Uncertainty, Networks and Real Options*

Isabelle Bajeux-Besnainou
Department of Finance, George Washington Univeristy

Sumit Joshi
Department of Economics, George Washington University

Nicholas Vonortas
Department of Economics,
Center for International Science and Technology Policy,
George Washington University

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Abstract

Two pervasive features of industries experiencing rapid technological progress are *uncertainty* (with regard to the technological feasibility and marketabilility of an innovation) and *networks* (the dense web of research alliances and joint ventures linking firms to each other). This paper connects the two disparate phenomena using the notion of *real options*. It visualizes firms as nodes and the links connecting them as call options that give each pair of interlinked firms the right, but not the obligation, to sink additional resources into a project at some future date conditional on favorable technical/market information. The formation of networks is endogenous as firms establish links with others by appraising their value using option pricing methods. Our model explains the following: why networks are particularly ubiquitous in industries that are subject to high uncertainty; why networks sometimes display an interconnected "hubs and spokes" architecture; why small (or peripheral spoke) firms often sink resources into relatively higher risk higher return investment projects (and those too with only large, or hub firms); and why so many research alliances are continuously formed and dissolved. Our paper also outlines the conditions

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under which ex-ante symmetric firms end up ex-post forming complex asymmetric networks.

1 Introduction

There are two pervasive features of industries experiencing rapid technological progress. The first is *uncertainty*, both technological (uncertainty regarding whether the investment will yield a successful innovation) and market (uncertainty regarding the marketability of the innovation). The second feature is *networks* which refers to the linkages among firms in the form of strategic alliances and joint ventures to jointly conduct R&D activities and share the benefits of cooperation. Recent examples of networks in such industries include the strategic partnerships of Sony, IBM and Toshiba to produce the sophisticated chips at the heart of Blu-ray and HD DVD formats, the partnerships of Boeing and of Airbus with multiple suppliers and buyers in developing their new, composite material airplanes, and the partnerships of large pharmaceutical companies with smaller biotechnology firms. This paper examines the relation between uncertainty and networks using the concept of *real options*.

There is already a significant literature that examines the endogenous formation of research networks (for e.g. Bloch 1995, Yi 1998, Yi and Shin 2000, Goyal and Moraga 2001 and Goyal and Joshi 2003). This literature casts network formation in an exclusively *deterministic* framework in which research alliances stimulate product/process innovations that reduce costs of production for participants non-randomly as a function of the alliance's size. The tension between the benefits from cost-reduction and the costs of enlarging the size of the alliance shapes the strategic incentives of firms and determines the equilibrium architecture of networks. Such a deterministic formulation however misses a number of important empirical facts:[1]

[1]Examples of technology-intensive alliance strategies across various sectors that exhibit such phenomena include the following: the alliance between Hewlett-Packard and Microsoft that pools the companies' systems integration and systems software skills, respectively, to create technology solutions for small and big customers; the alliance between the biotechnology firm Abgenix and the pharmaceuticals company AstraZeneca that combines the strengths of the former in discovering new drugs and the familiarity of the latter with the FDA approval process; Pfizer's alliance with Warner-Lambert for the cholesterol decreasing drug Lipitor in the mid-1990s, the first step of a buy-out; the FreeMove alliance between T-Mobile, Telefonica Moviles, Telecom Italia Mobile and Orange announced in 2003 for a "unified service offering" to both their business and consumer customers; the

1. High-tech fast-evolving competitive environments, such as those of biotechnology/pharmaceuticals and information technology, are characterized by uncertainties regarding both the technical feasibility of ideas for new products/processes and their economic viability in the market.

2. Research networks are particularly ubiquitous in industries characterized by such uncertainty.[2]

3. Firms choose projects that differ widely with respect to their risk characteristics. Firms that are smaller and more peripheral than larger and more central firms often pursue higher risk projects.

4. Research networks are characterized by a high degree of link formation and link destruction activity as the uncertainty resolves.

These empirical facts suggest that the incentives shaping the network architecture in industries characterized by rapid technological progress depend in a fundamental way on the underlying uncertainty. This link between uncertainty and network architecture is *a priori* excluded in the received deterministic literature on endogenous research networks.

The simple model of network formation that we propose captures the main empirical facts quite nicely. *The prevalence of networks in an environment of high uncertainty is explained by viewing research networks as a set of real options among firms.* In the presence of uncertainty, a firm cannot be sure whether any one investment in a new product/process will be successful. Firms diversify the risk by making relatively small initial investments in a number of R&D projects and then waiting to commit significant resources only into those projects that are deemed favorable on the basis of new information. This flexibility increases the ability of firms to better allocate scarce resources to profitable projects. Firms typically identify and enter promising new fields quickly, thus jumping early on the learning curve. All

Starmap alliance between O2, Amena, One, Pannon GSM, Sunrise, Telenor Mobile, and Wind to provide seamless, enhanced voice and data solutions for business and consumers across Europe; the joint ventures Alcatel Alenia Space and Telespazio Holding between Alcatel and Finmeccanica in 2005 to consolidate leadership in the telecommunication satellite systems and services, and to acquire a strong position in the most important European programmes such as Galileo and GMES.

[2] For example see Ahuja (2000), Ebers and Jarillo (1998), Gulati (1998), Gulati et al. (2000), Kogut (2000), Nohria and Eccles (1992), Powell et al. (1996), and Walker et al. (1997).

firms are, of course, limited in their ability to realize these objectives by internal resource constraints. This is precisely where networks play an important role. In high-tech sectors, research partnerships serve as technology search engines: firms unable to justify heavy investments in fluid, high-risk, high-potential technological areas can form multiple research partnerships to explore the field and create opportunities for more investment there in the future (Hemphill and Vonortas, 2003). In addition to learning about new opportunities, research partnerships also help share research costs, share technological and market risk, access complementary resources, access markets, and increase strategic flexibility.[3] In sum, networks allow firms to diversify and expand their technology search space collectively in terms of pursuing multiple and bolder (high risk, high return) research projects than what they otherwise could by operating alone due to paucity of resources.

In the uncertainty framework therefore, an alliance between any two firms may not actually reduce the costs of either. Rather, the alliance can be perceived as an agreement to pursue an R&D project jointly by making an initial investment and retaining the option of revisiting the project at a later date to sink more resources on the basis of new information. This view of two firms forging an alliance is analogous to two firms agreeing to buy a call option. By making an initial joint investment, the two firms have the right, but not the obligation, to commit to a joint R&D project (i.e. exercise the option) at some future date and buy the entitlement to the future stream of profits from this project. These call options, when applied to investment in new products/processes are called *real options*. The novel feature of our analysis is to combine uncertainty and networks by viewing the firms as nodes in a network and the links (or alliances) connecting them as real options. The value of a link to a firm is then appraised by the use of option-pricing methods.[4]

[3] For surveys of this literature see Caloghirou et al. (2004), Gomes-Casseres (1996), Hagedoorn et al. (2000), Jankowski et al. (2001), Nooteboom (1999), and Vonortas (1997). This networking view is also supported by the strategies of some leading companies. For example, in the ten years to 2004, Cisco had entered into more than 100 alliances (and had acquired 36 companies). Internal development of products, acquisitions and alliances are considered alternatives. When there is a high degree of uncertainty around technologies, or when they aren't critical, Cisco uses alliances. Moreover, Procter & Gamble Co. has transformed its traditional in-house R&D process into an open-source innovation strategy it calls "connect and develop". The new method can be described as embracing the collective brains of the world. It has made it a goal that 50% of the new products come from outside P&G's labs. For this purpose, it taps networks of inventors, scientists and suppliers for new products that can be developed in-house.

[4] See, for example, Berk et al. (2004), Childs and Triantis (1999), Davis and Owens

The underlying uncertainty dictates the option value of each link. Each link between two firms also requires an initial (relatively small as compared to the exercise price) precommitment of resources to the project. The difference between the option value of a link and its initial investment cost dictates the architecture of research networks. Significant evidence exists indicating that collaboration networks have a self-organizing architecture with highly uneven distribution of links among firms. In particular, a large number of firms have relatively few links whereas a minority of firms have a disproportionately large number of links. This more or less universal network feature is captured in our model through the notion of *interlinked stars* networks. This network architecture is composed of asymmetrically-sized *hubs* and *spokes* with the property that the hubs are connected to each other and to the spokes while the spokes are only connected to hubs but not to each other.

It is well known that the option value of a project increases with the riskiness of the project. It is also generally true that R&D investment is characterized by economies of scale (at least over some initial range). R&D projects within the same technological area have fairly similar requirements in terms of fixed inputs such as research facilities, laboratories, and specialized capital. Once a firm has made these basic investments for one project, then they do not have to be duplicated (at least not to the same scale) for additional projects. If the potential partner has also made similar investments, then it allows even more possibilities to effect cost reduction through an efficient sharing of resources. These considerations help explain why hub firms often choose to engage in *higher* risk (and higher return) projects with smaller, or more peripheral, spoke firms as compared to their projects with other hub firms. Consider a project between a hub firm and a small spoke firm. This project is relatively costly for the spoke firm because it has yet to realize the full benefits of its research investments from economies of scale. It is also relatively costlier for a hub firm than the same project with another hub because the opportunity to share fixed resources is smaller with a spoke. Thus both firms need to be compensated for their higher cost with a project that has greater option value.

A deterministic framework cannot explain how a large number of links or alliances can dissolve *in equilibrium*. If firms form links knowing exactly what benefits will accrue from each alliance, there is no incentive to form

(2003), Dixit and Pindyck (1994), Lee and Paxson (2001), Perlitz et al. (1999), Schwartz and Moon (2000), and Trigeorgis (1996).

or dissolve links in an equilibrium network. The options view of a link, on the other hand, explains this phenomenon quite easily. In any equilibrium network, there is a positive probability that a link that was formed will have zero option value at the exercise date. The dissolution of links in equilibrium is therefore the result of firms continuously adjusting their research "portfolios" in the light of new information.

We model the formation of networks as a link formation game that is similar to Dutta et al (1998). Firms announce links with other firms and only those links that are reciprocated are formed. Each link between a pair of firms is an agreement to pursue jointly a research project. We consider Nash networks in which no firm has an incentive to delete any subset of its links. We further refine the set of Nash networks by considering a strong stability concept due to Jackson and Nouweland (2001). This requires that no coalition of firms in a Nash network have any incentive to rearrange their links. We then attempt to characterize the architecture of strongly stable Nash networks. The technical methods that we employ are standard and follows the method of proof developed in Goyal and Joshi (2003, 2006).[5] The contribution of our paper rests mainly on providing a framework that unifies two relatively disparate fields: the theory of real options with the strategic formation of networks.

The paper is organized as follows. Section 2 describes the model and the evaluation of network links as real options. Section 3 describes the inter-linked star architecture. Section 4 offers a characterization of the equilibrium networks. Section 5 describes the choice of projects of hub and spoke firms. Section 6 discusses the dissolution of links in an equilibrium network. Section 7 concludes with avenues for future research. The longer mathematical proofs are relegated to an Appendix.

[5] There are some important differences however. In our framework, the marginal profit of a firm from forming a link depends on all the links that the firm has formed thus far. Further, all the partners of the firm in question are affected by the new link. Therefore, unlike other papers that consider the pairwise stability criterion of Jackson and Wolinsky (1996), we have had to consider the stronger notion of *strong stability.* This stronger concept creates problems of existence. Our subsequent characterization results therefore should be interpreted as what architectures would emerge if the set of strongly stable networks is non-empty. We also offer a numerical example in Section 4 showing the existence of an equilibrium star network.

2 The Model

Let $\mathcal{N} = \{1, 2, ..., N\}$ denote a set of ex-ante identical firms who wish to explore new opportunities within the same technological area. These technical opportunities are represented by a menu of R&D projects, parametrized by θ, where θ is drawn from a technology set Θ. The set Θ could either be discrete (with cardinality of at least $N(N-1)/2$) or continuous (a subset of the real line). In either case, let $\Theta = [\underline{\theta}, \overline{\theta}]$, $0 < \underline{\theta} < \overline{\theta} < \infty$, and it will be clear from the context whether Θ is discrete or continuous. The technology set Θ allows firms to explore a variety of product/process innovations. We make this assumption in order to focus as clearly as possible on the link between uncertainty and networks via the pricing of real options.[6]

Each project θ has a value V^θ which is uncertain. The initial value at date 0 is V_0^θ and the instantaneous volatility is σ^θ. The cost of pursuing the project to completion, (the exercise price) is denoted by K^θ. We would like to capture the notion that networks permit high return high volatility (or risk) projects. Therefore it is assumed that projects in Θ are ranked in increasing order of returns and volatility, i.e V_0^θ and σ^θ are continuously differentiable and strictly increasing in θ. In addition, K^θ is non-decreasing in θ. Let P^θ denote the option value of project θ at date 0. We will maintain that:

(A.1) The option value P^θ is strictly increasing in θ.

We now present three examples of stochastic processes that illustrate the conditions under which $(A.1)$ is satisfied.

Example 2.1: (*General stochastic process, exogenous exercise date*) The present value V_t^θ of the project follows a general stochastic process with an instantaneous volatility parameter σ^θ and initial value at date 0, V_0^θ. Suppose the exercise date, T, is exogenously given and the exercise price

[6] Consider the other extreme where there is only one technology available and those firms who adopt late get a lower payoff. In this case we would have to incorporate the possibility of preemption and therefore the issue of timing (for example see Fudenberg and Tirole 1985). Further, the pricing of the real options would be quite complicated due to these strategic considerations (Grenadier 2000a, 2000b, Huisman and Kort 1999). This would distract us from the main objective of this paper, which is to draw a simple link between uncertainty and networks. We therefore leave issues of preemption and timing to a future paper.

of pursuing the project to completion is independent of θ, the particular project that is chosen. From general option theory the option value always increases with the intial value, V_0^θ, and the volatility, σ^θ. Therefore $(A.1)$ is satisfied. ■

Example 2.2: (*Brownian motion process, exogenous exercise date, T*) The present value V_t^θ of the project follows a geometric Brownian motion with an instantaneous volatility parameter σ^θ and initial value at date 0, V_0^θ. Assuming risk-neutral probability, its dynamics can be described as:

$$\frac{dV_t^\theta}{V_t^\theta} = rdt + \sigma^\theta dW_t \tag{1}$$

where r is the instantaneous risk-free rate, assumed constant, and W_t is a standard Brownian motion. Suppose that V_0^θ and K^θ are proportional (i.e. the ratio V_0^θ/K^θ is constant). From the Black-Scholes formula, the real option value can be written as:

$$P^\theta = V_0^\theta \mathbf{N}\left(d_1\right) - K^\theta e^{-rT}\mathbf{N}\left(d_2\right) \tag{2}$$

where $\mathbf{N}(x)$ denotes the cumulative normal distribution of x and:

$$\begin{aligned} d_1^\theta &= \frac{\log\left(\frac{V_0^\theta}{K^\theta}\right) + \left(r + \frac{(\sigma^\theta)^2}{2}\right)T}{\sigma^\theta\sqrt{T}} \\ d_2^\theta &= d_1^\theta - \sigma^\theta\sqrt{T} \end{aligned} \tag{3}$$

depend only on the ratio V_0^θ/K^θ which is assumed constant. Then the option value is strictly increasing in θ and $(A.1)$ is satisfied. ■

Example 2.3: (*Brownian motion process, endogenous exercise date*) The value of the project θ is given by the geometric Brownian motion process:

$$\frac{dV_t^\theta}{V_t^\theta} = \alpha^\theta dt + \sigma^\theta dW_t \tag{4}$$

where α^θ represents the drift and is continuously differentiable and strictly increasing in θ. Let δ denote the discount rate. Then the option value is given by (see Dixit and Pindyck, 1994, for details):

$$P^\theta = \begin{cases} \frac{1}{\beta_1^\theta(\delta-\alpha^\theta)} \frac{\left(V^\theta\right)^{\beta_1^\theta}}{\left(V_\tau^\theta\right)^{\beta_1^\theta - 1}}, & V^\theta < V_\tau^\theta \\ \frac{V^\theta}{\delta-\alpha^\theta} - K^\theta, & V^\theta \geq V_\tau^\theta \end{cases} \tag{5}$$

where β_1^θ is the positive (and greater than 1) root of the quadratic equation $\left(\sigma^\theta\right)^2 \beta(\beta-1) + 2\alpha^\theta\beta - \delta = 0$ and V_τ^θ is the trigger value:

$$V_\tau^\theta = \frac{\beta_1^\theta}{\beta_1^\theta - 1}(\delta - \alpha^\theta)K^\theta \tag{6}$$

The option is exercised at time $T = \inf\{t : V_t^\theta \geq V_\tau^\theta\}$.

Now assume that K^θ does not vary with the project θ and normalize it to unity. Choose the parameters $\alpha^{\underline{\theta}}$, $\beta^{\overline{\theta}}$, and δ such that:

$$V_\tau^\theta = \frac{\beta_1^\theta}{\beta_1^\theta - 1}(\delta - \alpha^\theta) \leq \frac{\beta_1^{\overline{\theta}}}{\beta_1^{\overline{\theta}} - 1}(\delta - \alpha^{\underline{\theta}}) < 1$$

Consider the continuation range where the option is not exercised. Following Dixit and Pindyck (1994, pages 142-144), an increase in α^θ and σ^θ (following an increase in θ) decreases β_1^θ and increases $\frac{\beta_1^\theta}{\beta_1^\theta - 1}$. Since $V^\theta < V_\tau^\theta < 1$ in the continuation range, $\left(V^\theta\right)^{\beta_1^\theta}$ is increasing with θ. It can be verified that:

$$\frac{1}{\beta_1^\theta(\delta - \alpha^\theta)} \frac{1}{(V_\tau^\theta)^{\beta_1^\theta - 1}} = \frac{\left(\beta_1^\theta - 1\right)^{\beta_1^\theta - 1}}{\left(\beta_1^\theta\right)^{\beta_1^\theta}(\delta - \alpha^\theta)^{\beta_1^\theta}}$$

is decreasing in β_1^θ (and therefore increasing in θ). Under these parametric restrictions, $(A.1)$ is satisfied. ■

The fact that the option value is sensitive to the choice of a project θ will be exploited later when we put a joint restriction on this variation and the non-specific project costs in $(A.1)^*$. The model of research networks begins in period 0 when firms form a network of research alliances. Every firm makes an announcement of both intended links and the project it intends to pursue with each link. An announcement by firm i is of the form $s_i = (a_{ij}, \theta_{ij})_{j \neq i}$. The intended link $a_{ij} \in \{0, 1\}$, where $a_{ij} = 1$ means that i intends to form a link with j, while $a_{ij} = 0$ means that i intends no such link. The intended project is $\theta_{ij} \in \Theta$ if $a_{ij} = 1$. Let S_i denote the set of announcements, or strategies, of player i. A link between two players i and j is formed if and only if $a_{ij} = a_{ji} = 1$. This assumes that the two firms also agree on the choice of a project, i.e. $\theta_{ij} = \theta_{ji}$. We denote the formed link by $g_{ij} = 1$ and the absence of a link by $g_{ij} = 0$. A strategy profile $s = \{s_1, s_2, ..., s_n\}$, consisting of a strategy for each firm, therefore induces a network $g(s)$. To simplify the notation we shall often omit the dependence of the network on the underlying strategy profile. Note that $g_{ij} = g_{ji}$ and $g_{ii} = 1$. A *network* $g = (g_{ij})$ is a formal description of the pairwise links that exist between the firms. We also let $N_i(g) = \{j \in \mathcal{N} : j \neq i, g_{ij} = 1\}$ be the neighborhood of firm i; it is composed of the set of firms with whom firm i has a direct link in the network g. We will let $n_i(g) = |N_i(g)|$ denote the cardinality of this set.

A *path* in g connecting players i and j is a distinct set of players $\{i_1, \ldots, i_n\}$ such that $g_{ii_1} = g_{i_1 i_2} = g_{i_2 i_3} = \cdots = g_{i_n j} = 1$. We say that a network is *connected* if there exists a path between any pair $i, j \in \mathcal{N}$. A network, g', is a *component* of g if for all $i, j \in g'$, $i \neq j$, there exists a path in g' connecting i and j, and for all $i \in g$ and $j \in g$, $g_{ij} = 1$ implies $g_{ij} \in g'$. A component is essentially a self-contained sub-network within the larger network. We will say that a component g' is *complete* if $g_{ij} = 1$ for all $i, j \in g'$. An *empty* network g^e is one in which there are no links among firms. A *complete* network g^c is one in which a link exists between every pair of firms. We will let $g + g_{ij}$ denote the network obtained by replacing $g_{ij} = 0$ in g by $g_{ij} = 1$. Similarly, $g - g_{ij}$ will denote the network obtained by replacing $g_{ij} = 1$ in network g by $g_{ij} = 0$.

Each link/project θ requires an initial (small) investment with the option of revisiting the investment and incurring the set-up cost K^θ to move the project further. Alternatively, this investment can be interpreted as the cost of link formation. It is given by a function $C : \mathbb{Z}_+^2 \times \Theta \longrightarrow \mathbb{R}_+$ which is allowed to depend on the number of links of the two collaborating firms as well as the choice of project. In particular, the cost incurred by i to pursue a project θ with j is assumed to be of the additively separable form:

$$C(n_i, n_j, \theta) = c(n_i, n_j) + \psi(\theta), \qquad \forall i, j \in \mathcal{N} \tag{7}$$

The assumption here is that there are two seperable components to the initial cost of investing in a project. The first component is the cost to firm i of expanding the existing core R&D capability of a firm to accommodate a new project. This cost is not project-specific and would be incurred with any project. The second component is the additional investment by firm i which is specific to pursuing project θ. It seems reasonable to assume that this component is non-decreasing in θ since high return (and high risk) projects also generally require greater initial investment. The investment that is not project-specific is allowed to depend on the number of links of the participants. It is assumed that for all $i, j \in \mathcal{N}$:

(A.2) $c(n_i+1, n_j) < c(n_i, n_j)$, $c(n_i, n_j+1) < c(n_i, n_j)$, $\quad 1 \leq n_i, n_j < N-1$.

(A.3) $c(n_i, n_j)$ is concave in n_i for each $n_j \geq 0$ and $0 < n_i < N-1$, i.e. $2c(n_i, n_j) > c(n_i-1, n_j) + c(n_i+1, n_j)$.

(A.4) $c(n_i, n_j+1) - c(n_i+1, n_j+1)) > c(n_i, n_j) - c(n_i+1, n_j)$ for $1 \leq n_i, n_j < N-1$.

The rationale behind these assumptions is as follows. All projects within the same technological area (narrowly defined) generally have fairly similar requirements in terms of core research facilities and equipment. Once a firm has already made an initial investment in core R&D facilities and equipment for one project, then additional projects will usually not require the same duplication of fixed inputs. Firms can buy real options to more technological opportunities by making smaller initial investments. Therefore non-specific investment costs are assumed to be decreasing in the number of links of the participants in $(A.2)$. A similar reasoning applies to $(A.3)$. From the concavity property, $c(n_i, n_j) - c(n_i+1, n_j) > c(n_i-1, n_j) - c(n_i, n_j)$ indicating that the marginal cost of an additional link is falling with the number of links. These could be due to economies of scale that can be harnessed by forming more links as well as knowledge spillovers from existing projects that can be applied to new projects (economies of scope). A firm could also gain from the economies of scale and knowledge spillovers of its partners. This is captured by $(A.4)$ which is the well-known property of increasing differences. It states that the reduction in cost from an additional link is greater when the potential partner is more connected.

An important motive behind network formation is *diversification*: forming links allows a firm to simultaneously explore a number of diverse projects – above and beyond those it would have explored on its own – thereby increasing the probability of having at least one successful project. In the presence of a technology space allowing numerous technological opportunities, it would seem intuitive that a firm will not limit itself to choosing the same project with all partners. In the case of a continuous technology space, a simple continuity argument establishes that a firm will choose different projects with different partners. With a discrete technology space, however, there could be some overlap of projects. In either case, since all firms are ex-ante symmetric, it seems reasonable to suppose that the option value of a joint project will be equally shared by the participating firms. Formally:

(A.5) Consider a network g and let $j_1, j_2, ..., j_l \in N_i(g)$. If $\theta_{ij_1} = \theta_{ij_2} = \cdots = \theta_{ij_l} = \theta$, then the option value of θ for each of these $l+1$ firms is given by $\frac{P^\theta}{l+1}$.

We can now establish:

Lemma 1 *Suppose Θ is continuous. Consider a network g and let $j, k \in N_i(g)$. Under (A.5), $\theta_{ij} \neq \theta_{ik}$.*

In the discrete case, it is possible that $\frac{P^{\theta'}}{3} - \mu(\theta') > \frac{P^\theta}{2} - \mu(\theta)$ for all $\theta \neq \theta'$. Thus it is possible for a firm to engage in the same project with more than one partner. Note that it is also possible that two or more distinct pairs of firms finance the same project. In this case, given the ex-ante symmetry of firms, we assume that the monopoly right to the future stream of profits from the project is randomly allocated to one pair. Therefore, if h distinct pairs of firms pursue the same project θ, then the option value to a firm in any one of the pairs is assumed to be $\frac{1}{h}\left(\frac{P^\theta}{2}\right)$. Now suppose that any two firms choose a project θ' in a network g. Let $\xi(\theta', g)$ denote the total number of distinct firms that pursue project θ'. Then the option value of θ' to each participating firm in the network g is $\frac{P^{\theta'}}{\xi(\theta', g)}$. In the following discussion we will let:

$$\lambda(\theta', g) = \frac{P^{\theta'}}{\xi(\theta', g)} - \psi(\theta'), \quad \theta' \in \Theta \tag{8}$$

where $\psi(\theta)$ is the project-specific cost of θ. With a continuous technology space we know from Lemma 1 that firms will choose different projects in equilibrium and there will be no miscoordination. Therefore (8) takes the following form when Θ is continuous:

$$\lambda(\theta', g) \equiv \frac{P^{\theta'}}{2} - \psi(\theta'), \quad \theta' \in \Theta \tag{9}$$

since $\xi(\theta', g) = 2$. We will refer to $\lambda(\theta', g)$ as the *net* option value of θ' in the network g.

We are now ready to look at a firm's payoffs. Consider a network g and any firm i. Then:

$$\pi_i(g) = \sum_{j \in N_i(g)} \left[\lambda(\theta_{ij}, g) - c\left(n_i(g), n_j(g)\right)\right] \tag{10}$$

where θ_{ij} denotes i's project with j. Recall that the network g is a function of the underlying strategy profile $s = \{s_1, s_2, ..., s_N\}$. A strategy profile $s^* = \{s_1^*, s_2^*, ..., s_n^*\}$ is Nash if and only if $\pi_i(g(s_i^*, s_{-i}^*) \geq \pi_i(g(s_i, s_{-i}^*))$, $\forall s_i \in S_i$, $\forall i \in N$, where s_{-i} is the strategy profile of all firms other than i. The corresponding network is referred to as a Nash network. The Nash criterion is however not discriminating enough. For this purpose we will employ a *strong stability* property due to Jackson and Nouweland (2001) to refine the Nash equilibrium. Let $S \subset \mathcal{N}$ denote a coalition of firms. A network g' can be obtained from a network g through deviations by a coalition $S \subset \mathcal{N}$ if:

1. $g_{ij} = 1$ in g' and $g_{ij} = 0$ in g implies that $i, j \in S$. In words, any new links added in the movement from g to g' can only be formed by firms in the coalition S.

2. $g_{ij} = 1$ in g and $g_{ij} = 0$ in g implies that $\{i, j\} \cap S \neq \varnothing$. In words, if any links are deleted in the movement from g to g', then at least one of the firms severing the link should be from the coalition S.

A network g is said to strongly stable if for any coalition S and any g' that can be obtained from g through deviations by S, $\pi_i(g') > \pi_i(g)$ for some $i \in S$ implies that $\pi_j(g') < \pi_j(g)$ for some $j \in S$. We are now ready to define:

Definition 1 *A network g is an equilibrium network if:*

1. *There is a Nash strategy profile supporting g.*

2. *The network g is strongly stable.*

In the network setting, each firm can unilaterally sever links. However forming a link is a bilateral decision requiring agreement by both firms. The equilibrium property of a network g states that no firm has any incentive to delete any subset of links and, for any coalition of firms, the member firms have no incentive to bilaterally form links that did not exist in g. [7]

In an equilibrium network there could be isolated firms, i.e. those who do not form any link. We will normalize the payoffs of isolated firms to zero and assume:

[7] Please see Jackson and Nouweland (2001) on the existence and characterization of such strongly stable networks.

(A.6) $c(0,0)+\psi(\theta)>P^{\theta}$ for all $\theta\in\Theta$.

The term $c(0,0)$ is the cost component that is not project-specific to a firm with no links. We are assuming that these costs (plus any specific costs) are sufficiently high to preclude firms from pursuing any project in isolation. This is in keeping with the rationale behind networks as a means to collaboratively explore technological opportunities that are otherwise impossible due to individual resource constraints. We also note that this normalization is mainly to streamline the exposition and does not entail any essential loss of generality.

3 Interlinked Stars

In this section we describe the interlinked stars architecture. Consider a partition of firms $\mathcal{H}=\{H_1(g),H_2(g),...,H_m(g)\}$ according to increasing number of links. In particular, if $i,j\in H_h(g)$, then i and j have the same number of links. We will also sometimes refer to differences among firms in the number of links as differences in their *size*. Note that h denotes the ordering in the partition according to the number of links and does not mean that the two firms have h number of links. An interlinked star network g is characterized by "hub" and "spoke" firms of different sizes. Let us consider the spoke firms first arranged in increasing order of size. Assume for the sake of argument that m is even. If not, then the largest set of spoke firms is $H_{\frac{m+1}{2}}$.

Table 1: Spoke Firms

Spoke Firms In	**Linked to Firms In**
$H_1(g)$	$H_m(g)$
$H_2(g)$	$H_m(g),H_{m-1}(g)$
$H_3(g)$	$H_m(g),H_{m-1}(g),H_{m-2}(g)$
$\cdots$	$\cdots$
$H_{\frac{m}{2}}(g)$	$H_m(g),H_{m-1}(g),H_{m-2}(g),\cdots,H_{\frac{m}{2}+1}(g)$

The set of firms in $H_1(g)$ are the smallest, or most "peripheral", of the spoke firms. If they have any links at all, then they are only connected to firms in $H_m(g)$. The largest set of spoke firms, or the most connected, are those in $H_{\frac{m}{2}}(g)$. They are connected to all the hub firms. In between are spoke firms in increasing order of connectedness. Let us now consider the hub firms in decreasing order of size:

Table 2: Hub Firms

Hub Firms In	**Linked to Firms In**
$H_m(g)$	$H_1(g), H_2(g), \cdots, H_m(g)$
$H_{m-1}(g)$	$H_2(g), H_3(g), \cdots, H_m(g)$
$H_{m-2}(g)$	$H_3(g), H_4(g), \cdots, H_m(g)$
$\cdots$	$\cdots$
$H_{\frac{m}{2}+1}(g)$	$H_{\frac{m}{2}}(g), H_{\frac{m}{2}+1}(g), \cdots, H_m(g)$

Firms in $H_m(g)$ form the largest, or the most central, hubs who are connected to all the firms. The smallest hubs are firms in $H_{\frac{m}{2}+1}(g)$. A feature shared by all spoke firms is that they are only connected to hub firms and not to each other. Hub firms, on the other hand, are connected to all other hub firms. They differ only with regard to the spoke firms to whom they are connected. The largest hubs in $H_m(g)$ are connected to all the spoke firms, the next smaller hubs in $H_{m-1}(g)$ are connected to all but spokes in $H_1(g)$ and so on. A special case is the *star* network, $\mathcal{H} = \{H_1(g), H_2(g)\}$, in which $H_1(g)$ consists of $N-1$ spoke firms that have one link each with the single hub in $H_2(g)$. The empty network corresponds to the extreme case $\mathcal{H} = \{H_1(g)\}$ where $H_1(g) = \{1, 2, ..., N\}$ is the set of all singleton spokes and there are no hubs; the complete network $\mathcal{H} = \{H_1(g)\}$ corresponds to the case where $H_1(g) = \{1, 2, ..., N\}$ is the set of all interconnected hubs and there are no spokes.

The following figures illustrate the structure of some interlinked stars.

– Figures 1 and 2 somewhere here –

In Figure 1 we see networks of the form $\mathcal{H} = \{H_1(g), H_2(g)\}$. The set $H_1(g)$ is that of spoke firms and the set $H_2(g)$ is that of hub firms. The hubs are connected to other hubs and the spokes while the spokes are only connected to the hubs. In the first network, $H_2(g) = \{1\}$, indicating that firm 1 is the only hub and the remaining firms are spokes. This is referred to as the *star* network. In the next network, $H_2(g) = \{1, 2\}$, indicating that we have two symmetrically sized hubs and the remaining firms are spokes. Therefore we have an *interlinked star* network, i.e. one in which two star networks are connected by a link between the hubs and links between hubs and spokes.

The remaining two networks show symmetric hubs that are increasing in size with $H_2(g) = \{1, 2, 3\}$ and $H_2(g) = \{1, 2, 3, 4\}$ respectively.

In Figure 2 we see interlinked stars with asymmetric hubs and spokes. The first network has an architecture of the form $\mathcal{H} = \{H_1(g), H_2(g), H_3(g)\}$. The set $H_1(g) = \{5, 6\}$ are the spoke firms connected only to the largest hub $H_3(g) = \{1\}$. The set $H_2(g) = \{2, 3, 4\}$ is an intermediate-sized hub of firms connected to each other and to firm 1 but not to the two spokes. Firm 1 constitutes the largest hub. The second network is of the form $\mathcal{H} = \{H_1(g), H_2(g), H_3(g), H_4(g)\}$. Firm 1 in $H_4(g)$ is once again the largest hub that is connected to all firms. The set $H_3(g) = \{4, 5\}$ consist of smaller hubs who, in addition to firm 1, are connected to each other and the larger spoke firms in $H_2(g) = \{6, 7, 8\}$. The smallest spoke firms are $H_1(g) = \{2, 3\}$ who are only connected to the largest hub. It is worth reiterating the distinguishing characteristic of interlinked stars: hub firms are always connected to all other hubs (whether small or large) and sufficiently large spokes; spoke firms, on the other hand, are never connected to other spokes and are linked to only sufficiently large hubs.

4 Equilibrium Networks

In this section we show that equilibrium networks take the form of interlinked stars. We start with an important property of equilibrium networks: if firm i has fewer links than j in an equilibrium network, then all firms who are connected to i are also connected to j.

Lemma 2 *Assume* $(A.1) - (A.3)$ *and* $(A.5) - (A.6)$ *hold. Suppose* g *is a non-empty equilibrium network. If* $n_i(g) \leq n_j(g)$*, then* $N_i(g) \subseteq N_j(g)$*.*

Proof: See Appendix. ■

The intuition behind this result is fairly simple and exploits the assumptions on costs. Suppose some firm k finds it profitable to link with firm i for a project θ_{ik}. Then k should also find it profibable to link up with firm j who has more links than i. The cost to k of linking with j is lower than that of linking to i. The same is true of j because if the higher cost firm i could profitably link up with k, then the lower cost firm j certainly can. Moreover we show that both can find a project θ_{jk} that is mutually profitable. Therefore all partners of i will be partners of j as well.

With the help of Lemma 2 we can now show the interlinked stars characterization. We start with the following result showing that if firms have at least one link (they are not isolated), then they must have a link with the largest hub firms in $H_m(g)$.

Proposition 1 *Assume* $(A.1) - (A.3)$ *and* $(A.5) - (A.6)$ *hold. If the equilibrium network is non-empty and connected (there are no isolated firms), then for each* $j \in H_m(g)$, $N_j(g) = \{1, 2, ..., N\}$. *Thus the largest hubs are connected to each other and all other firms.*

Proof: Let $j \in H_m(g)$ and assume to the contrary that there exists a firm k such that $g_{jk} = 0$. Since the network is connected, k must have a link with at least one firm l, i.e. $k \in N_l(g)$. Since j belongs to the set of the largest hubs, $n_j(g) \geq n_l(g)$. It follows from Lemma 2 that $k \in N_l(g) \subseteq N_j(g)$ contradicting $g_{jk} = 0$. ■

Consider the spoke firms with the fewest links in an equilibrium network. Note that spoke firms must have at least one link or otherwise they would be isolated. The next result shows that these firms can only be linked to the largest hub firms in $H_m(g)$. An example is the star network in which $N - 1$ spoke firms have links with one hub only but not with each other.

Proposition 2 *Assume* $(A.1) - (A.3)$ *and* $(A.5) - (A.6)$ *hold and consider a non-empty equilibrium network* g. *For each* $i \in H_1(g)$, *the neighborhood of* i *is* $N_i(g) = \varnothing$ *or* $N_i(g) = \{j : j \in H_m(g\}$.

Proof: From Proposition 1, $\{j : j \in H_m(g)\} \subseteq N_i(g)$. Since firm i is the smallest spoke in the sense of having the smallest number of links, every other firm l satisfies $n_l(g) \geq n_i(g)$. From Lemma 2 it follows that $N_i(g) \subseteq N_l(g)$ for each l. Now suppose that $k \in H_h(g)$, $h \neq m$, and $g_{ik} = 1$. In other words, the smallest spoke i has a link with k who does not belong to the set of the largest hubs. Then $k \in N_l(g)$ for each l, i.e. k belongs to the neighborhood of each firm. But then $k \in H_m(g)$, contradicting the hypothesis that k is not the largest hub. ■

Note that if $N_i(g) = \varnothing$ for all $i \in H_1(g)$, then $H_1(g)$ is the set of isolated firms and $H_2(g)$ must be the set of spoke firms with the fewest links. In this case, the statement of Proposition 2 applies to firms in $H_2(g)$. Having dealt with the smallest spokes and the largest hubs, we now turn attention to the intermediate-sized spokes and hubs and a characterization of their partners.

Proposition 3 *Assume* $(A.1) - (A.3)$ *and* $(A.5) - (A.6)$ *hold and consider a non-empty equilibrium network* g*. For any* $i \in H_{h+1}(g)$*,* $1 \leq h < \frac{m}{2}$ *(if* m *is odd, then for any* $h < \frac{m+1}{2}$*), the neighborhood of* i *is:*

$$N_i(g) = H_{m-h}(g) \cup H_{m-h+1}(g) \cup \cdots \cup H_{m-1}(g) \cup H_m(g) \qquad (11)$$

For any $j \in H_{m-h}(g)$*:*

$$N_j(g) = H_{h+1}(g) \cup H_{h+2}(g) \cup \cdots \cup H_{m-1}(g) \cup H_m(g) \qquad (12)$$

Proof: See Appendix. ■

According to Proposition 3, corresponding to each spoke firm of a given size, there exists a threshold size for hub firms so that the spoke firm is linked to all hubs whose size is at least as great as the threshold. This threshold is decreasing in the size of the spoke firm. Thus larger spokes are distinguished from smaller spokes in that they are connected to a larger range of asymmetrically sized hubs. We can equivalently view this result from the perspective of the hub firms. For each hub of a given size, there is a threshold size for spoke firms so that the hub firm is connected to all those spokes whose size exceeds the threshold. This threshold is decreasing in the size of the hub firms. Therefore, as the size of hub firms increase, they have an incentive to connect with spokes of smaller size. We can now collect all the results to prove the following:

Proposition 4 *Assume* $(A.1) - (A.3)$ *and* $(A.5) - (A.6)$ *hold and* g *is an equilibrium network. In the class of connected networks, an equilibrium network is either complete or an interlinked star. In the class of unconnected networks, an equilibrium network can be empty or have at most one non-singleton component; further, this component is either complete or an interlinked star.*

Proof: The interlinked star characterization follows from Propositions 1-3. We only have to show that an unconnected network can have at most one non-singleton component. If $\frac{P^\theta}{2} - \psi(\theta) > c(1,1)$ for some $\theta \in \Theta$, then the equilibrium network will be non-empty. Let us suppose that there are two non-singleton components in g. The above arguments imply that they must be complete or interlinked stars. In either case we can identify players i, j, k such that $g_{ij} = 1$, $g_{ik} = g_{jk} = 0$ and $n_k(g) \geq n_i(g)$. However, from Lemma 2, $j \in N_i(g) \subseteq N_k(g)$ contradicting $g_{jk} = 0$. ■

We now provide an example to show that a star equilibrium network exists.

Example 4.1: Assume that $\mathcal{N} = \{1, 2, 3, 4\}$ and Θ is given by a discrete set with the following net option values:

Project	θ_1	θ_2	θ_3	θ_4	θ_5	θ_6
P^θ	6	7	8	9	10	11

Assume that $\psi(\theta) = 0$ for all θ. Non-specific costs are given by:

(n_i, n_j)	$(0,0)$	$(1,1)$	$(2,1)$	$(3,1)$	$(1,2)$	$(1,3)$	$(2,2)$	$(3,2)$	$(2,3)$	$(3,3)$
$c(n_i, n_j)$	12	6	5.25	4.2	5.75	4.25	5	4.1	4.15	4

Non-specific costs satisfy all the assumptions maintained in this section. It can be checked that the only equilibrium network is the star g^s where (say) firm 1 is the hub who engages in projects θ_6, θ_5, and θ_4 respectively with firms 2, 3, and 4. The payoffs are $\pi_1(g^s) = 2.4$, $\pi_2(g^s) = 1.25$, $\pi_3(g^s) = 0.75$, and $\pi_4(g^s) = 0.25$. No firm has an incentive to delete any of its links (recall that an isolated firm has a payoff of 0). It can be checked that no coalition of firms can do better by reorganizing their links. For example, if the two spokes, firms 2 and 3, add a link to the star by choosing project θ_3, then $\pi_2(g^s + g_{23}) = 0.5 < \pi_2(g^s)$ and similarly for firm 3. Suppose the spoke firms delete their link with 1 and form a link among themselves (network g') with firm 2 choosing projects θ_6 and θ_4 with firms 1 and 3 while the latter two firms choose θ_5. It suffices to see that firm 3 is worse off since $\pi_3(g') = 0.5 < \pi_3(g^s)$. Similarly, it can be verified that if all the firms formed a complete network, then at least one firm is worse off relative to the star. For example, consider the allocation of projects under g^s and allocate projects that give the highest payoff to firm 4. In particular, suppose firm 4 is connected to firm 3 through the project θ_3 and to firm 2 through the project θ_6. Note that $\frac{P^{\theta_6}}{3} = 3.67 > 3.5$ so it is better for firm 4 to share θ_6 with two other firms rather than engage in θ_2 with one other firm. Then $\pi_4(g^c) = 0.17 < \pi_4(g^s)$. ■

5 Choice of Projects

We now turn to a characterization of the research projects that are chosen by hub firms with the spoke firms. The main result of this section is as follows. When comparing the projects of a hub with two spokes, one small and the other large, the hub firm chooses projects with higher returns and higher risk with the smaller spoke. Similarly, when comparing the project of a hub with another smaller hub and a spoke, the hub chooses a higher return and higher variance project with the spoke. In general smaller, or more peripheral, spokes engage in riskier investments with hub firms than relatively larger spokes or other hubs.

If more than one pair of firms pursue the same project, then it is difficult to characterize the risk characteristics of these projects. We therefore limit ourselves in this section to the case of a continuous technology space. By virtue of Lemma 1 we know that each pair of firms will choose a distinct project. This means that, as long as we explicitly state the project that is chosen, we can drop reference to the network g in the net option value function. If any two firms choose the same project θ in networks g and g', then $\lambda(\theta, g) = \lambda(\theta, g')$. We now replace $(A.1)$ by putting a joint restriction on the net option value and non-specific project costs. Recall that the option value increases as we choose projects with a higher index θ because returns and variance are increasing with θ. However, since project-specific investment cost is also non-decreasing with θ, it is not clear how net option value changes with θ. Moreover, the net option values have to be compared to non-specific costs. The following assumption simply requires net option values in the case of a continuous technology space to be sufficiently responsive to a change in θ.

(A.1)* For any 3-tuple (n, n', θ), where $1 \leq n, n' < N - 1$ and $\theta \in \Theta$, there exists $\theta' \in \Theta$ such that:

$$\lambda(\theta') - \lambda(\theta) \geq c(n + 1, n') - c(n, n' + 1) \tag{13}$$

In other words, the above assumption places some (minimum) bounds on the variation in the net option value.[8]

Proposition 5 *Assume* $(A.1)^* - (A.6)$ *hold and* g *is an equilibrium interlinked star network. Consider a hub firm* $j \in H_q(g)$ *which has links with spoke firms* $i \in H_{q''}(g)$ *and* $k \in H_{q'}(g)$ *where* $q'' < q' < q$*. Then* $\lambda(\theta_{ij}) > \lambda(\theta_{kj})$*, i.e.* j *chooses a project with a greater net option value with* i *relative to its project with* k*.*

[8] Note that this assumption does not state that $\lambda(\theta') > \lambda(\theta)$. Since no assumption is placed on the RHS of (13), it is possible that $\lambda(\theta') \leq \lambda(\theta)$.

Corollary 1 *Suppose that project-specific costs are constant, or small, so that net option value is strictly increasing in* θ*. Under the assumptions of Proposition 5,* $\theta_{ij} > \theta_{kj}$*.*

Proof: Since the net option value $\lambda(\theta) = \frac{P^\theta}{2} - \psi(\theta)$ is strictly increasing in θ, $\lambda(\theta_{ij}) > \lambda(\theta_{kj})$ implies the result. ■

As an illustration of Corollary 1, Table 3 shows the project chosen by the largest hub with spoke firms of different sizes. The hub and smaller spokes choose relatively more risky projects because the cost of linking is greater for both firms. The cost for the hub is greater because it is linking with a peripheral (less connected) firm and thus needs to contribute relatively greater resources to their joint project. The cost for smaller spokes is greater as well because they have not been able to harness the scale economies afforded by having more links. Thus both need to be compensated with a project with a higher option value and hence they choose higher return higher risk projects.

Table 3: Project of Largest Hub Firm j with Spoke Firms

Project with	**Project**
$i_1 \in H_1(g)$	θ_{ji_1}
$i_2 \in H_2(g)$	$\theta_{ji_2} < \theta_{ji_1}$
$i_3 \in H_3(g)$	$\theta_{ji_3} < \theta_{ji_2} < \theta_{ji_1}$
$\cdots$	$\cdots$
$i_{\frac{m}{2}} \in H_{\frac{m}{2}}(g)$	$\theta_{ji_{\frac{m}{2}}} < \theta_{ji_{\frac{m}{2}-1}} < \cdots < \theta_{ji_2} < \theta_{ji_1}$

The same argument also shows that a hub firm will choose a relatively lower risk lower return project with another hub as compared to its project choice with a spoke firm. This is shown next:

Proposition 6 *Assume* $(A.1)^* - (A.6)$ *hold and* g *is an equilibrium interlinked network. Consider hub firms* j *and* k *and a spoke firm* i *such that* $n_j(g) > n_k(g)$*,* $g_{ij} = 1$ *and* $g_{ki} = 0$*. Then* $\lambda(\theta_{ij}) > \lambda(\theta_{kj})$*. If the net option value is strictly increasing in* θ*, then* $\theta_{ij} > \theta_{kj}$*.*

Proof: Since j and k are hub firms, it follows that $g_{jk} = 1$ and $n_k(g) > n_i(g)$. The proof now follows the same argument as Proposition 5 and Corollary 1. ■

6 Dissolution of Links

Finally, consider the issue of why we are likely to see many links that have been formed in period 0 dissolving at some future time. To fix ideas, consider the Black-Scholes pricing of real options with exogenous exercise date in Example 2.2. Since the returns are random, there is a positive probability that the real option will have zero value at the exercise date T. In particular, the probability that the real option θ ends up "out of the money" (zero price at maturity) is given by:

$$g(\theta) = \mathbf{N}(d_2) \tag{14}$$

Each pair of firms will therefore dissolve all those links whose option value is zero at maturity.

It would have been interesting to see what kinds of links are more likely to be dissolved. In particular, is it more likely to see dissolution of links between hubs and spokes relative to those between hubs? Unfortunately we were unable to obtain any concrete results in this direction. Note that the variation in the probability of dissolving a link with respect to θ can be determined by examining the variation of d_2 with respect to θ:

$$\frac{\partial d_2^\theta}{\partial \theta} = \left(\frac{\partial d_2^\theta}{\partial V_0^\theta} \times \frac{\partial V_0^\theta}{\partial \theta} \right) + \left(\frac{\partial d_2^\theta}{\partial K^\theta} \times \frac{\partial K^\theta}{\partial \theta} \right) + \left(\frac{\partial d_2^\theta}{\partial \sigma^\theta} \times \frac{\partial \sigma^\theta}{\partial \theta} \right)$$

However, it can be verified that:

$$\frac{\partial d_2^\theta}{\partial V_0^\theta} = \frac{1}{\sigma^\theta V_0^\theta \sqrt{T}} > 0, \quad \frac{\partial d_2^\theta}{\partial \sigma^\theta} = -\frac{d_1^\theta}{2\sigma^\theta} < 0, \quad \frac{\partial d_2^\theta}{\partial K^\theta} = -\frac{1}{\sigma^\theta V_0^\theta \sqrt{T}} < 0$$

Therefore, no general results can be derived without additional assumptions on the variations of V_0^θ, K^θ, and σ^θ.

7 Conclusion

This paper explored the incentives of firms to form networks of research partnerships in their pursuit of new technology opportunities in contexts of high uncertainty. Our model explained the following: why networks are particularly ubiquitous in industries that are subject to high uncertainty; why networks sometimes display an interconnected "hubs and spokes" architecture; why small (or peripheral spoke) firms often sink resources into relatively higher risk higher return investment projects with only hub firms;

and why so many research alliances are continuously formed and dissolved. Our paper also delineated the conditions under which ex-ante symmetric firms ended up ex-post forming complex asymmetric networks. Firms were assumed to view collaborative links (research partnerships) as vehicles to create opportunities and evaluated them as real options to new technologies and, accordingly, new markets. As such, the paper addressed the intersection of strategic networks and real options theory. It formalized a process through which firms partnered with others to expand their technology search space collectively in terms of pursuing bolder research projects (high risk and high return). It therefore provided an explanation of why strategic alliances are particularly prevalent in high uncertainty industries. The assumptions on option values and the cost of initial investment in a project helped explain the existence and architecture of research networks that have been observed in industries experiencing rapid technological change. In particular, the paper demonstrated that when the initial investment cost for any project between two firms was falling in the number of links of the firms, then the equilibrium network assumed a hub-and-spoke architecture. Therefore, even though firms were ex-ante symmetric, the equilibrium network was ex-post asymmetric. The paper further demonstrated that each hub firm chose a relatively higher risk (and higher return) project with a more peripheral (or smaller spoke) than with another hub or a larger spoke. Evaluating the value of each link as a real option also helped explain why firms dissolved links even in equilibrium.

This paper provided firms with a menu of technological opportunities so that any pair of firms could choose a different project and assume monopoly control over non-overlapping technological areas. An issue of great interest is the other extreme where there is only one technological innovation possible and a partnership of firms which is the first to be successful can patent it for monopoly use. This is the kind of framework that has been envisaged by Huisman and Kort (2000) and Grenadier (2000a,b) who have noted that standard option price calculations would change if the strategic behavior of agents, and in particular the possibility of *preemption*, was taken into account. Their analyis is within a 2-player framework and looks at the option value of waiting and the optimal exercise strategy under threat of preemption. Introducing the possibility of preemption and monopoly control over a technology in the network framework could have interesting consequences. Since the threat of preemption would affect the option value of links, it would also then impact the architecture of the research networks. Combining real options and strategic network formation in an environment of preemption should provide a fertile area for future research.

8 Appendix

Proof of Lemma 1: Suppose i and j have chosen the project θ'. If i and k also choose θ', then the option is only worth $\frac{P^{\theta'}}{3}$ from $(A.5)$. Suppose $\frac{P^{\theta}}{2} - \psi(\theta)$ is non-decreasing at θ'. Then for some $\theta'' > \theta'$, $\frac{P^{\theta''}}{2} - \psi(\theta'') \geq \frac{P^{\theta'}}{2} - \psi(\theta') > \frac{P^{\theta'}}{3} - \psi(\theta')$. Since the non-specific cost c depends only on $n_i(g)$ and $n_k(g)$ and is independent of θ, both i and k have an incentive to choose θ''. If $\frac{P^{\theta}}{2} - \psi(\theta)$ is non-increasing at θ', then the same argument applies for some $\theta'' < \theta'$. ■

Proof of Lemma 2: Suppose to the contrary that $n_i(g) \leq n_j(g)$ in an equilibrium network g but $N_i(g)\backslash N_j(g) \neq \varnothing$. Index the firms such that:

$$\begin{array}{rcl} 1, 2, ..., L & \in & N_i(g)\backslash N_j(g) \\ L+1, L+2, ...L' & \in & N_i(g) \cap N_j(g) \\ L'+1, L'+2, ..., L'' & \in & N_j(g)\backslash N_i(g) \end{array}$$

Let $g' = g - \sum_{l=1}^{L} g_{il}$ denote the network in which i has deleted all the links in $N_i(g)\backslash N_j(g)$. Since i has no incentive to delete any subset of links:

$$\pi_i(g) - \pi_i(g') = \sum_{l=1}^{L} \left[\lambda(\theta_{il}, g) - c\left(n_i(g), n_l(g)\right)\right] + \sum_{l=L+1}^{L'} \left[c\left(n_i(g'), n_l(g')\right) - c\left(n_i(g), n_l(g)\right)\right] \geq 0$$

Now consider the coalition $S = \{j\} \cup N_i(g)\backslash N_j(g)$ and let $g'' = g' + \sum_{l=1}^{L} g_{jl}$ denote the network in which each firm $l \in N_i(g)\backslash N_j(g)$ deletes its link with i and forms a link with j by choosing the project $\theta_{jl} = \theta_{il}$, i.e. the same project it pursued with i in g. Note that $n_l(g'') = n_l(g) = n_l(g') + 1$ for

$l \in N_i(g) \backslash N_j(g)$. For firm j:

$$\pi_j(g'') - \pi_j(g) = \sum_{l=1}^{L} \left[\lambda(\theta_{jl}, g'') - c\left(n_j(g''), n_l(g'')\right)\right]$$
$$+ \sum_{l=L+1}^{L'} \left[c\left(n_j(g), n_l(g)\right) - c\left(n_j(g''), n_l(g'')\right)\right]$$
$$+ \sum_{l=L'+1}^{L''} \left[c\left(n_j(g), n_l(g)\right) - c\left(n_j(g''), n_l(g'')\right)\right]$$

Note from (A.2) that for $l = 1, 2, ..., L$, $c\left(n_j(g''), n_l(g'')\right) < c\left(n_i(g), n_l(g)\right)$ since $n_j(g'') > n_i(g)$ and $n_l(g'') = n_l(g)$. From the choice of the project $\theta_{jl} = \theta_{il}$, it follows that $\lambda(\theta_{il}, g) = \lambda(\theta_{jl}, g'')$. Therefore:

$$\sum_{l=1}^{L} \left[\lambda(\theta_{jl}, g'') - c\left(n_j(g''), n_l(g'')\right)\right] > \sum_{l=1}^{L} \left[\lambda(\theta_{il}, g) - c\left(n_i(g), n_l(g)\right)\right]$$

Note that $n_j(g) < n_j(g'')$. Further $n_l(g'') = n_l(g)$ for $l \in N_j(g) \backslash N_i(g)$. Therefore from (A.2):

$$\sum_{l=L'+1}^{L''} \left[c\left(n_j(g), n_l(g)\right) - c\left(n_j(g''), n_l(g'')\right)\right] > 0$$

Finally, note that $n_l(g) = n_l(g') = n_l(g'')$ for $l \in N_i(g) \cap N_j(g)$ and:

$$c\left(n_j(g), .\right) - c\left(n_j(g''), .\right) = \left[c\left(n_j(g), .\right) - c\left(n_j(g) + 1, .\right)\right]$$
$$+ \left[c\left(n_j(g) + 1, .\right) - c\left(n_j(g) + 2, .\right)\right] + \cdots + \left[c\left(n_j(g) + L - 1, .\right) - c\left(n_j(g''), .\right)\right] \tag{15}$$

$$c\left(n_i(g'), .\right) - c\left(n_i(g), .\right) = \left[c\left(n_i(g'), .\right) - c\left(n_i(g') + 1, .\right)\right]$$
$$+ \left[c\left(n_i(g') + 1, .\right) - c\left(n_i(g') + 2, .\right)\right] + \cdots + \left[c\left(n_i(g') + L - 1, .\right) - c\left(n_i(g), .\right)\right] \tag{16}$$

From (A.3):

$$c\left(n_j(g) + x, .\right) - c\left(n_j(g) + x + 1, .\right) > c\left(n_j(g) + x - 1, .\right) - c\left(n_j(g) + x, .\right)$$
$$> \cdots > c\left(n_i(g') + x, .\right) - c\left(n_i(g') + x + 1, .\right)$$

Therefore each term within the square parentheses in (15) is strictly greater than the corresponding term in (16). It follows that:

$$\sum_{l=L+1}^{L'} \left[c\left(n_j(g), n_l(g)\right) - c\left(n_j(g''), n_l(g'')\right)\right] >$$

$$\sum_{l=L+1}^{L'} \left[c\left(n_i(g'), n_l(g')\right) - c\left(n_i(g), n_l(g)\right)\right]$$

and we have shown that $\pi_j(g'') - \pi_j(g) > \pi_i(g) - \pi_i(g') \geq 0$. Therefore j has an incentive to form links with all the firms in $N_i(g)\backslash N_j(g)$ and move from g to g''.

We now show that each firm k in $N_i(g)\backslash N_j(g)$ has an incentive to reciprocate the link with j. From the equilibrium property of g, k would not delete the link with i:

$$\begin{aligned} \pi_k(g) - \pi_k(g') = {} & \lambda(\theta_{ik}, g) - c\left(n_k(g), n_i(g)\right) \\ & + \sum_{l \in N_k(g')} \left[c\left(n_k(g'), n_l(g')\right) - c\left(n_k(g), n_l(g)\right)\right] \geq 0 \end{aligned} \tag{17}$$

Recall that each $k \in \{1, 2, ..., L\}$ forms a link with j by choosing a project $\theta_{kj} = \theta_{ik}$.

$$\begin{aligned} \pi_k(g'') - \pi_k(g') = {} & \lambda(\theta_{kj}, g'') - c\left(n_k(g''), n_j(g'')\right) \\ & + \sum_{l \in N_k(g')} \left[c\left(n_k(g'), n_l(g')\right) - c\left(n_k(g''), n_l(g'')\right)\right] \end{aligned} \tag{18}$$

From (A.2) and the fact that $n_i(g'') < n_i(g)$, $c\left(n_k(g''), n_j(g'')\right) < c\left(n_k(g), n_i(g)\right)$. Therefore $\lambda(\theta_{kj}, g'') - c\left(n_k(g''), n_j(g'')\right) > \lambda(\theta_{ik}, g) - c\left(n_k(g), n_i(g)\right)$. Note that for $l \in N_k(g')\backslash S$ we have $n_l(g) = n_l(g'') = n_l(g')$ while for $l \in N_k(g') \cap S$ we have $n_l(g'') = n_l(g) = n_l(g') + 1$. Therefore, the last terms on the RHS of (17) and (18) are the same. It follows that $\pi_k(g'') - \pi_k(g') > \pi_k(g) - \pi_k(g') \geq 0$. Therefore, from the network g', all $k \in N_i(g)\backslash N_j(g)$ are strictly better off forming a link with j than with i. Thus these firms will jointly delete their links with i and form a link with j. Since j does better as well by reciprocating these links (relative to g), this contradicts the starting hypothesis that g is an equilibrium network. ■

Proof of Proposition 3: In the following proof it will be convenient to let $l_1, l_2, ..., l_m$ denote a representative firm from the sets $H_1(g), H_2(g), ..., H_m(g)$

respectively. Consider $l_2 \in H_2(g)$. From Proposition 1, $H_m(g) \subset N_{l_2}(g)$. It is a *proper* subset because from Proposition 2 $H_m(g)$ is the neighborhood for the smallest spoke firms in $H_1(g)$ and l_2 has strictly more links than the smallest spokes. We now argue that the additional links of l_2 must be with hub firms in $H_{m-1}(g)$. Suppose not and let $k \in N_{l_2}(g)$ but $k \notin H_{m-1}(g) \cup H_m(g)$. Then $n_k(g) < n_{l_{m-1}}(g)$. From Lemma 2, $k \in N_{l_2}(g) \subseteq N_{l_3}(g) \subseteq \cdots \subseteq N_{l_m}(g)$ and therefore $N_k(g) = H_2(g) \cup H_3(g) \cup \cdots \cup H_m(g)$. From Proposition 2, $N_{l_{m-1}}(g) \cap H_1(g) = \varnothing$ and therefore $N_{l_{m-1}}(g) \subseteq H_2(g) \cup H_3(g) \cup \cdots \cup H_m(g) = N_k(g)$. Thus $n_{l_{m-1}}(g) \leq n_k(g)$, a contradiction. It follows that (11) and (12) hold for $h = 1$.

Now suppose that (11) and (12) are true for any $h' \geq 1$. We will show that they hold for $h'+1$. By induction, $H_{m-h'}(g) \cup H_{m-h'+1}(g) \cup \cdots \cup H_m(g) \subset N_{l_{h'+2}}(g)$. We now show that the additional links of $l_{h'+2}$ must be with firms in $H_{m-h'-1}(g)$. Suppose not and let $j \in N_{l_{h'+2}}(g)$ but $j \notin H_{m-h'-1}(g) \cup H_{m-h'}(g) \cup \cdots \cup H_m(g)$. Then $n_j(g) < n_{l_{m-h'-1}}(g)$. From induction, $N_{l_{m-h'-1}}(g) \subseteq H_{h'+2}(g) \cup H_{h'+1}(g) \cup \cdot\cdot \cup H_m(g)$. From Lemma 2, $j \in N_{l_{h'+2}}(g) \subseteq N_{l_{h'+3}}(g) \subseteq \cdots \subseteq N_{l_m}(g)$ and thus $N_j(g) = H_{h'+2}(g) \cup H_{h'+1}(g) \cup \cdot\cdot \cup H_m(g)$. But then $n_j(g) \geq n_{l_{m-h'-1}}(g)$, a contradiction. ■

Proof of Proposition 5: Since i does not want to delete a link with j in an equilibrium network g:

$$\begin{aligned} \pi_i(g) - \pi_i(g - g_{ij}) = {} & \lambda(\theta_{ij}) - c(n_i(g), n_j(g)) \\ & + \sum_{l \in N_i(g-g_{ij})} [c(n_i(g)-1, n_l(g)) - c(n_i(g), n_l(g))] \geq 0 \end{aligned} \tag{19}$$

Consider k and note that $g_{ik} = 0$ since i and k are spoke firms. Both spokes belong to $N_j(g)$. Consider a coalition $S = N_j(g)\backslash\{i\}$ and connect each pair of unlinked firms $l, h \in S$ with some project θ_{lh}. Each firm in S now has $n_j(g) - 1$ links. Extend the coalition to $S \cup \{i\}$ and connect i to k through some project θ_{ik}. Call the network obtained from g by the stated deviations of the coalition $S \cup \{i\}$ as g'. Note that $n_j(g) = n_k(g') > n_l(g') > n_i(g')$ for all $l \in S\backslash\{k\}$. We now claim the following:

Claim: If $\pi_i(g') \geq \pi_i(g)$, then $\pi_l(g') > \pi_l(g)$ for all $l \in S$.
In words, suppose firm i with the lowest number of links in g' finds it profitable through a suitable project to move from g to g'. Then all other firms in S who have more links than i in g' will be able to find suitable projects

that makes this move profitable as well. We prove by contradiction. Suppose there exists a project θ_{ik} such that $\pi_i(g') \geq \pi_i(g)$, i.e.

$$\lambda(\theta_{ik}) - c(n_i(g'), n_k(g')) + \sum_{h \in N_i(g)} \big[c(n_i(g), n_h(g)) - c(n_i(g'), n_h(g'))\big] \geq 0 \tag{20}$$

However for some $l \in S$, and for all choice of projects θ_{lh}, we have $\pi_l(g') \leq \pi_l(g)$:

$$\sum_{h \in S \backslash N_l(g)} \big[\lambda(\theta_{lh}) - c(n_l(g'), n_h(g'))\big] + \sum_{h \in N_l(g)} \big[c(n_l(g), n_h(g)) - c(n_l(g'), n_h(g'))\big] \leq 0 \tag{21}$$

In order to show a contradiction, we will show that the LHS of (21) is strictly greater than the LHS of (20). Note that without loss of generality we can assume that $\lambda(\theta_{ik}) - c(n_i(g'), n_k(g')) \geq 0$. There are 2 possible cases:

Case 1: Let l be such that $n_l(g) \geq n_i(g)$. Note that $n_h(g') \leq n_k(g')$, $h \in S \backslash N_l(g)$. From $(A.1)^*$ there exists a θ' such that:

$$\begin{aligned} \lambda(\theta') - \lambda(\theta_{ik}) &\geq c(n_l(g'), n_h(g')) - c(n_l(g') - 1, n_h(g') + 1) \qquad (22) \\ &\geq c(n_l(g'), n_h(g')) - c(n_i(g'), n_k(g')) \end{aligned}$$

where the second inequality follows from $(A.2)$. From the continuity of Θ it is possible to choose projects θ_{lh}, $h \in S \backslash N_l(g)$, such that:

$$\frac{1}{|S \backslash N_l(g)|} \sum_{h \in S \backslash N_l(g)} \lambda(\theta_{lh}) = \lambda(\theta')$$

Substitute in (22) and note that $|S \backslash N_l(g)| c(n_l(g'), n_h(g')) \geq \sum_{h \in S \backslash N_l(g)} c(n_l(g'), n_h(g'))$ since all firms in $S \backslash \{k\}$ have the same number of links and one less link than k. Rearranging it follows that $\sum_{h \in S \backslash N_l(g)} [\lambda(\theta_{lh}) - c(n_l(g'), n_h(g'))] \geq \lambda(\theta_{ik}) - c(n_i(g'), n_k(g'))$.

Note from Lemma 2 that $N_i(g) \subseteq N_l(g)$. For each $h \in N_i(g)$, write the second term in (20) as:

$$[c(n_i(g), n_h(g)) - c(n_i(g) + 1, n_h(g))] + \big[c(n_i(g) + 1, n_h(g)) - c(n_i(g) + 1, n_h(g'))\big] \tag{23}$$

where we have used the fact that $n_i(g') = n_i(g) + 1$. Consider the second term in (21). It is positive for all $h \in N_l(g) \backslash N_i(g)$ from $(A.2)$. For all

$h \in N_i(g)$, we can write it as:

$$\begin{aligned}
&[c(n_l(g), n_h(g)) - c(n_l(g)+1, n_h(g))] + [c(n_l(g)+1, n_h(g)) - c(n_l(g)+2, n_h(g))] \\
&\qquad + \cdots + \left[c(n_l(g'), n_h(g)) - c(n_l(g'), n_h(g'))\right] \\
&\qquad > [c(n_l(g), n_h(g)) - c(n_l(g)+1, n_h(g))] \\
&\qquad\qquad + \left[c(n_l(g'), n_h(g)) - c(n_l(g'), n_h(g'))\right] \quad (24)
\end{aligned}$$

since the intermediate terms are positive by virtue of $(A.2)$. Consider the first term in (24). Using $(A.3)$ repeatedly shows that it is greater than the first term in (23):

$$\begin{aligned}
c(n_l(g), n_h(g)) - c(n_l(g)+1, n_h(g)) &\geq c(n_l(g)-1, n_h(g)) - c(n_l(g), n_h(g)) \\
&\geq \cdots \geq c(n_i(g), n_h(g)) - c(n_i(g)+1, n_h(g))
\end{aligned}$$

It now remains to compare the last terms in the two expressions. The last term in (23) can be expanded as:

$$\begin{aligned}
&\left[c(n_i(g'), n_h(g)) - c(n_i(g'), n_h(g)+1)\right] + \left[c(n_i(g'), n_h(g)+1) - c(n_i(g'), n_h(g)+2)\right] \\
&\qquad + \cdots + [c(n_i(g'), n_h(g')-1) - c(n_i(g'), n_h(g'))] \quad (25)
\end{aligned}$$

The last term in (24) can be expanded similarly:

$$\begin{aligned}
&\left[c(n_l(g'), n_h(g)) - c(n_l(g'), n_h(g)+1)\right] + \left[c(n_l(g'), n_h(g)+1) - c(n_l(g'), n_h(g)+2)\right] \\
&\qquad + \cdots + [c(n_l(g'), n_h(g')-1) - c(n_l(g'), n_h(g'))] \quad (26)
\end{aligned}$$

Each term within the square parentheses in (26) dominates the corresponding term in (25) by applying $(A.4)$:

$$\begin{aligned}
&c(n_l(g'), n_h(g)+x) - c(n_l(g'), n_h(g)+x+1) > c(n_l(g')-1, n_h(g)+x) - c(n_l(g')-1, n_h(g)+x+1) \\
&\qquad > \cdots > c(n_i(g'), n_h(g)+x) - c(n_i(g'), n_h(g)+x+1)
\end{aligned}$$

Collecting all the above results, $\pi_l(g') - \pi_l(g) > \pi_i(g') - \pi_i(g) \geq 0$ contradicting the hypothesis that $\pi_l(g') \leq \pi_l(g)$.

Case 2: Let l be such that $n_l(g) < n_i(g)$ so that $N_l(g) \subset N_i(g)$. We can rewrite (20) as:

$$\begin{aligned}
&\lambda(\theta_{ik}) - c(n_i(g'), n_k(g')) + \sum_{h \in N_i(g) \setminus N_l(g)} \left[\lambda(\theta_{ih}) - c(n_i(g'), n_h(g'))\right] \\
&- \sum_{h \in N_i(g) \setminus N_l(g)} [\lambda(\theta_{ih}) - c(n_i(g), n_h(g))] - \sum_{h \in N_l(g)} \left[c(n_i(g), n_h(g)) - c(n_i(g'), n_h(g'))\right] \geq 0
\end{aligned} \quad (27)$$

Note that since g is an equilibrium network:

$$\sum_{h\in N_i(g)\backslash N_l(g)} [\lambda(\theta_{ih}) - c(n_i(g), n_h(g))] \geq 0$$

otherwise i would have an incentive to delete all links in $N_i(g)\backslash N_l(g)$ and maintain the same number of links as l. Using the same argument employing $(A.1)^*$ for $h \in N_i(g)\backslash N_l(g)$ as in Case 1:

$$\sum_{h\in S\backslash N_l(g)} \big[\lambda(\theta_{lh}) - c(n_l(g'), n_h(g'))\big] > \lambda(\theta_{ik}) - c(n_i(g'), n_k(g')) + \sum_{h\in N_i(g)\backslash N_l(g)} \big[\lambda(\theta_{ih}) - c(n_i(g'), n_h(g'))\big]$$

It now remains to compare the last terms in (21) and (27). For each $h \in N_l(g)$, with the help of $(A.2)$:

$$\begin{aligned}
&c(n_l(g), n_h(g)) - c(n_l(g'), n_h(g')) \\
&= [c(n_l(g), n_h(g)) - c(n_l(g)+1, n_h(g))] \\
&+ \cdots + [c(n_i(g), n_h(g)) - c(n_i(g'), n_h(g))] + [c(n_i(g'), n_h(g)) \\
&- c(n_i(g'), n_h(g'))] + [c(n_i(g'), n_h(g')) - c(n_l(g'), n_h(g'))] \\
&> [c(n_i(g), n_h(g)) - c(n_i(g'), n_h(g))] + [c(n_i(g'), n_h(g)) - c(n_i(g'), n_h(g'))] \\
&= c(n_i(g), n_h(g)) - c(n_i(g'), n_h(g'))
\end{aligned}$$

Collecting all the above results, $\pi_l(g') - \pi_l(g) > \pi_i(g') - \pi_i(g) \geq 0$ contradicting the hypothesis that $\pi_l(g') \leq \pi_l(g)$.

We now return to the main proof. From the equilibrium property the movement from g to g' must leave at least one firm in S worse off. From the claim above, firm i must be worse off (since $\pi_l(g') \leq \pi_l(g)$ for some $l \in S$ implies $\pi_i(g') < \pi_i(g)$). Then for all θ:

$$\pi_i(g') - \pi_i(g) = \lambda(\theta) - c(n_i(g)+1, n_k(g')) + \sum_{l\in N_i(g)} \big[c(n_i(g), n_l(g)) - c(n_i(g)+1, n_l(g'))\big] < 0 \quad (28)$$

In particular (28) holds for $\theta = \theta_{kj}$. Then subtracting (28) from (19):

$$\begin{aligned}\lambda(\theta_{ij}) - \lambda(\theta_{kj}) + \big[c(n_i(g)+1, n_k(g')) - c(n_i(g), n_j(g))\big] \\ - \left[c(n_i(g), n_j(g)) - c(n_i(g)+1, n_j(g))\right] \\ + \sum_{l \in N_i(g-g_{ij})} \left[c(n_i(g)-1, n_l(g)) - c(n_i(g), n_l(g))\right. \\ \left. - c(n_i(g), n_l(g)) + c(n_i(g)+1, n_l(g'))\right] > 0\end{aligned}$$

Note that $n_k(g') = n_j(g)$. Therefore:

$$\begin{aligned}c(n_i(g)+1, n_k(g')) - c(n_i(g), n_j(g)) < 0 \\ c(n_i(g), n_j(g)) - c(n_i(g)+1, n_j(g)) > 0 \\ c(n_i(g)-1, n_l(g)) - c(n_i(g), n_l(g)) - c(n_i(g), n_l(g)) + c(n_i(g)+1, n_l(g')) < 0\end{aligned}$$

where the first two inequalities are a consequence of $(A.2)$. The third inequality follows from $(A.2)$, $(A.3)$ and $(A.4)$ as follows:

$$\begin{aligned}c\left(n_i(g), n_l(g)\right) - c\left(n_i(g)+1, n_l(g')\right) &= c\left(n_i(g), n_l(g)\right) - c\left(n_i(g), n_l(g')\right) \\ &\quad + c\left(n_i(g), n_l(g')\right) - c\left(n_i(g)+1, n_l(g')\right) \\ &> c\left(n_i(g), n_l(g')\right) - c\left(n_i(g)+1, n_l(g')\right) \\ &> c\left(n_i(g), n_l(g')-1\right) - c\left(n_i(g)+1, n_l(g')-1\right) \\ &> ... > c\left(n_i(g), n_l(g)\right) - c\left(n_i(g)+1, n_l(g)\right) \\ &> c\left(n_i(g)-1, n_l(g)\right) - c\left(n_i(g), n_l(g)\right)\end{aligned}$$

Thus $\lambda(\theta_{ij}) - \lambda(\theta_{kj}) > 0$. ■

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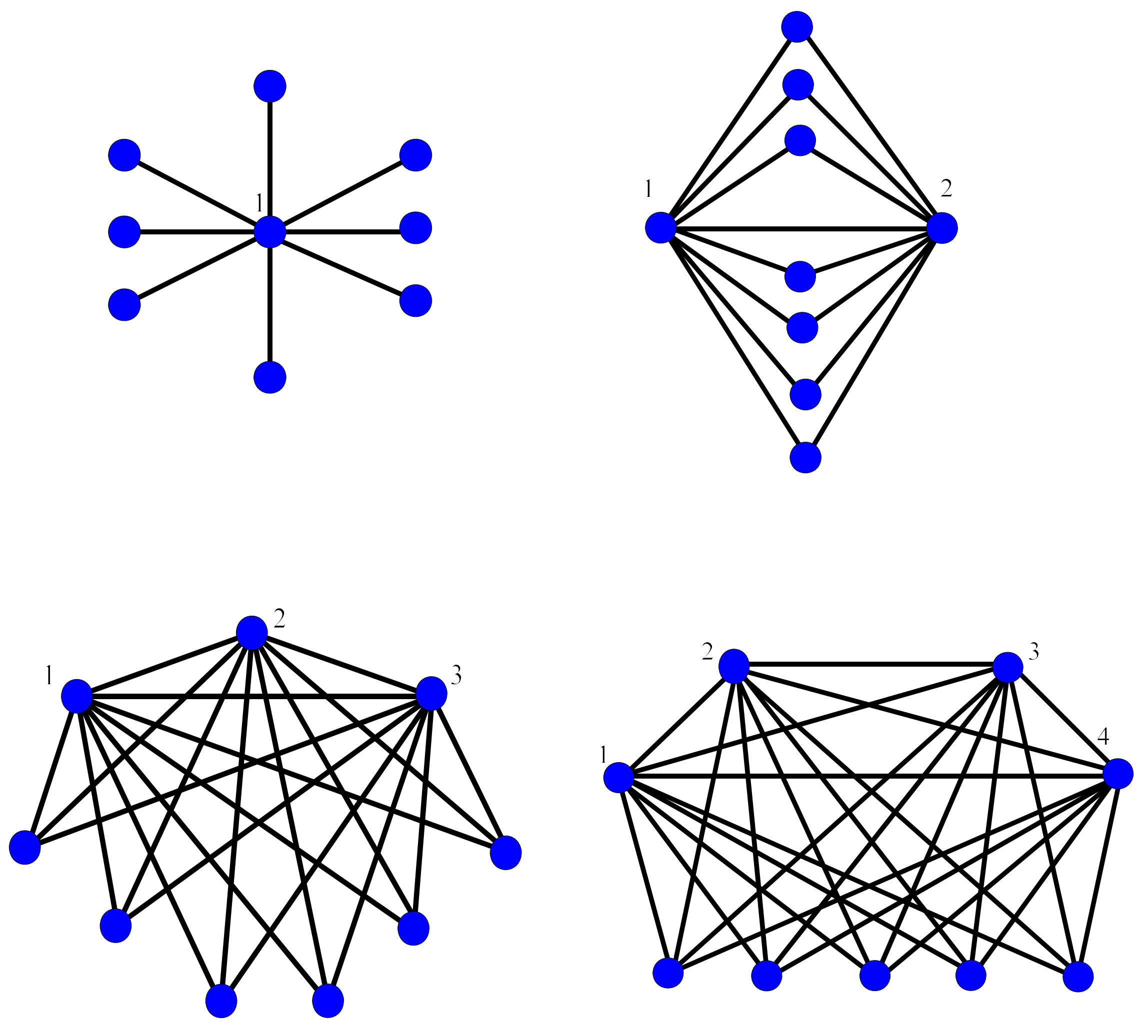

Figure 1: Star and Interlinked Stars Networks

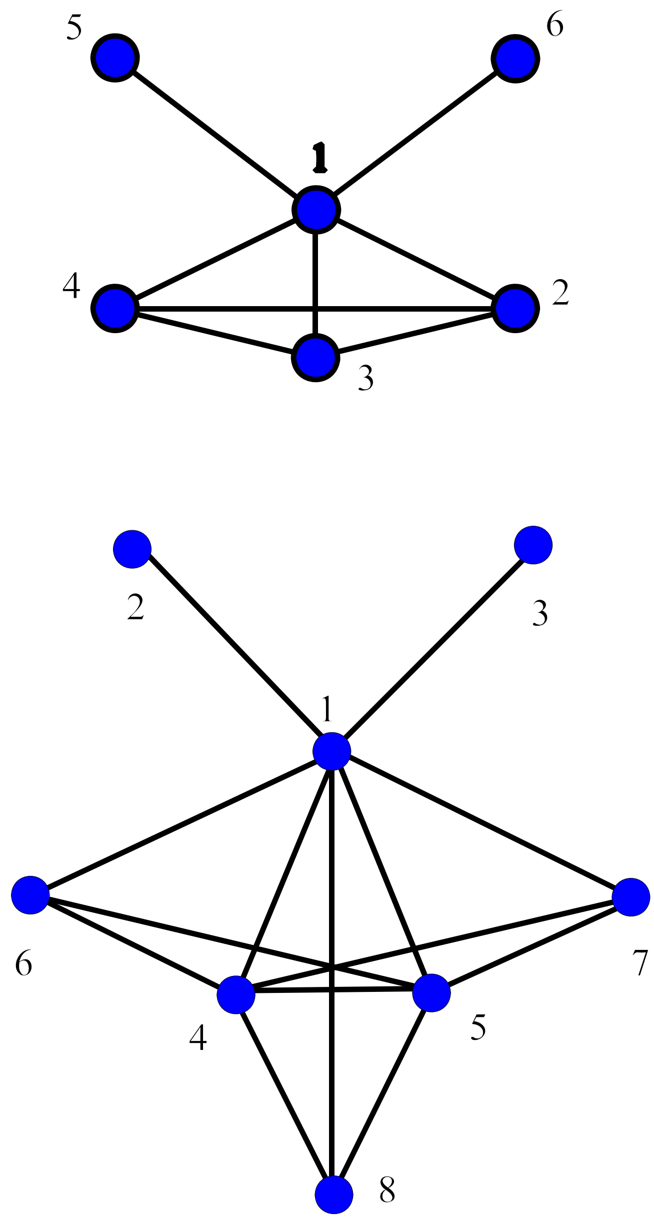

Figure 2: Interlinked Stars with Asymmetrically-Sized Hubs