bar{x} = \left\{x \neq 0: \frac{C'(0)}{G'(0)} < V \ \& W(x) = 0\right\}$. It can easily be shown that for $\frac{C'(0)}{G'(0)} \geq V$ the common winning curve will start from zero and is always negative. Following similar analysis as in section 2.3, we derive the following proposition.

Proposition 3. A pure strategy equilibrium for the game $\Gamma(1,2; \widetilde{R})$ with $V_1 = V_2 = V$ exists if and only if the condition $\frac{C'(0)}{G'(0)} \geq V$ holds. Moreover, under the specified condition, there exist unique equilibrium strategies $(x_1^*, x_2^*) = (0,0)$.

It can also be shown that for $\frac{C'(0)}{G'(0)} < V$ the common winning curve will start from zero and is inverted U-shaped. Following similar analysis as in section 2.4, we derive the following proposition. The proofs of Propositions 3 and 4 are obvious and are omitted.

Figure 5.1 Common Value Payoff functions

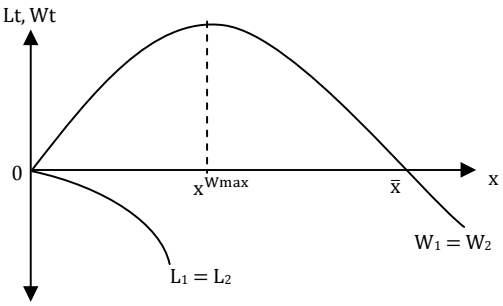
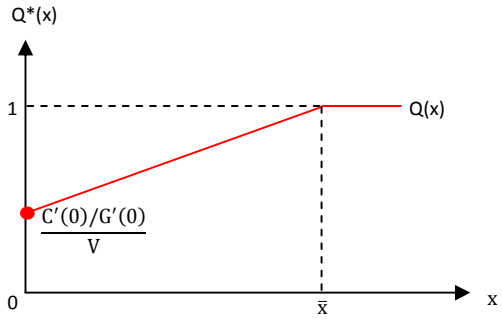


Figure 5.2 Common Value Equilibrium CDFs



Proposition 4. The unique mixed strategy Nash equilibrium of the game $\Gamma(1,2;\widetilde{R})$ with restriction $\frac{C'(0)}{G'(0)} < V_1 = V_1 = V$ is characterized by the common CDF $Q^*(s)$, where

$$Q^*(s) = \begin{cases} \frac{C'(0)/G'(0)}{V} & \text{for} \quad s = 0 \\ \frac{C(s)}{VG(s)} & \text{for} \quad s \in (0, \bar{x}] \\ 1 & \text{for} \quad s \geq \bar{x} \end{cases}$$

i.e., the common support is $[0, \bar{x}]$ and unlike the standard all pay auction results, in the equilibrium both the bidders place the same amount of mass $\left(\frac{C'(0)}{VG'(0)}\right)$ at 0. Both bidders earn zero payoff in the equilibrium.

In the standard 2-bidder all-pay auctions, because the winning payoff is positive at a zero-bid and is monotonically decreasing, bidders do not place atoms at the same point in equilibrium. For example, under the Baye et al. (1996) structure, if both bidders place mass points at zero, then shifting mass to a positive bid is a strictly dominant strategy for both the bidders. In contrast, under the current case the winning curve starts from the origin and is continuous. Hence, if a bidder shifts a mass of $\varepsilon > 0$ above zero, then its marginal payoff

remains zero and as a result the bidders do not have incentive to shift mass from zero. Hence, this explains the real life observation that in a contest between bidders with similar abilities sometimes some bidders opt out of the contest. In the standard common value all-pay auction, bidders do not place a mass point in equilibrium strategy, and as a result it is not possible to explain the observation by standard all-pay auction results.

2.6. Overall Characterization of Equilibria

Theorem. Propositions 1 to 4 fully characterize the equilibria for the all-pay auction with non-monotonic payoff described by the game $\Gamma(1,2;\widetilde{R})$.

Propositions 1 to 4 analyze mutually exclusive and exhaustive cases of the game $\Gamma(1,2;\widetilde{R})$. The proof of the theorem is essentially the collection of the proofs of the four propositions. This Theorem shows clear differences in strategies with the all-pay auction with monotonic payoff and nests the existing results of all-pay auction with non-monotonic payoffs. This also resembles the strategies obtained in the capacity constrained pricing games.

3. Discussion

This study is one of the first attempts to fully analyze the all-pay auction under complete information with bid-dependent prize schemes, where the winning payoff is not monotonic. This is also the first attempt to analyze an all-pay auction where it is possible that none of the bidders win the prize. We fully characterize the equilibria and show that the bidders' equilibrium strategies are strikingly different from that of the standard all-pay auction with monotonic payoff. In this study we offer several significant results. First, we determine the conditions for the existence of pure strategy equilibria and show a case when both bidders might abstain from bidding; second, we prove the existence of multiple mass points in the

initial high value bidder's equilibrium mixed strategy; third we show that the equilibrium mixed strategy of the low value bidder is discontinuous; and finally we discover bidders' common mass points at the same point of the support in the initial common value case.[10]

The results indicate that the monotonocity of the payoff is not necessary for the existence of any type of equilibrium, either in pure or in mixed strategies. This leads us to the following Folk theorem:

Folk Theorem: If the winning payoffs of the bidders in an all-pay auction are non-monotonic, but eventually become negative and remain negative forever with increase in own bid, then at least one equilibrium exists.

This area is of high interest in that it resembles several real life situations. This study partially explains the real life observations such as why, even in a two-firm industry, one or both firms might stay out of a patent race. The common support in mixed strategies that starts from a positive bid explains the reason why only sufficiently high amounts of R&D investment are observed across different industries; or why only sufficiently high quality of procurement samples in a procurement auction are submitted. Also, the existence of two mass points in the high-value bidder's equilibrium strategy can partially explain the dichotomous behavior (Barut et al, 2002; Gneezy and Smorodinsky, 2006) of bidders in an all-pay auction experiment. Here the bidders choose either very low or moderately high bids more often than predicted; and the empirical bid function seems to be drawn from a bimodal distribution. It might be the case that the bidders' believe that the payoff function is non-monotonic and that is why they place two mass points at the corners of the support.

[10] The results also show that under pure strategy equilibria, the payoff characterization results of Siegel (2009a) do not hold. But even in this non-monotonic structure, for mixed strategies, the payoff characterization results of Siegel (2009a) hold quite well. Hence, the results of Siegel (2009a) are more robust than claimed in the article.

The current study is focused on characterizing the equilibria of the all-pay auction with non-monotonic payoffs. One obvious idea for further research would be to extend the model to n-bidders. Another interesting extension would be to analyze the revenue generated in the auction and the corresponding effects of changes in initial prize values. The aforementioned extension would be important in policy issues. Another policy related extension would be to show the effects of caps on bidding. Finally, it will be intriguing to design non-monotonic payoff all-pay auction experiments using the structure of this study.

References

[1] Amegashie, J.A. (1999). The number of rent-seekers and aggregate rent-seeking expenditures: an unpleasant result. Public Choice, 99, 57–62.

[2] Amegashie, J.A. (2001). An all-pay auction with a pure-strategy equilibrium. Economics Letters. 70, 79-82.

[3] Araujo, A., de Castro, L.I., & Moreira, H. (2008). Non-monotoniticies and the all-pay auction tie-breaking rule, Economic Theory, 35(3), 407-440.

[4] Arora, A., Ceccagnoli, M., & Cohen, W. M. (2008). R&D and the patent Premium. International Journal of Industrial Organization, 26, 1153–1179

[5] Barut, Y., Kovenock, D., & Noussair, C.N. (2002). A Comparison of Multiple-Unit All-Pay and Winner-Pay Auctions Under Incomplete Information. International Economic Review, 43, 675-708.

[6] Bastock, M. (2007). Courtship: an ethological study. Aldine Transaction, Chicago.

[7] Baye, M. R., Kovenock, D., & de Vries, C. G. (1993). Rigging the Lobbying Process: An Application of the All-Pay Auction. American Economic Review, 83, 289-294.

[8] Baye M.R., Kovenock D., & de-Vries C.G. (1996). The all-pay auction with complete information. Economic Theory, 8, 1-25

[9] Bertoletti, P. (2006). On the Reserve Price in All-Pay Auctions with Complete Information and Lobbying Games. Munich Personal RePEc Archive Paper no.1083

[10] Bos, O., & Ranger, M. (2008). All-Pay Auctions with Endogenous Rewards. Paris School of Economics working paper

[11] Che, Y., & Gale, I. (2003). Optimal Design of Research Contests. American Economic Review, 93(3), 646—71.

[12] Che, Y., & Gale, I. (2006). Caps on Political Lobbying: Reply. American Economic Review, 96(4), 1355-60.

[13] Daly, M. (1978). The cost of mating. American Naturalist, 112, 771–774.

[14] Dasgupta, P., & Maskin, E.S. (1986). The Existence of Equilibrium in Discontinuous Economic Games, I: Theory. Review of Economic Studies, 83(1), 1-26

[15] Deneckere R.J., & Kovenock, D. (1996). Bertrand-Edgeworth duopoly with unit cost asymmetry. Economic Theory , 8(1), 1-25

[16] Gneezy, U., & Smorodinsky, R. (2006). All-Pay Auctions – An Experimental Study. Journal of Economic Behavior and Organization, 61, 255-275.

[17] Griliches, Z., and Mairesse, J. (1990). R&D and Productivity Growth: Comparing Japanese and US Manufacturing Firms. In Charles Hulten (Ed.), Productivity Growth in Japan and the United States, University of Chicago Press: Chicago.

[18] Hillman, A., & Riley, J. (1989). Politically contestable rents and transfers. Economics and Policy, 1, 17–39.

[19] Kaplan, T. R., Luski, I., Sela, A., & Wettstein, D. (2002). All-pay auctions with variable rewards. Journal of Industrial Economics, 4, 417–430.

[20] Kaplan, T.R., Luski I., & Wettstein D. (2003). Innovative activity and sunk cost. International Journal of Industrial Organization, 21, 1111 - 1133

[21] Loury, G. C. (1979). Market Structure and Innovation. The Quarterly Journal of Economics, 93 (3), 395-410.

[22] Mairesse, J., and M. Sassenou (1991), R&D and Productivity: A Survey of Econometric Studies at the Firm Level. STI Review (OECD Paris), 8, 9-43.

[23] Nti K.O.(1997). Comparative statics of contests and rent seeking games. International Economic Review, 38, 43-59.

[24] Osborne, M., & Pitchik, C. (1986). Price competition in a capacity-constrained duopoly. Journal of Economic Theory, 38, 238-260

[25] Sacco, D., & Schumtzler, A. (2008). All-Pay Auctions with Negative Prize Externalities: Theory and Experimental Evidence. The Berkeley Electronic Press working paper

[26] Siegel R. (2009a). All-Pay Contests. Econometrica, 77(1), 71-92.

[27] Siegel R. (2009b). Asymmetric Contests with Conditional Investments. Mimeo

[28] Sozou P.D., and Seymour R.M. (2005). Costly but worthless gifts facilitate courtship. Proceedings of the Royal Society of London, Series B, 272, 1877–1884.

[29] Tullock, G. (1980). Efficient Rent Seeking. In James M. Buchanan, Robert D. Tollison, Gordon Tullock, (Eds.), Toward a theory of the rent-seeking society. College Station, TX: Texas A&M University Press, 97-112.

[30] Tullock, G. (1997). Where is the rectangle?. Public Choice, 91, 149-159.

[31] Wakelin, K. (2001). Productivity growth and R&D expenditure in UK manufacturing firms. Research Policy, 30, 1079-1090.

[32] Wilson, D. H. (2007). Where's My Jetpack?: A Guide to the Amazing Science Fiction Future that Never Arrived. Bloomsbury USA

Appendices

Appendix 1. Characterization of the Shape of the Winning Curves

Claim 1. The winning curves for both bidders start from the origin and are strictly concave.

Proof: Given the assumptions $G(0) = C(0) = 0$ we get $W_t(0)=0$, for $t = 1, 2$ i.e., the winning curves start from the origin. Further, note that $\frac{d^2W_t(x_t)}{dx_t^2} = G"(x_t)V_t - C"(x_t) < 0$ as $G"(.)<0$ and $C"(.) \geq 0$. Hence, the winning curves are strictly concave. ■

Claim 2. If $\frac{C'(0)}{G'(0)} \geq V_t$ then any winning payoff is non-positive.

Proof: The slope of the winning curve is $\frac{dW_t(x_t)}{dx_t} = G'(x_t)V_t - C'(x_t)$. Recall from Claim 1 that the winning curves start from the origin. If $\frac{C'(0)}{G'(0)} \geq V_t$ then starting from the origin (Claim 1) the slope of the winning curve is non-positive throughout the bid range and consequently any winning payoff is also non-positive. ■

Claim 3. If $\frac{C'(0)}{G'(0)} < V_t$, then starting from the origin the winning curve is inverted U-shaped with unique maximum and as bid increases, eventually the winning curve cuts the X-axis at a unique point and winning payoff becomes negative.

Proof: Starting from the origin (Claim 1), as $\frac{C'(0)}{G'(0)} < V_t$, the winning curve has positive slope at the origin (Claim 2). But as winning curves are strictly concave (Claim 1) slope declines as bid increases; also as $G"(.)<0$ and $C"(.) \geq 0$, eventually at some unique point $\frac{C'(x_t)}{G'(x_t)} = V_t$ (follows from the uniqueness of a maximizer of a strictly concave function) and the winning curve reaches a unique maximum. After that point, $\frac{C'(x_t)}{G'(x_t)} > V_t$ and winning curve

has a strictly negative slope. As a result, as x_t increases winning curve declines and cuts the X-axis at a unique point and as x_t increases further, W_t becomes negative. ■

Claim 4. Starting with no-difference, W_1 and W_2 diverge away from each other and the difference tends to the initial value difference $(V_1 - V_2)$ as bid increases to infinity.

Proof: From the properties of G(x), $(W_1 - W_2)$=G(x)$(V_1 - V_2)$. Hence, $(W_1\text{-}W_2)|_0$=$(V_1 - V_2)G(x)|_0 = 0$. Also, $\frac{d(W_1-W_2)}{dx_t} = G'(x_t)(V_1 - V_2)>0$ and $\frac{d^2(W_1-W_2)}{dx_t^{\,2}} = G''(x_t)(V_1 - V_2)<0$. Finally, $\lim_{x\to\infty}(W_1 - W_2) = (V_1 - V_2)\lim_{x\to\infty} G(x) = (V_1 - V_2)$. ■

Claim 5. If $\frac{C'(0)}{G'(0)} < V_2$, define $x_t^{Wmax} = \text{argmax}\big(W_t(x_t)\big)$; then $x_1^{Wmax} \geq x_2^{Wmax}$.

Proof: From Claim 3, $\text{argmax}\big(W_t(x_t)\big)$ is the solution to the first order condition $\frac{dW_t(x_t)}{dx_t} = G'(x_t)V_t\text{-}C'(x_t) = 0$ or $\frac{C'(x_t)}{G'(x_t)} = V_t$. Define $K(x_t)$=$\frac{C'(x_t)}{G'(x_t)}$. Note that $\frac{d\left(\frac{C'(x_t)}{G'(x_t)}\right)}{dx_t} > 0$, hence the inverse of $K(x_t)$ exists and is also monotonically increasing function. Define $K^{-1}(.)$=H(.); thus, $\text{argmax}\big(W_t(x_t)\big) = H(V_t)$. By assumption $V_1 \geq V_2$ and by construction H(.) is a monotonically increasing function, hence $x_1^{Wmax} \geq x_2^{Wmax}$. ■

Claim 6. $\max W_1 \geq \max W_2$.

Proof: From Claim 5, $\max W_t = G(H(V_t))V_t - C(H(V_t))$. Hence $\frac{d\max W_t}{dV_t} = G'(.)H'(.)V_t + G(.) - C'(.)H'(.) = H'(.)[G'(.)V_t - C'(.)] + G(.) = G(.) > 0$ as $[G'(.)V_t - C'(.)] = 0$ for maximization and $V_t > 0$ *for* $t = 1, 2$. Given $V_1 \geq V_2$, we confirm $\max W_1 \geq \max W_2$. ■

Claim 7. Define $\bar{x}_t = \left\{x_t \neq 0: \frac{C'(0)}{G'(0)} < V_t \,\&\, W_t(x_t) = 0\right\}$ i.e., $\bar{x}_t$ is the unique positive bid by bidder t (Claim 3) for which his/her winning payoff is zero.[11] Then $\bar{x}_1 > \bar{x}_2$.

[11] $\bar{x}_t$ is defined as the 'reach of bidder *t*' in Siegel (2009 a, b)

Proof: From Claim 4, $(W_1(x) - W_2(x)) > 0 \; \forall \; x > 0$ and by definition $W_t(\bar{x}_t) = 0$. Consequently, $W_1(\bar{x}_2) > W_2(\bar{x}_2) = 0 = W_1(\bar{x}_1)$. Hence, the inverted U-shape of winning curve $W_1(.)$ (Claim 3) confirms $\bar{x}_1 > \bar{x}_2$. ■

Appendix 2. Proofs of Lemmas 1 and 2.

Lemma 1. An equilibrium in pure strategies for the game $\Gamma(1,2;\widetilde{R})$ exists under condition (i) $\left\{\frac{C'(0)}{G'(0)} \geq V_1\right\}$ or (ii) $\left\{\frac{C'(0)}{G'(0)} \in [V_2, V_1)\right\}$ or (iii) $\left\{\frac{C'(0)}{G'(0)} < V_2 \text{ and } (x_1^{Wmax} \geq \bar{x}_2)\right\}$. Moreover, under condition (i) there exist unique equilibrium strategies $(x_1^*, x_2^*) = (0,0)$, whereas under condition (ii) or (iii) the unique equilibrium strategies are $(x_1^*, x_2^*) = \left(x_1^{Wmax}, 0\right)$.

Proof: (i) If $\frac{C'(0)}{G'(0)} \geq V_1$ then by Claim 2 the winning payoffs are always non-positive and bidding any positive amount with positive probability ensures loss. So, in equilibrium both the bidders bid zero, i.e., $x_1^* = x_2^* = 0$.

(ii) If $\frac{C'(0)}{G'(0)} \in [V_2, V_1)$ then bidder 2's winning payoff is always non-positive and following the same logic as in (i), $x_2^* = 0$. Bidder 1's winning curve is inverted U-shaped and given bidder 2 bids 0 with certainty, bidder 1 maximizes its payoff by always bidding $x_1^* = \text{argmax}\left(W_1(x_1)\right) = x_1^{Wmax} > 0$.

(iii) If $\frac{C'(0)}{G'(0)} < V_2$, then in some sufficiently small neighborhood of zero bid, the winning payoff is positive for both bidders. When $(x_1^{Wmax} \geq \bar{x}_2)$ then knowing bidder 2 never bids on or over $\bar{x}_2$ (as that will result in a negative payoff whereas a zero-bid ensures a zero payoff), any bid between $\bar{x}_2$ and $\bar{x}_1$ gives a sure positive payoff to bidder 1. The sure payoff reflected by the winning curve is maximized at x_1^{Wmax}. Hence, bidder 1 bids at $x_1^* = x_1^{Wmax} > \bar{x}_2$ with certainty. Knowing this, bidder 2 bids $x_2^* = 0$ with certainty. ■

Lemma 2. If $\frac{C'(0)}{G'(0)} < V_2$ and $(x_1^{Wmax} < \bar{x}_2)$ then there exists no pure strategy Nash equilibrium for the game $\Gamma(1,2;\widetilde{R})$.

Proof: We prove the non-existence of PSNE by following the same procedure as in Kaplan et al. (2003), however, in their structure the payoff is monotonically decreasing in own bid. A pure strategy Nash equilibrium in this game is a set of bids $\{x_1^*, x_2^*\}$ where bidder t cannot increase its payoff by deviating from x_t^* given rival bid x_{-t}^*. Suppose there exists PSNE for the game $\Gamma(1,2;\widetilde{R})$ under the stated condition in Lemma 2. Also let $\{\tilde{x}_t\}$ be the set of maximum bids among the PSNE bids. Therefore, either $\{\tilde{x}_t\}$ is a singleton set or $\tilde{x}_1 = \tilde{x}_2$.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) > 0$ then bidder 1 is the highest bidder as bidder 2 never bids more than $\bar{x}_2 < \bar{x}_1$ and bidding $\bar{x}_2$ gives bidder 1 a sure payoff of $W_1(\bar{x}_2)$. Because $x_1^{Wmax} < \bar{x}_2$, bidding more than $\bar{x}_2$ decreases payoff for bidder 1. But if bidder 1 bids $\bar{x}_2$ then the best response for bidder 2 would be to bid zero. Consequently, if bidder 2 bids zero, then the best response for bidder 1 is to bid at x_1^{Wmax}. As $x_1^{Wmax} < \bar{x}_2$ bidder 2 can overbid bidder 1 and make a positive payoff by bidding $x_1^{Wmax} < x_2 < \bar{x}_2$(by the continuity of the payoff functions). Hence there exists no pure strategy Nash equilibrium when $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) > 0$.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) = 0$ then by construction the highest bidder, say bidder t, bids at $\bar{x}_t$. Bidder 1 never bids at $\bar{x}_1$ as placing a bid $x_1 \in (\bar{x}_2, \bar{x}_1)$ strictly increases payoff. Bidder 2 also never bids at $\bar{x}_2$ as bidder 1 can always place a bid $(\bar{x}_1 + \varepsilon)$ where $\varepsilon > 0$ and that will result in negative payoff for bidder 2. So, there exists no PSNE in this case.

If $\{\tilde{x}_t\}$ is singleton and $\pi_t(\tilde{x}_t) < 0$ then the highest bidder can always make a zero payoff by bidding zero; implying no PSNE. Therefore, there exists no PSNE with $\{\tilde{x}_t\}$ being singleton.

If $\{\tilde{x}_t\}$ is not singleton and $\tilde{x}_1 = \tilde{x}_2 = 0$ then bidder t can improve payoff by placing a bid of x_t^{Wmax}. If $\tilde{x}_1 = \tilde{x}_2 \neq 0$ then $\tilde{x}_t < \bar{x}_2$ as placing a bid more than or equal to $\bar{x}_2$ ensures loss for bidder 2 (recall the tie breaking rule). Finally, when $\tilde{x}_t \in (0, \bar{x}_2)$ then from the tie breaking rule $\pi_1(\tilde{x}_1) = (G(\tilde{x}_1)V_1 - C(\tilde{x}_1))>0$ and $\pi_2(\tilde{x}_2) = -C(\tilde{x}_2) < 0$. But bidder 2 can always bid $(\tilde{x}_1 + \varepsilon)$ (where $\varepsilon > 0$) and earn $\pi_2(\tilde{x}_1 + \varepsilon) = W_2(\tilde{x}_1 + \varepsilon)>0>= -C(\tilde{x}_2)$. Hence, again, there exists no PSNE when $\{\tilde{x}_t\}$ is not singleton. ■

Appendix 3. Proofs of Lemmas 3 to 20..

Lemma 3. Denote $\bar{s}_t = \inf\{x: F_t(x) = 1\}$ and $\underline{s}_t = \sup\{x: F_t(x) = 0\}$, then $0 \leq \underline{s}_t \leq \bar{s}_t \leq \bar{x}_t$.

Proof: $0 \leq \underline{s}_t$, as by construction bid cannot be negative. $\underline{s}_t \leq \bar{s}_t$ comes from the definitions of $\underline{s}_t$ and $\bar{s}_t$ and strict inequality holds if there is no pure strategy for any of the bidders. By definition $\bar{s}_t = \inf\{x: F_t(x) = 1\}$. If bidder t places a mass on any bid more than $\bar{x}_t$ then it will make a sure negative payoff for that mass and as a result the expected payoff will fall. He can always increase the expected payoff by placing that mass at 0. $\therefore \bar{s}_t \leq \bar{x}_t$. ■

Lemma 4. $\bar{s}_t \leq \bar{x}_2$.

Proof: From Lemma 3, $\bar{s}_2 \leq \bar{x}_2$. At $\bar{x}_2$ bidder 1 earns a sure payoff of $W_1(\bar{x}_2)$.[12] No PSNE case implies $x_1^{Wmax} < \bar{x}_2$, hence $W_1(.)$ is falling at $\bar{x}_2$ and placing any bid above $\bar{x}_2$ with positive probability strictly reduces expected payoff for bidder 1. So, bidder 1 never places a bid above $\bar{x}_2$, i. e. , $\bar{s}_1 \leq \bar{x}_2$ as well. ■

Lemma 5. Define $x_1' = \{x \neq \bar{x}_2 : W_1(x) = W_1(\bar{x}_2)\}$, then $x_1' < x_1^{Wmax} (< \bar{x}_2)$.

Proof: From Claim 3, $W_1(x)$ curve is inverted U shaped and from the definition of x_1', $W_1(x_1') = W_1(\bar{x}_2)$. Given the stated condition of no PSNE $x_1^{Wmax} < \bar{x}_2$, we must have $x_1' < x_1^{Wmax}$. Note that the strict concavity property of $G(.)$ function ensures a unique x_1'. ■

[12] $W_1(\bar{x}_2)$ is the 'power' of bidder 1 as in Siegel (2009 a, b).

Lemma 6. $\underline{s}_1 \geq x_1'$.

Proof: Bidder 1 can always bid $\bar{x}_2$ to earn a sure payoff of $W_1(\bar{x}_2)$. $x_1' < x_1^{Wmax}$ (Lemma 5); i.e., at x_1', $W_1(.)$ is increasing. Hence, if bidder 1 bids $x_1 < x_1'$, then $W_1(x_1) < W_1(x_1') = W_1(\bar{x}_2)$. i.e., even winning the bid provides less payoff to bidder 1 than the sure payoff. Thus, bidder 1 never places a positive probability to bid less than x_1' i.e., $\underline{s}_1 \geq x_1'$. ■

Lemma 7. Support for $2 \in \{0, [x_1', \bar{x}_2]\}$.

Proof: From Lemma 6, $\underline{s}_1 \geq x_1'$, i.e., bidder 1 never places a bid less than x_1' with positive probability. Knowing this, bidder 2 also never places a positive probability of bidding in $(0, x_1')$ as that ensures a negative payoff. So, bidder 2 places positive probability of bidding in either 0 or between $[x_1', \bar{x}_2]$, i.e., bidder 2's support is in the set $\{0, [x_1', \bar{x}_2]\}$. ■

Lemma 8. The possible equilibrium payoff of bidder 1, $\pi_1^* \geq W_1(\bar{x}_2) > 0$ and the possible equilibrium payoff of bidder 2, $\pi_2^* \geq 0$.

Proof: The sure payoff of bidder 1 is $W_1(\bar{x}_2) > 0$. Therefore, if the expected payoff of bidder 1 is not at least as high as $W_1(\bar{x}_2)$, then it is not an equilibrium. So, $\pi_1^* \geq W_1(\bar{x}_2) > 0$. Similarly, bidder 2 can always earn a zero payoff by not submitting any bid, hence $\pi_2^* \geq 0$. ■

Lemma 9. $\exists t \in \{1,2\}$ such that $\underline{s}_t \leq \underline{s}_{-t}$ and $F_{-t}(\underline{s}_t) = 0$.

Proof: Suppose not, i.e., $\nexists t \in \{1,2\}$: $\underline{s}_t \leq \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) = 0$. Then $\forall t \in \{1,2\}$ either (i) $\underline{s}_t \leq \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) > 0$ or (ii) $\underline{s}_t > \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) = 0$ or (iii) $\underline{s}_t > \underline{s}_{-t}$ & $F_{-t}(\underline{s}_t) > 0$. Cases (i) and (ii) cannot be true from the definition of $\underline{s}_t$ and case (iii) cannot be simultaneously true for bidder 1 and 2. Hence we arrive at a contradiction. In consequence, $\exists t \in \{1,2\}$ such that $\underline{s}_t \leq \underline{s}_{-t}$ and $F_{-t}(\underline{s}_t) = 0$. ■

Lemma 10. $\exists k \in \{1,2\}$ s. t. $\pi_k^* = 0$.

Proof: Suppose not. Then at $\underline{s}_k$, $\pi_k^*|_{\underline{s}_k} > 0$ i.e., $\underline{s}_k > 0$ (as $\pi_1^* > 0$ and $\pi_2^* \geq 0$ from Lemma 8). But from Lemma 9, if $\underline{s}_t \leq \underline{s}_{-t}$ then $F_t(\underline{s}_{-t}) = 0$ i.e., $\pi_k^*|_{\underline{s}_k} < 0$ for some $k \in \{1,2\}$: a contradiction. Hence, we must have some $k \in \{1,2\}$ such that $\pi_k^*=0$. ■

Lemma 11. $\pi_2^*=0$ i.e., $k=2$ and $\underline{s}_2=0$.

Proof: Combining Lemma 8: $\pi_1^* > 0$ and Lemma 10: $\exists k \in \{1,2\}$ s.t. $\pi_k^*=0$ we conclude $\pi_2^*=0$. Combining Lemma 9 with $\pi_2^*=0$ and the fact that bidder 1 must win with positive probability over the whole support to attain $\pi_1^* > 0$ we must have $\underline{s}_2=0$. ■

Lemma 12. $\bar{s}_1 = \bar{x}_2$ and $\pi_1^* = W_1(\bar{x}_2)$.

Proof: From Lemma 4, $\bar{s}_1 \leq \bar{x}_2$. Suppose $\bar{s}_1 < \bar{x}_2$ then bidding any $x_2 \in (\bar{s}_1, \bar{x}_2)$ ensures bidder 2 a strictly positive payoff; which is a contradiction with Lemma 11. Also, $\bar{s}_1 = \bar{x}_2$ implies $\pi_1(\bar{s}_1) = W_1(\bar{x}_2)$. Hence, in equilibrium $\pi_1^* = W_1(\bar{x}_2)$ throughout the support. ■

Lemma 13. $\bar{s}_2 = \bar{x}_2$.

Proof: From Lemma 4, $\bar{s}_2 \leq \bar{x}_2$ and from Lemma 12, $\bar{s}_1 = \bar{x}_2$. Suppose $\bar{s}_2 < \bar{x}_2$ then because $W_1(.)$ is decreasing at $\bar{x}_2$, placing any bid $x_1 \in [\bar{s}_2, \bar{x}_2)$ ensures bidder 1 a sure payoff of $W_1(x_1) > W_1(\bar{x}_2)$: a contradiction with Lemma 12. Hence $\bar{s}_2 = \bar{x}_2$. ■

Lemma 14. $F_2(\underline{s}_1) = F_2(0)$.

Proof: Suppose not. Then bidder 2 places a positive probability of bidding in the semi-open interval $(0, \underline{s}_1]$. But that ensures a negative payoff which is contradictory to Lemma 11. ■

Lemma 15. If $\alpha_t(s)$ is the amount of mass bidder t places at point s, then $\alpha_2(0) \in (0,1)$.

Proof: From Lemma 14, $\text{Prob}(x_2 < \underline{s}_1) = \alpha_2(0)$. $\therefore \pi_1(\underline{s}_1) = \alpha_2(0)G(\underline{s}_1)V_1 - C(\underline{s}_1)$. Hence, if $\alpha_2(0) = 0$ then $\pi_1(\underline{s}_1) = -C(\underline{s}_1) < W_1(\bar{x}_2)$ and if $\alpha_2(0) = 1$ then bidding any $x_1 \in (x_1', \bar{x}_2)$ ensures bidder 1 a payoff $W_1(x_1) > W_1(x_1') = W_1(\bar{x}_2)$ both of which are contradictory with Lemma 12. $\therefore \alpha_2(0) \in (0,1)$. ■

Lemma 16. $\alpha_1(\underline{s}_1) \in (0,1)$.

Proof: From Lemma 6, $\underline{s}_1 > 0$ and from Lemma 15, $\alpha_2(0) > 0$. If $\alpha_1(\underline{s}_1) = 0$, then at $\underline{s}_1$ bidder 2 loses with certainty and payoff of bidder 2 becomes $-C(\underline{s}_1) < 0$; and if $\alpha_1(\underline{s}_1) = 1$ then for any small $\varepsilon > 0$, bidding $(\underline{s}_1 + \varepsilon)$ gives bidder 2 a sure payoff of $W_2(\underline{s}_1 + \varepsilon) > 0$: both of which are contradictory with Lemma 11. $\therefore \alpha_1(\underline{s}_1) \in (0,1)$. ■

Lemma 17. A No-arbitrage Bid Function (NBF) of bidder 1 to keep bidder 2 indifferent is: $F_1(s) = \frac{C(s)}{V_2 G(s)}$, and a No-arbitrage Bid Function of bidder 2 to keep bidder 1 indifferent is: $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1 G(s)}$.

Proof: To keep bidder 1 indifferent, bidder 2 places the bid function $F_2(.)$ in a way such that $F_2(s)V_1G(s) - C(s) = \pi_1^* = W_1(\bar{x}_2)$. Solving for $F_2(.)$ yields the NBF of bidder 2: $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1 G(s)}$. Similarly, bidder 1 places the bid function $F_1(.)$ in a way such that $F_1(s)V_2G(s) - C(s) = \pi_2^* = 0$. Solving for $F_1(.)$ yields NBF of bidder 2 $F_1(s) = \frac{C(s)}{V_2 G(s)}$. ■

Lemma 18. $\lim_{s\to 0} F_1(s), F_1(x_1'), F_1(\bar{x}_2) < 1$.

Proof: Using L'Hospital rule: $\lim_{s\to 0} F_1(s) = \frac{\lim_{s\to 0}\frac{dC(s)}{ds}}{\lim_{s\to 0}\frac{d(V_2 G(s))}{ds}} = \frac{C'(0)}{V_2 G'(0)} < 1$ as $\frac{C'(0)}{G'(0)} < V_2$.

Also, $F_1(x_1') = \frac{C(x_1')}{V_2 G(x_1')} = \frac{C(x_1')}{\{V_2 G(x_1') - C(x_1')\} + C(x_1')} = \frac{x_1'}{W_1(x_1') + C(x_1')} < 1$ as $W_1(x_1') = W_1(\bar{x}_2) > 0$.

And $F_1(\bar{x}_2) = \frac{C(\bar{x}_2)}{V_2 G(\bar{x}_2)} = \frac{C(\bar{x}_2)}{\{V_2 G(\bar{x}_2) - C(\bar{x}_2)\} + C(\bar{x}_2)} = \frac{C(\bar{x}_2)}{W_1(\bar{x}_2) + C(\bar{x}_2)} < 1$ as $W_1(\bar{x}_2) > 0$. ■

Lemma 19. $F_1(s)$ is monotonically increasing in the closed interval $[x_1', \bar{x}_2]$.

Proof: From Lemma 18, $\frac{1}{F_1(x_1')} = \frac{W_1(x_1') + C(x_1')}{C(x_1')} = 1 + \frac{W_1(\bar{x}_2)}{C(x_1')}$, and similarly $\frac{1}{F_1(\bar{x}_2)} = 1 + \frac{W_1(\bar{x}_2)}{C(\bar{x}_2)}$. We know $W_1(x_1') = W_1(\bar{x}_2)$. And $\bar{x}_2 > x_1'$ implies $C(\bar{x}_2) > C(x_1')$, hence we obtain $\frac{1}{F_1(x_1')} > \frac{1}{F_1(\bar{x}_2)}$, i.e., $F_1(\bar{x}_2) > F_1(x_1')$. If there exists no extreme point of $F_1(.)$ within the open interval $(x_1', \bar{x}_2)$ then it means that $F_1(s)$ is monotonically increasing in $(x_1', \bar{x}_2)$. In any

extreme point of $F_1(s)$, $\frac{dF_1(s)}{ds} = \frac{V_2G(s)C'(s) - C(s)V_2G'(s)}{[V_2G(s)]^2} = 0$, i.e., $[G(s)C'(s) - C(s)G'(s)] = 0$. But, $[G(s)C'(s) - C(s)G'(s)]$ is a strictly upward rising curve from origin.[13] Hence there is no solution except origin for $[G(s)C'(s) - C(s)G'(s)] = 0$, i.e., there exists no extreme point for $F_1(s)$ within the interval$(x_1', \bar{x}_2)$. Thus, $F_1(s)$ is monotonically increasing in $[x_1', \bar{x}_2]$. ■

Lemma 20. $F_2(s)$ starts from infinity, monotonically decreases to 1 at $s = x_1'$, reaches unique minimum within the open interval $(x_1', \bar{x}_2)$ and then monotonically increases to 1 at $s = \bar{x}_2$.

Proof: From Lemma 17, $F_2(s) = \frac{C(s)+W_1(\bar{x}_2)}{V_1G(s)}$. $\therefore F_2(0) = \infty$. And $F_2(x_1') = \frac{C(x_1')+W_1(\bar{x}_2)}{V_1G(x_1')} = \frac{C(x_1')+W_1(x_1')}{V_1G(x_1')} = \frac{C(x_1')+\{V_1G(x_1')-C(x_1')\}}{V_1G(x_1')} = 1$. Also, $F_2(\bar{x}_2) = \frac{C(\bar{x}_2)+W_1(\bar{x}_2)}{V_1G(\bar{x}_2)} = \frac{C(\bar{x}_2)+\{V_1G(\bar{x}_2)-C(\bar{x}_2)\}}{V_1G(\bar{x}_2)} = 1$. If we prove that $F_2(.)$ is decreasing at x_1' then there will be at least one minimum point of $F_2(.)$ in the open interval $(x_1', \bar{x}_2)$. Note that $\frac{dF_2(s)}{ds} = \frac{V_1G(s)C'(s) - \{C(s)+W_1(\bar{x}_2)\}V_1G'(s)}{[V_1G(s)]^2}$. Hence $Sign\left(\frac{dF_2(s)}{ds}\right) = Sign\big(G(s)C'(s) - \{C(s) + W_1(\bar{x}_2)\}G'(s)\big)$. At point x_1', it can be shown that $\text{Sign}\left(\frac{dF_2(s)}{ds}|_{x_1'}\right) = \text{Sign}\big(C'(x_1') - V_1G'(x_1')\big) = \text{Sign}\left(-\frac{dW_1(s)}{ds}|_{x_1'}\right) < 0$ as $W_1(.)$ is upward rising at point x_1' (Claim 3 and Lemma 5). Thus $F_2(.)$ is decreasing at x_1' and consequently, there exists at least one minimum point of $F_2(.)$ in the open interval $(x_1', \bar{x}_2)$.

Now, if we show that the minimum is unique then we will prove that (i) $F_2(.)$ decreases from infinity to 1 at x_1' and (ii) $F_2(.)$ the minimum in the interval $(x_1', \bar{x}_2)$. At a minimum, $\frac{dF_2(s)}{ds} = 0$, which implies $\frac{\big(G(s)C'(s) - C(s)G'(s)\big)}{G'(s)} = W_1(\bar{x}_2)$. Here RHS is a positive constant whereas LHS is an upward rising curve from origin.[14] Thus there exists unique solution for $\frac{\big(G(s)C'(s) - C(s)G'(s)\big)}{G'(s)} = W_1(\bar{x}_2)$, i.e., there exists unique minimum for $F_2(.)$. Because $F_2(x_1') = F_2(\bar{x}_2) = 1$ and $F_2(s)$ is decreasing at x_1', $\therefore \text{argmin}\big(F_2(s)\big) = x_2^{\min} \in (x_1', \bar{x}_2)$. ■

[13] $[G(s)C'(s) - C(s)G'(s)]|_0 = 0$, and $\frac{d[G(s)C'(s) - C(s)G'(s)]}{ds} = G(s)C''(s) - C(s)G''(s) > 0 \ \forall s > 0$.

[14] Note that $\frac{\big(G(s)C'(s) - C(s)G'(s)\big)}{G'(s)}|_0 = 0$ and $\frac{d\big[\big(G(s)C'(s) - C(s)G'(s)\big)/G'(s)\big]}{ds} = \frac{G(s)[G'(s)C''(s) - C'(s)G''(s)]}{[G'(s)]^2} > 0$