The Structural Transformation of Innovation

Deliverable D2.1: Interim report on multi-sector extension of the baseline model. **Revised**

Authors: Diego Comin, Danial Lashkari, and Marti Mestieri

Version: 1.0

Quality review: All Partners

Date: 31 January, 2018

Grant Agreement number: 727073

Starting Date: 01/04/2017

Duration: 24 months

Coordinator: Dr. Georg Licht, ZEW

Email: licht@zew.de

This project has received funding from the European Union's Horizon 2020 research and innovation programme under the grant agreement No 727073

Contents

Project Information Summary i

Deliverable Documentation Sheet i

Disclaimer ii

Acknowledgment iii

Executive Summary 1

1 Introduction 2

2 Motivating Evidence 7

3 Model 10

3.1 Demand and Income Elasticities 10
3.2 Production 13
3.3 Innovation and Technology-Push 14
3.4 Market Equilibrium 16
3.5 Equilibrium Dynamics 18
3.6 Constant Growth Path 22

4 Conclusion 28

References 28

Appendix 31

Project Information Summary

Table 1: Project Information Summary

Project Acronym	FRAME
Project Full Title	Framework for the Analysis of Research and Adoption Activities and their Macroeconomic Effects
Grant Agreement	727073
Call Identifier	H2020 - SC6 – CO-CREATION – 2016 -1
Topic	CO-CREATION-08-2016/2017: Better integration of evidence on the impact of research and innovation in policy making
Funding Scheme	Medium-scaled focused research project
Project Duration	1st April 2017 – 31st March 2019 (24 months)
Project Officer(s)	Hinano SPREAFICO (Research Executive Agency)
	Roberto MARTINO (DG Research and Innovation)
Co-ordinator	Dr. Georg Licht, Zentrum für Europäische Wirtschaftsforschung GmbH
Consortium Partners	Centre for Economic Policy Research
	Lunds Universitet
	Università Luigi Bocconi
	Universitat Pompeu Fabra
	London Business School
Website	http://www.h2020frame.eu/frame/home.html

Deliverable Documentation Sheet

Table 2: Deliverable Documentation Sheet

Number	**D2.1**
Title	Interim report on multi-sector extension of the baseline model
Related WP	WP2
Lead Beneficiary	CEPR
Author(s)	Diego Comin (CEPR), Danial Lashkari (Harvard University), Marti Mestieri (Northwestern University)
Contributor(s)	
Reviewer(s)	All partners
Nature	R (Report)
Dissemination level	PU (Public)
Due Date	31.01.2018
Submission Date	
Status	

Disclaimer

The opinion stated in this report reflects the opinion of the authors and not the opinion of the European Commission.

Acknowledgment

This document is a deliverable of the FRAME project, which has received funding from the European Union's Horizon 2020 Programme for Research, Technological Development and Demonstration under Grant Agreement number 727073.

Revision

The paper delivered is not the interim report but an early draft of the final scientific paper connected to the issues discussed in WP2. The latter aims at completing D2.2, and extending its modelling approach to better fit the needs of the DG RTD (presented during the MONROE-FRAME meeting, September 2017).

Executive Summary

The goal of the second Work Package is to develop a multi-sector model of innovation that is rich enough to offer new insights about the effects of various innovation policies on productivity growth but also translates the shift from manufacturing to services industries. To do so, the deliverable D2.1 develops a unique modelling approach which conceptualizes endogenously the direction of technical change with a demand system that allows for non-homotheticities in consumption that differ across sectors. This setting is capable of reproducing structural transformation (i.e., reallocation of economic activity across sectors as they develop) we observe in the data. Similarly, this setting is capable of producing a reallocation of innovation activity towards higher income-elastic sectors, structural transformation of innovation, similar to what we have observed in the data.

Our theory provides a framework to study the general equilibrium interactions between demand-pull and technology-push forces in determining the equilibrium rates of innovation and productivity growth across industries. We provide simple and intuitive characterizations for the evolution of R&D intensity across sectors. In the model, the asymptotic rates of innovation and productivity growth are pinned down by the income elasticities of sectoral outputs. We show that the rates of growth of R&D expenditure and patents in the US show sizable correlations with the income elasticities of their outputs, estimated using micro-level household consumption surveys.

We document a change in the direction of innovation both in terms of sectoral composition of R&D investment and patent applications that we call the structural transformation of innovation. A significant driver of the structural transformation of innovation is the income elasticity of demand for the value added produced in the sector. For example, a one standard deviation in the sectoral income elasticity is associated with over 50% of the standard deviation of the change in the sectoral R&D share. We develop a multi-sector endogenous growth model in which the direction of innovation across sectors is endogenous. A calibrated version of the model simultaneously accounts for the transformation of the sectoral distribution of employment as well as of the structural transformation of innovation.

1. Introduction

Modern economies go through immense structural transformations as their incomes grow: the shares of manufacturing industries in employment and output dramatically decline, while they continuously rise for services. What are the implications of these transformations for understanding the future of growth? Historically, manufacturing industries have shown the fastest rates of technological growth, have hired the most number of R&D workers, and have created the highest volumes of patents, our best available indicator of innovation. In contrast, service industries are often associated with slow productivity growth and low intensity of innovation activity. These observations have caused concerns about the sustainability of innovation and productivity growth among academics and policy makers.[1] These concerns stem from a view of growth that emphasizes the role of technological possibilities as the key determinants of the direction of innovation. If technological possibilities determine which industries are innovation-intensive and which are not, growth will inevitably slow down when the share of economic activity in the former group falls.

As early as Schmookler (1966), however, economists have noted that, due to the non-rivalry of ideas, there are strong incentives for innovation efforts to shift to sectors with larger market demand. Accordingly, to the extent that consumer preferences act as drivers of sectoral market size, they may also act as determinants of the sectoral intensity of innovation. In particular, changes in consumer income can result in shifts in patterns of consumption across sectors. Indeed, recent work has shown that the income elasticities of goods and services produced by different industries show robust and persistent differences (Aguiar and Bils, 2015), and that these differences explain a major part of the long-run reallocations of output and employment across sectors (Boppart, 2014; Comin et al., 2015). These observations provide us with an alternative view for the future of growth: one in which innovation will follow the patterns of structural transformation, and the pull force of demand will raise the rates of innovation and productivity growth in highly income elastic sectors such as services.

In this paper, we study how demand nonhomotheticity and technological possibilieis together shape, first, cross-sectoral variations in market size and, second, the sectoral direction of innovation. We first empirically document that the income elasticities of the outputs of US industries significantly correlate with the rates of growth

[1] Baumol (1967) famously coined the term "cost disease" to describe a chronic problem of the personal services industry that limits the potential for productivity growth in this sector. More recently, Gordon (2016) has forcefully argued that the main innovations driving the productivity growth of the manufacturing sector in the 20th century were unique to this era and will not be matched by the more recent innovations in the information and telecommunication industries (see also The Economist's special issue *Innovation pessimism: Has the ideas machine broken down?*, 2013). Clark (2016) discusses the close connection between the phenomenon of structural transformation and Gordon's projections about the future of growth.

of patenting and R&D expenditure, as proxies for the output and inputs of innovation. For this exercise, we use income elasticity estimates that Aguiar and Bils (2015) have obtained using household-level Consumption Expenditure Survey (CEX) data in the US. We further develop a multi-sector model of endogenous growth with nonhomothetic demand and persistent variations in relative income elasticities of sectoral outputs. We find that for, a general class of innovation possibilities frontiers, the income elasticities of sectoral outputs ultimately determine technological growth across industries. The asymptotic equilibrium paths of our economy give rise to the types of correlations that we empirically document in the paper.

To see the core mechanism of our theory and to better appreciate potential interactions between the pull force of demand and push force of technology, consider a static closed economy with two sectors: services (s) and manufacturing (m). Relative output of the two sectors Y_s/Y_m equals the relative demand, which in general depends both on relative prices (P_s/P_m) and income (real consumption C_{tot}). Equalization of the marginal product of productive factors imply that relative sectoral prices (negatively) depend on the relative states of technologies (N_s/N_m) in the two sector. We can show these relationships as

$$\frac{Y_s}{Y_m} = \mathfrak{D}\left(\frac{P_s}{P_m}; C_{tot}\right) \qquad \text{and} \qquad \frac{P_s}{P_m} = \mathfrak{P}\left(\frac{N_s}{N_m}\right).$$

A body of evidence shows that broad categories of goods are gross complements; therefore, the relative expenditure of households $\mathfrak{D}$ is increasing in relative prices P_s/P_m. Together, these demand-side relations suggest relative output is negatively related to relative sectoral technologies. In addition, we can capture the forces of technological possibilities through an innovation supply function

$$\frac{N_s}{N_m} = \mathfrak{T}\left(\frac{Y_s}{Y_m}\right),$$

that shows how relative technology may (positively) respond to the relative size of output and demand. Substituting for the relative prices and intersecting supply and demand, the model endogenizes equilibrium technology (N_s^*/N_m^*) as a function of income C_{tot}. Figure 1 shows how a rise in income C_{tot} affects the equilibrium bias of technology when sector s is more income elastic relative to m. So long as the market size elasticity of the supply of innovation exceeds zero, as income rises the relative state of technology improves in sector s.

Our theory shows how the core mechanism of this static model generalizes to a dynamic setting, and endogenously determines both the *rates of growth* and the levels of technology in an infinite-horizon multisector growth model.[2] We use nonhomothetic

[2] In this sense, our model can be distinguished from theories of factor-biased technical change (Ace-

CES preferences (Hanoch, 1975; Comin et al., 2015) to account for variations in the income elasticity of sectoral outputs and formulate the pull forces in demand $\mathfrak{D}$. We assume that entrepreneurs innovate to create novel sector-specific varieties, and that the masses of these varieties determine the states of sectoral technologies in a Romer-style setting. We account for the push forces in our formulation of $\mathfrak{T}$ through heterogeneous costs of innovation for different sectors and allow for intersectoral knowledge spillovers.

Along the equilibrium path of our model, two conditions fully characterize the allocation of sectoral innovation, offering simple and intuitive expressions for their dynamics. First, the entrepreneurial arbitrage condition, or simply the free entry condition, equates the value of entry across sectors. This implies that the productivity of R&D (in creating new varieties) and the value of each new variety are inversely related across sectors. Second, the R&D investment arbitrage equation equates the return to investing in the value of monopoly assets across different sectors. This implies that the sum of growth in value of monopolies and their dividend to value ratios are equalized across sectors. Together, these two conditions imply that the allocation of R&D at any point in time depends on two distinct distributions across sectors: the distribution of (production) market size, and the distribution of (knowledge) asset values. The distribution of market size captures the role of demand-side pull forces, and the distribution of asset values captures that of technology-side push forces. The allocation of R&D workers across sectors is given by a linear combination of these two distributions at any point in time.

We further characterize the asymptotic rates of technological growth for different sectors, along equilibrium paths that converge to a constant rate of growth in real income. We find that relative income elasticities asymptotically pin down both the relative shares of R&D investments and the relative rates of technological growth across sectors. Sectors that produce more income elastic goods asymptotically grow fasters in terms of innovation and productivity. We show that these asymptotic results explain the empirical regularities that we document using firm patenting and R&D expenditure, as proxies for the growth of innovation across sectors. In Appendix B, we show that our asymptotic results generalize to a broad class of potential patterns of inter-industry knowledge spillovers.[3]

Our paper contributes to a literature that investigates the determinants of the cross-industry variations in innovation activity. This literature in particular has attempted

moglu, 2002, 2007), in which sectors reach endogenously different levels of technologies, but still grow at the same rates. See below for further discussions of the distinctions between the two models.

[3]We also show that the predictions of our theory generalize to alternative formulations of the process of innovation and endogenous growth. While we present the theory in the main paper based on an expanding varieties model to simplify the exposition, Appendix C shows that all of our main results also generalize to a Schumpeterian model of innovation and growth.

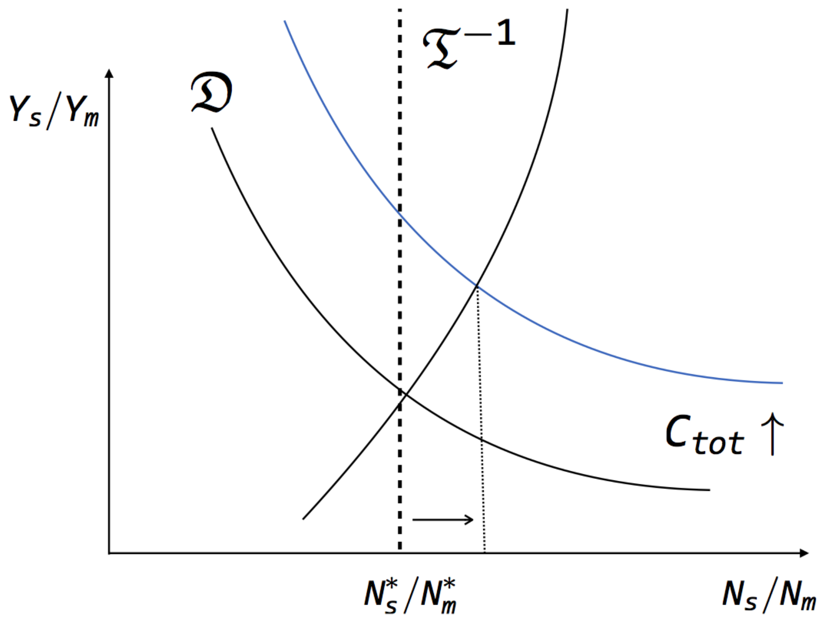

Figure 1: Determination of relative technologies in a static two-sector model of endogenous growth and nonhomothetic demand. The $\mathfrak{D}$ curve shows the relative sectoral consumer demand, and the $\mathfrak{T}^{-1}$ curve the relative supply of innovation. So long as the innovation supply function is not perfectly inelastic (the dashed line), when real income C_{tot} grows, it shifts up the relative demand for the income elastic sector s, and the equilibrium level of relative technology shifts to the right.

to determine whether the ultimate driver of innovation across industries is the incentive pull of the size of demand or the push force of technological possibilities. A number of early and groundbreaking studies documented an industry-level relationship between market size and innovation (Griliches and Schmookler, 1963; Scherer, 1982). However, these studies did not rely on a general equilibrium framework and were therefore marred with conceptual issues. For instance, the empirical exercise favored in this literature, which consisted of correlating patents (or other measures of innovation) with sectoral outputs, did not account for the potential reverse causality between innovation and market size. In particular, the size of output is shaped by sectoral prices, which in turn reflect the state of sectoral technologies. Despite some further work in the literature (Pakes and Schankerman, 1984; Cohen, 2010), we still lack a theoretical framework to study the interactions between technology and demand in shaping the sectoral direction of innovation in a general equilibrium setting. More recent theoretical work on growth has largely disregarded the empirical regularities documented by this literature. In particular, even though the endogenous growth theory (Romer, 1986, 1990; Grossman and Helpman, 1990) relies on Schmookler's concept of private, profit-driven R&D, but it has abandoned the question of sectoral bias of innovation in favor of single sector models that study aggregate outcomes.

We argue that the gap in the literature on the sectoral direction of innovation in part stems from the lack of demand systems that feature persistent heterogeneity in income elasticities. In this paper, we employ nonhomothetic CES preferences, introduced first by Hanoch (1975) and Sato (1975) and recently used by Comin et al. (2015), to address the difficulty in formulating such demand systems.[4] We focus on constant growth paths where persistent variations in income elasticities, in combination with income growth, imply large variations in market sizes across sectors.[5]

Our theory shares some of the core features of the theory of factor-biased technical change (Acemoglu, 2002, 2007), but is also distinct in important ways. First, Acemoglu (2002) focuses on the bias of innovation toward different factors of production, e.g., skilled versus non-skilled labor, in a two-factor and one-good model of growth. Instead, we focus on the direction of innovation across different industries in a single-factor and multiple-good model of growth. In Acemoglu's model, the underlying heterogeneity stems from the variations in the stocks (endowments) of factors of production.[6] In contrast, the underlying heterogeneity in our model is driven by the (plausibly exogenous) differences in the income elasticity of goods and services produced by different sectors. There is no a priori reason to believe that factor intensities across different sectors necessarily produce biases similar to those we derive here based on income elasticities.[7] Even if such correlations exist, the mechanism identified in our paper would operate distinctly from that of a theory of factor-biased technical change.

On the empirical side, very few papers have attempted to carefully identify the response of innovation to market size. The few exceptions are Acemoglu and Linn (2004), who report in a study of pharmaceutical industry a response in terms of the number of drugs developed, but not for patents (see also Cerda, 2007; Budish et al., 2015). More recently, Jaravel (2017) provides evidence that product innovations are disproportionately biased toward goods consumed by high-income households, using barcode-level scanner data from the US retail sector.

The paper is organized as follows. Section 2 discusses our empirical exercise. Section 3 presents the model, and Section 4 concludes the paper. Most proofs are contained in Appendix A.

[4]We note another recent theory that has studied sectoral composition of innovation in a two-sector model for the production of goods and services (Boppart and Weiss, 2013).

[5]Relatedly, previous work in the field of industrial organizations (see, e.g., Cohen et al., 1987) has used variations in the income elasticity of demand to account for variations in R&D intensity of firms.

[6]We note that, importantly, Acemoglu (2002) abstracts away from factor accumulation in his theory of factor-biased technical change, assuming exogenous and constant factor endowment across the economy. It is likely that accounting for factor accumulation may modify some of the predictions of the benchmark theory of factor-biased technical change.

[7]We note that a recent paper by Caron et al. (2014) in fact documents a correlation between skill intensity and income elasticity of demand for outputs across sectors, based on trade flows data.

2. Motivating Evidence

Let us begin by revisiting the cross-sectoral relationship between innovation and market size. As we mentioned in the introduction, an earlier literature has documented strong correlations between innovation activity and the size of output across sectors, but we cannot readily interpret such relations as evidence for the effect of latter on the former. Here, we instead aim to document the relationship between innovation activity and plausibly exogenous measures of the income elasticity of sectoral outputs. This allows us to examine the extent to which the variations in the dynamics of innovation outputs and inputs may be driven by the dynamics of sectoral demand differences as income grows.

As for the measures of income elasticities for different industries, we rely on the micro estimates of Aguiar and Bils (2015) based on the Consumption Expenditure Survey (CEX) for the period of 1980-2010. Aguiar and Bils (2015) provide estimates for the income elasticity of 20 different categories of consumption expenditures. Their choices of these 20 categories are driven by the availability of consistent observations in the CEX. We first map these estimates to 69 standard Personal Consumption Expenditure (PCE) product types (BEA table 2.4.5). We then use the crosswalk between PCE categories and NAICS industry codes at the 2 and 3 levels of classifications. We associate all industry codes corresponding to the same PCE commodity type with the same elasticity.

We use two different proxies for innovation activity at the industry level: total number of patents and total R&D expenditures of private firms. Our patent data includes all patents issued by the USPTO between 1976 to 2015. We employ the following strategy to assign patents to different industries. We focus attention on the set of patents that are assigned to companies included in the compustat dataset. We then associate each patent to the industry class (NAICS or SIC) of the company based on the compustat data. This procedure leaves us with a total of 778,558 patents. For the R&D expenditure, we rely on the confidential Business R&D and Innovation Survey (BRDIS) and its predecessor Survey of Industrial Research and Development (SIRD) administered by NSF, matched to the LBD data. We aggregate the R&D expenditure of the firms that belong to the same industry code.

Our measures of income elasticity vary at the industry level, but the relation between our proxies of innovation activity and the underlying innovation may also vary across industries. To deal with this problem, we focus on the *rates of growth* of our proxies as the variables of interest. To the extent that the mapping between our proxies and the underlying innovation activity is linear and invariant over time, using the rates of growth as the outcome variable allows us to compare the differences in the

	Raw Patents			Patent Weighted by Citations		
	(1)	(2)	(3)	(4)	(5)	(6)
Elasticity	0.024	.024***	.016***	.025	.025***	.016***
	(.020)	(.007)	(.007)	(.021)	(.006)	(.008)
Year FE	No	Yes	Yes	No	Yes	Yes
Broad Ind. FE	No	No	Yes	No	No	Yes
R^2	.0004	.91	.91	.0005	.90	.90

Obs. 3002, s.e.: robust, clustered at year and NAICS 1, respectively.

Table 1: Regressions of annual patent growth on the income elasticity parameters from Aguiar and Bils (2015). The three columns on the left include the total number of patents, while the three columns on the right weight patents by their number of citations.

dynamics of the underlying activity across sectors.[8] Accordingly, we run regressions of the form

$$InnovGrowth_{it} = \alpha + \beta\,\varepsilon_i + FE_t + FE_I + error_{it},$$

where $InnovGrowth_{it}$ is growth in R&D and patents in sector i at time t, ε_i is the proxy for the income elasticity of goods and services produced by industry i, FE_t is a time fixed effect, FE_I is a fixed effect for a broad industry code (NAICS 1-digit).

Table 2 shows the relation between the growth in the number of patents and the income elasticity across 3 digit NAICS codes. We find relatively robust and sizable correlations between the growth in patenting and our measures of income elasticity. An estimated value of 0.24 implies that the rate of patenting grows 1.5% faster in an industry at the 95 percentile of income elasticity ($\varepsilon_i = 1.48$) relative to one at the 5 percentile ($\varepsilon_i = 0.82$). As we can see, the estimates are fairly robust to the inclusion of fixed effects for time and broad industry classes (NAICS 1 digit). Note, however, that some of the variations across broad industries are also driven by the differences in the income elasticity of the outputs of these sectors, e.g., the difference between manufacturing and agriculture. Nevertheless, it is reassuring that we can find similar estimates when relying only on the variations of income elasticities within broader sectors (NAICS 1 digit industries).

For further robustness, the three columns on the right of the same table present the regressions weighted by the patents' the numbers of citations, to account for the differences in the quality of the underlying innovation. We find that the estimates do not show sensitivity to the choice of weighting of patents.

Table 2 shows the relation between the growth of R&D expenditure, as a proxy for

[8] In Section 3 below, we discuss how these proxies relate to the predictions of the model.

	(1)	(2)	(3)
Elasticity	0.001	.136***	.496***
	(.069)	(.06)	(.127)
Year FE	No	Yes	Yes
Broad Ind. FE	No	No	Yes
R-squared	.004	.257	.349

Number obs. is 1120. Robust standard errors in parenthesis. Weighted regression by number of obs. by industry.

Table 2: Regressions of the rate of R&D expenditure growth on the income elasticity parameters from Aguiar and Bils (2015).

innovation inputs, and the income elasticity across industries using the BRDIS and SIRD data. In this case, we need to include the fixed effects to uncover the relationship. However, the estimates appear at least an order of magnitude larger than those obtained when we use patents as the proxy for innovation outputs.

We conclude from these results that micro-level estimates of income elasticity of industry outputs, drawn independently based on consumer surveys, are correlated with the growth in different proxies of inputs and outputs of innovation across industries. In the next section, we offer a theoretical framework to understand and interpret the empirical relationships uncovered in this section.

3. Model

Consider an economy endowed by one factor of production, labor, that is inelastically supplied by a mass H of households. Households consume goods produced by a fixed set of I distinct industries, which we may interchangeably refer to as sectors. Goods in each industry are produced by perfectly competitive producers who use as their inputs a set of industry-specific intermediate goods. Innovators invent new varieties of intermediate inputs for each industry.

The innovation side of the economy has a simple Romer-style expanding varieties structure. R&D firms hire workers and direct their research and development efforts to a given industry and develop novel intermediate goods intended for production in that specific industry. Upon success in creating a new intermediate product in a given sector, the R&D firm earns perpetual monopoly rights on the invented blueprint. Equity shares in firms that produce the resulting intermediate inputs are the only investment instrument in this economy.

This simple set up allows us to focus attention on our main channel of interest, which is how the differential market size effect implied by preference nonhomotheticity determines the direction of firms' R&D efforts. In Appendix 2 we show that all of our results also generalize to a standard Schumpeterian structure for the innovation side of the model.

Notation We denote logarithms of variables with lower case letters, for instance, $p_i(t)$, $q_i(t)$, and $y_i(t)$ denote logarithms of $P_i(t)$, $Q_i(t)$, and $Y_i(t)$, respectively. We denote vectors and matrices with bold face notation, for instance $\boldsymbol{p}(t)$ is a vector with elements $[p_i(t)]_{i=1}^{I}$. The only exception to this rule is the real interest rate, which we denote by $r(t)$ to maintain the notation most familiar for the readers.

3.1. Demand and Income Elasticities

There is a mass H of identical households with preferences characterized by the standard intertemporal utility

$$\int_0^\infty e^{-\rho t} \frac{C(t)^{1-\theta} - 1}{1-\theta} dt, \tag{1}$$

where $C(t)$ aggregates a bundle of sectoral consumption goods $\boldsymbol{C}(t) \equiv \{C_i(t)\}_{i=1}^{I}$, according to the implicitly defined function

$$\sum_{i=1}^{I} \Xi_i^{\frac{1}{\sigma}} \left(\frac{C_i(t)}{C^{\epsilon_i}(t)} \right)^{\frac{\sigma-1}{\sigma}} = 1. \tag{2}$$

In Equation (2), $\epsilon_i > 0$ is a parameter specifying the income elasticity of demand for sector i, while $\Xi_i > 0$ is a sector specific taste parameter and σ is the elasticity of substitution.

The aggregator in Equation (2) belongs to the family of nonhomothetic CES preferences, introduced first by Hanoch (1975) and Sato (1975) and used recently by Comin et al. (2015) in a theory of structural change. This aggregator has the unique property that it allows for heterogeneity in income elasticities of sectoral goods for all levels of income, while maintaining the constancy of elasticity of substitution as in the standard CES preferences. Moreover, as we will see below, these preferences give rise to a log-linear demand system that closely parallels recent empirical work documenting robust variations in the income elasticity of demand across sectors (see Young, 2012, 2013; Aguiar and Bils, 2015). Different features of nonhomothetic preferences have been extensively discussed in (Comin et al., 2015).

Solving the expenditure minimization problem for the households, we find that sectoral demand is given by

$$\frac{C_i(t)}{C(t)} = \left(\frac{P_i(t)}{P(t)} \right)^{-\sigma} C(t)^{(1-\sigma)(\epsilon_i - 1)}, \tag{3}$$

where $P(t)$ indicates the aggregate price index, which is a function of real consumption $C(t)$ and sectoral prices. This function is defined by

$$P(t) = \mathcal{P}(C(t), \boldsymbol{P}(t)) \equiv \left(\sum_{i=}^{I} \Xi_i \left(P_i(t) \, C(t)^{\epsilon_i - 1} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \tag{4}$$

Equations (3) and (4) characterize the optimal allocation of consumption across goods from different sectors for aggregate real consumption $C(t)$ and vector of sectoral prices $\boldsymbol{P}(t)$. We normalize the household wage (labor income) to 1 in every period.

The next proposition characterizes the allocation of aggregate real consumption $C(t)$ over time for a given path of real interest rate $[r(\cdot)]_{t=0}^{\infty}$ and sectoral goods prices $[\boldsymbol{P}(\cdot)]_{t=0}^{\infty}$. The proof of the proposition in included in Appendix A.

Proposition 1. *(Household Intertemporal Problem) Consider the problem of a household that chooses time paths of aggregate consumption and assets $[C(\cdot), A(\cdot)]_{t=0}^{\infty}$ maximizing (1) for given paths of sectoral prices and real interest rate $[\boldsymbol{P}(\cdot), r(\cdot)]_{t=0}^{\infty}$, where the per-capita stock of assets evolves according to*

$$\dot{A}(t) \leq 1 + r(t) \, A(t) - P(t) \; C(t), \tag{5}$$

where 1 is the wage of household, taken to be the numeraire, and $P(t)$ is given by Equation (4). The household's allocation should further satisfy the No-Ponzi condition

$$\lim_{t \to \infty} A(t) \exp\left(- \int_0^t r(t') \, dt' \right) \geq 0. \tag{6}$$

Assume an interior solution exists for a household starting with initial level of assets $A(0)$. Then, the path of consumptions and assets should satisfy the following Euler equation

$$\dot{c}(t) = \frac{\dot{C}(t)}{C(t)} = \frac{r(t) - \rho - \bar{\dot{p}}(t) \left[1 + (1-\sigma) \, Cov\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}, \frac{\dot{p}_i(t)}{\bar{\dot{p}}(t)}; t \right) \right]}{\theta + \bar{\epsilon}(t) \left[1 + (1-\sigma) \, Var\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}; t \right) \right] - 1}, \tag{7}$$

as well as the transversality condition

$$\lim_{t \to \infty} e^{-\rho t} \frac{A(t)}{P(t)} C(t)^{-\theta} \, \frac{1}{\bar{\epsilon}(t)} = 0. \tag{8}$$

In the Euler equation above, $\bar{\dot{p}}(t)$ and $\bar{\epsilon}(t)$ denote the economy-wide expenditure-weighted averages of sectoral price growth $\dot{p}_i(t)$ income elasticity parameters ϵ_i and the rate of growth of prices where the expenditure shares are given by $\Omega_i(t) \equiv P_i(t) \, C_i(t) \, / \, P(t) \, C(t)$. The variance and covariance terms are also similarly defined under the sectoral distribution defined by expenditure shares $\boldsymbol{\Omega}(t)$.

Furthermore, assume that $\sigma \in (0,1]$ and $\epsilon_i > 1-\theta$ for all i. Then, the household problem has a unique solution characterized by the Euler equation and the transversality condition above.

Henceforth, we assume the conditions specified in the proposition to ensure the uniqueness of the solution are satisfied in our economy.

Proposition 1 and the static demand allocation Equation (3) jointly characterize the paths of sectoral consumption for consumers with preferences that feature long-run income nonhomotheticity. First, we observe that in the special case where $\epsilon_i \equiv 1$ for all sectors i, the preferences reduce to the standard CES and the Euler equation (7) reduces to the familiar form of $\theta \cdot \dot{c} = r - \rho - \dot{p}$.[9]

In the more general case of nonhomothetic CES where ϵ_i's vary across sectors, the Equation (7) deviates from the standard Euler equations in two important ways. First, the term $\bar{\epsilon}(t)\left[1 + (1-\sigma)\, Var\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}; t\right)\right] - 1$ in the denominator accounts for the fact that in presence of nonhomotheticity, the concavity of the instantaneous utility function and, correspondingly, the elasticity of intertemporal substitution both vary with time and real income. Second, the term $(1-\sigma)\, Cov\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}, \frac{\dot{p}_i(t)}{\bar{\dot{p}}(t)}; t\right)$ in the numerator shows that consumption grows faster to the extent that prices are falling relatively faster for more income-elastic goods, since households would be more inclined to substitute future consumption with current consumption.

The following corollary highlights a key implication of Proposition 1 that we will heavily use in our characterization of the general equilibrium.

Corollary 1. *Along an optimal consumption path for households, the growth rates of aggregate real consumption, aggregate consumption expenditure satisfy*

$$\overline{\epsilon_i}(t)\, \dot{c}(t) = \dot{e}(t) - \overline{\dot{p}_i}(t), \tag{9}$$

where $\dot{e}(t)$ denotes the growth rate of consumption expenditure $E(t) \equiv \mathcal{P}(C(t), \boldsymbol{P}(t)) \times C(t)$.

The growth in the share of a given sector i in consumption expenditure is given by:

$$\begin{aligned} \dot{\omega}_i(t) \equiv \frac{\dot{\Omega}_i(t)}{\Omega_i(t)} &= (1-\sigma)\left(\epsilon_i \dot{c}(t) + \dot{p}_i(t) - \dot{e}(t)\right), \\ &= (1-\sigma)\left[\left(\epsilon_i - \overline{\epsilon_i}(t)\right)\dot{c}(t) + \dot{p}_i(t) - \overline{\dot{p}_i}(t)\right]. \end{aligned}$$

This expression shows two distinct forces that shape the evolution of sectoral share of consumption expenditure: 1) the difference between the sector's income elasticity

[9] Note that the equality $\dot{p}(t) = \sum_i \Omega_i(t)\, \dot{p}_i(t) = \overline{\dot{p}_i}(t)$, which we used to find $\theta \cdot \dot{c} = r - \rho - \dot{p}$, only holds in the special case of homothetic preferences where the changes in the aggregate price index are *only* driven by changes in prices.

parameter and the share-weighted average of all sectors, and 2) the difference between the sector's rate of growth in prices and the share-weighted average rate among all sectors. The relation ensures that the identity $\sum_i \dot{\omega}_i(t) = 0$ is satisfied for all t.

3.2. Production

Labor is the only factor of production in the economy. Production in each sector involves two groups of producers. First, a continuum of monopolistically competitive firms produce sector-specific intermediate goods. Second, perfectly competitive final good producers at each sector combine sector-specific intermediate goods to produce output consumed by the households.

Let us assume that sector i at time t has a continuum of $N_i(t)$ varieties of intermediate goods. Second-stage final good producers in this sector produce the final sectoral output using the production function

$$Y_i(t) = \left(\int_0^{N_i} X_{iv}(t)^{\frac{\zeta}{\zeta+1}} di \right)^{\frac{\zeta+1}{\zeta}}, \tag{10}$$

where $X_{iv}(t)$ is the sector-specific intermediate input of type v. We assume $\zeta > 0$ so that the elasticity of substitution among different intermediate goods $\zeta + 1$ exceeds 1.

The producer of product v in sector i has an eternal monopoly over the market for that good. This monopolist firm produces intermediate goods at constant marginal cost Φ in units of labor, and sells it to the final producers of the sectoral good. Since the elasticity of demand faced by the monopolist for her particular variety is $1 + \zeta$, all monopolists charge the same price $P_{iv}(t) = \frac{\zeta+1}{\zeta}\Phi$. We normalize the price of intermediate goods to 1 by assuming that the value of the marginal cost satisfies $\Phi \equiv \zeta / (1 + \zeta)$.

Given that the markets for final sectoral goods are competitive, price $P_i(t)$ of final goods in sector i equals the marginal cost of production. Since all intermediate goods sell at the same unit price, sectoral good prices depend solely on the number of available intermediate good varieties $P_i(t) = N_i(t)^{-\frac{1}{\zeta}}$. Accordingly, we can write the profits $\Pi_{iv}(t)$ accrued to a monopolist operating the variety v in sector i as

$$\begin{aligned} \Pi_{iv}(t) = \frac{1}{1+\zeta} X_{iv}(t) &= \frac{1}{1+\zeta} \left(\frac{P_i(t)}{P_{iv}(t)} \right)^{1+\zeta} Y_i(t), \\ &= \frac{1}{1+\zeta} \frac{1}{N_i(t)} \left(\frac{P_i(t) Y_i(t)}{Y(t)} \right) Y(t), \end{aligned} \tag{11}$$

where we have defined $Y(t)$ as the total value of output in the economy, that is, $Y(t) \equiv \sum_i P_i(t) Y_i(t)$. Note that the expression within the parentheses on the second line corresponds to the share of final production in sector i.

Equation (11) already contains the core mechanisms driving the direction of innovation in our model. In particular, we can identify two different components that influence the size of profits reaped by a monopolist in a given sector. First, there is the *price effect* $\left(\frac{P_i}{P_{iv}}\right)^{\frac{1}{\zeta}} = \frac{1}{N_i}$ that declines as the number of varieties rises and the technology advances in a given sector. Better technologies reduce the prices charged on a given item and reduce the potential profits for a given level of output. The second term $P_i(t) Y_i(t) / Y(t)$ is the share of industry i in the total output of the economy. With a given technology at the sectoral level, if the share of that sector in the economy rises, the profits of monopolists in that sector follows suit. This is the *market size* effect. As we will see below, in our model the income elasticities together with sectoral prices determine the market size for a given level of total output.

3.3. Innovation and Technology-Push

R&D firms hire workers to pursue research and develop new varieties. Since varieties are sector-specific, they have to decide the sector to which they direct their activities. If R&D firms hire Z_i workers to invent intermediate goods in sector i, new varieties arrive a flow rate

$$\dot{N}_i(t) = S_i(\boldsymbol{N}(t)) Z_i(t), \tag{12}$$

where S_i is the efficiency of R&D workers in sector i. The efficiency $S_i(\cdot)$ of R&D workers depends on the state of technology in all sectors $\boldsymbol{N}(t)$ and specifies an *inter-sectoral innovation spillover function*. The idea is that innovators in sector i benefit from the available stock of knowledge in other sectors as well as that in their own sector, and therefore costs of inventing new varieties should vary as the number of varieties change over time. The function S_i characterizes the technology-push side of our model. Formally, we assume these functions satisfy the conditions stated in Assumption 1 below.

Assumption 1. *Let $\boldsymbol{S} : \mathbb{R}_+^I \to \mathbb{R}_+^I$ be a 2nd order differentiable and positive valued function such that $\partial S_i / \partial N_j \geq 0$ for all pairs i and j. We assume that each element S_i is homogenous of degree 1 in its arguments for all i. Moreover, we assume that the following limit always exists and is bounded away from zero*

$$\lim_{N_i \to \infty} \frac{S_i(\boldsymbol{N})}{N_i} > 0, \tag{13}$$

for all sectors i and for all vectors of technological states $\boldsymbol{N} \equiv (N_i, \boldsymbol{N}_{-i})$.

The first part of the assumption merely imposes the intuitive constraint that the rise in knowledge stock of a sector j *cannot* reduce the productivity of innovators in any other sector i. The homogeneity of degree 1 is required in order to allow for sustained growth in our economy, in which the scarce factor of labor is used in the innovation sector. The intuition behind the constraint (13) is also rather straightforward. From Equation (12), we can see that the fraction $\frac{S_i}{N_i}$ gives the technological growth generated in sector i per unit of R&D labor. The condition (13) then states that we can sustain growth in any given sector i by investing in R&D within that sector, regardless of the state of technology in all other sectors. This condition, in particular, rules out innovation spillover functions $S_i(\cdot)$ functions that are asymptotically Cobb-Douglas.[10]

We generally attempt to provide characterizations of the behavior of our economy for general intersectoral innovation spillover functions $\boldsymbol{S}$ satisfying Assumption 1. However, in order to provide clear predictions from the model, we need to make more specific functional form assumptions. Below, we present one particular functional form, which we later employ for characterizing the equilibrium paths of our model.

Definition 1. Consider the following choice for the innovation spillover functions

$$S_i(\boldsymbol{N}(t)) = \frac{1}{\eta_i} N_i(t)^{\delta} S(\boldsymbol{N}(t))^{1-\delta}, \tag{15}$$

where $\delta \in [0,1]$, and $S(\boldsymbol{N}(t))$ is homogenous of degree 1 and monotonically increasing in its arguments. Note that in the special case of $\delta = 1$, we reach the case with no cross-industry innovation spillovers. Function S provides an index of the general-purpose stock of knowledge in the economy that is the main source of innovation spillovers to all other sectors. The spillover functions defined by Equations (15) allows for potential asymmetry across sectors in the way they *contribute* to the economy-wide source of spillovers. However, it assumes perfect symmetry across sectors in the way they *receive* spillovers from the economy-wide source of ideas S. To ensure condition (13), function S has to satisfy

$$\lim_{\boldsymbol{N}_{-i} \to \boldsymbol{o}} S(\boldsymbol{N}) \bigg|_{N_i = 1} > 0,$$

where $\boldsymbol{N}_{-i} \in \mathbb{R}^{I-1}$ is the vector of all sectoral technologies with sector i removed. An

[10]To see why, note that the asymptotic R&D employment in any given sector i has to satisfy

$$\lim_{t\to\infty} Z_i(t) = \lim_{t\to\infty} \frac{\dot{n}_i(t)}{S_i(t)/N_i(t)}. \tag{14}$$

If condition (13) is violated for a given sector, then sustaining asymptotically constant technological growth $\dot{n}_i(t)$ requires unbounded growth in R&D employment for that sector.

example of such a function is given by

$$S(\boldsymbol{N}) \equiv \left(\sum_{i=1}^{I} \vartheta_i^{1-\varsigma} N_i^{\varsigma} \right)^{\frac{1}{\varsigma}}, \qquad \varsigma > 0, \tag{16}$$

where ς a parameter specifying the degree of substitutability between different types of sectoral knowledge in contributing to the general-purpose stock of knowledge.

3.4. Market Equilibrium

We first review the conditions that characterize equilibrium allocations of the economy and, along the way, identify the set of variables that together characterize an equilibrium allocation in our economy.

Market Clearing Since we have assumed a closed economy, the output in each sector has to equal the total sectoral demand, $Y_i(t) = H C_i(t)$. Thus, the share of a given industry in total output value equals the household expenditure shares on products from that sector, i.e., $\frac{P_i Y_i}{Y} = \Omega_i$, which is in turn given by the demand equation (3). Substituting the share of each industry in total value in the expressions for the monopoly profits (11), we find the following relation for the relative profits of the monopolists in two different sectors i and j

$$\begin{aligned} \frac{\Pi_i}{\Pi_j} &= \underbrace{\frac{N_j}{N_i}}_{\text{price effect}} \times \underbrace{\left(\frac{N_j}{N_i}\right)^{\frac{1-\sigma}{\zeta}} C^{(1-\sigma)(\epsilon_i - \epsilon_j)}}_{\text{total market size effect}}, \\ &= \underbrace{\left(\frac{N_j}{N_i}\right)^{1+\frac{1-\sigma}{\zeta}}}_{\text{technology effect}} \times \underbrace{C^{(1-\sigma)(\epsilon_i - \epsilon_j)}}_{\text{income effect}}. \end{aligned} \tag{17}$$

The second equality above shows the decomposition of relative profits into the *total* effect of technology (directly through the price effect and indirectly through the market size effect) as well as the income effect. In particular, we see that a rise in the number of varieties of intermediates in sector i relative to sector j results in a greater than proportional fall in the relative profits of monopolists in that sector. As a result, the technology effect acts as a force bringing the technology levels in different sectors toward convergence: better technology in a given sector erodes the profits of producers in that sector, thereby reducing the incentives to invest in further innovation. The second term in Equation (17) distinguishes our model from previous models of directed technical change, by highlighting the force of income elasticities in shaping innovation incentives. If the demand for output of sector i is more income-elastic compared to sector j, the demand for the output of this sector grows relative to sector j as the households' aggregate consumption grows.

Clearing the goods markets further implies that the total consumption expenditure of households should equate the total value of output, that is,

$$H \cdot E(t) = Y(t) = \sum_i P_i(t) Y_i(t), = \left(1 + \frac{1}{\zeta}\right) L(t), \quad (18)$$

where E stands for the expenditure of a household. In the second equality we have used the fact that the final goods markets are competitive, revenues are equal to the costs, which in turn equate the total revenues of intermediate goods monopolists, and the wage is normalized to unity. The total revenues of all monopolists are $(1+\zeta)/\zeta$ times their costs, which is equal to total wages paid, $L(t) \equiv \sum_i L_i(t)$.

Finally, labor markets clear when

$$\sum_{i=1}^{I} (L_i(t) + Z_i(t)) = L(t) + Z(t) = H, \quad (19)$$

where we have defined total employment of R&D firms as $Z(t) = \sum_i Z_i(t)$. The economy's sole resource, total labor H, can be employed in R&D, in the form of $Z(t)$, or in production, in the form of $L(t)$. We assume that households maintain a balanced portfolio of equity shares in all intermediate goods producers at all times. Therefore, R&D employment is an effective instrument of investment. Equation (18) shows that the employment share of production $L(t)/H$ changes linearly with the per-capita nominal expenditure of households. Therefore, in the aggregate, we can think of the decomposition of H into Z and L to be reflective of the household's allocation of available resources between investment and consumption.

Free Entry Condition The value of owning the monopoly rights on different intermediate products is the same within a given sector i, given by the net present discounted value of all future profits

$$V_i(t) \equiv \int_t^{\infty} e^{-\int_t^s r(t')dt'} \Pi_i(t+t')\, dt'. \quad (20)$$

We can write the value $V_i(t)$ of owning the monopoly rights over an intermediate good in sector i as the solution to the Bellman equation

$$r(t) V_i(t) - \dot{V}_i(t) = \Pi_i(t). \tag{21}$$

Along any equilibrium path, the marginal product of labor in R&D firms in any sector i cannot exceed the unit costs of labor (wages)

$$1 \geq S_i(t) V_i(t), \tag{22}$$

where the expression is satisfied with equality for any sector i in which firms actively engage in R&D, that is, $Z_i(t) > 0$.

Equilibrium We define an *allocation* as a collection of the time paths of aggregate and sector consumptions of households $[C(t), \boldsymbol{C}(t)]_{t=0}^{\infty}$, employment in production and R&D in each sector $[\boldsymbol{L}(t), \boldsymbol{Z}(t)]_{t=0}^{\infty}$, masses of varieties in each sector $[\boldsymbol{N}(t)]_{t=0}^{\infty}$, and the price, quantity, and net present value of monopoly rights for each intermediate good in each sector $\left\{[P_{iv}(t), X_{iv}(t), V_{iv}(t)]_{v\in[0,N_i(t)],t=0}^{\infty}\right\}_{i=1}^{I}$.

An *equilibrium* is an allocation that corresponds to the combination of constraints imposed by household utility maximization, monopolist profit maximization, and the free entry condition everywhere along the time paths. The sectoral and aggregate consumption of households should satisfy the sectoral demand Equation (3) and the Euler Equations (7) and (8), where household assets satisfy $A_i(t) = \frac{1}{H}\sum_i \left(\int_0^{N_i(t)} V_{iv} dv\right)$ and aggregate and sectoral prices indices are given by Equations (4) and $P_i(t) = \left(\int_0^{N_i(t)} P_{iv}^{-\zeta} dv\right)^{-1/\zeta}$ for all i. Employment allocations satisfy $Z_i(t) = \dot{N}_i(t) / S_i(\boldsymbol{N}(t))$ and $L_i(t) = \frac{1}{\Phi}\int_0^{N_i(t)} X_{iv}(t) dv$ for all i, as well as the labor market clearing condition (19). Prices and quantities of intermediate goods satisfy $P_{iv}(t) = \left(1 + \frac{1}{\zeta}\right)\Phi$ and $X_{iv}(t) = \frac{1}{\zeta}\frac{P_i(t)C_i(t)}{N_i(t)}$. Finally, stocks of varieties and firm values satisfy the free entry condition (22).

3.5. Equilibrium Dynamics

In this section, we characterize the dynamics of the allocations along an equilibrium path, in terms of the vector of technological state variables $\boldsymbol{N}(t)$ and the aggregate consumption (control variable) $C(t)$. Throughout this section, we will assume an *interior* equilibrium path, along which R&D and innovation is carried on in all sectors, that is, $Z_i > 0$ for all i.

We begin by defining two sectoral distributions for the allocations of output and assets $\{\boldsymbol{\Omega}(t), \boldsymbol{\Lambda}(t)\}$. First, note that from the assets market clearing condition, the total value of assets in the economy is given by

$$H \cdot A(t) = \sum_{i=1}^{I} N_i(t) V_i(t) = \sum_{i=1}^{I} \frac{N_i(t)}{S_i(\boldsymbol{N}(t))}, \tag{23}$$

where in the second equality we have used the free entry condition (22). Now, define the share of equity values of all corporate assets held in sector i as

$$\begin{aligned} \Lambda_i(t) &\equiv \frac{N_i(t) V_i(t)}{\sum_{i'} N_{i'}(t) V_{i'}(t)}, \\ &= \frac{1}{H \cdot A(t)} \frac{N_i(t)}{S_i(\boldsymbol{N}(t))}, \end{aligned} \tag{24}$$

Equations (23) and (24) show how the free entry condition directly links the value of assets held in a sector to the vector of technological states across different sectors. Therefore, both the total value of assets and the sectoral distribution of assets are pinned down at time t by the vector of state of sectoral technology $\boldsymbol{N}(t)$.[11] The economic logic is the usual force of free entry: higher innovation spillovers to a sector reduces the cost of innovation in the sector, which in turn lowers the value of owning intellectual property rights on an innovation in that sector. Note that Assumption 1 ensures that this value always remains finite along any allocation path.

The second sectoral allocations are the shares of sectors in output, consumption expenditure, and production employment already defined by Equations (3) and (4) as

$$\begin{aligned} \Omega_i(t) = \frac{E_i(t)}{E(t)} &= \frac{L_i(t)}{L(t)}, \\ &= \Xi_i \left(\frac{N_i(t)^{-\frac{1}{\zeta}} C(t)^{\epsilon_i}}{E(t)} \right)^{1-\sigma}, \end{aligned} \tag{25}$$

where total consumption expenditure, or the total production employment from Equation (18), is given by

$$E(t) = \frac{\zeta + 1}{\zeta} \frac{L(t)}{H} = \left(\sum_{i=1}^{I} \Xi_i \left(N_i(t)^{-\frac{1}{\zeta}} C(t)^{\epsilon_i} \right)^{1-\sigma} \right)^{\frac{1}{1-\sigma}}. \tag{26}$$

Both sets of allocations $\{\boldsymbol{\Omega}(t), \boldsymbol{\Lambda}(t)\}$ defined above are functions only of aggregate consumption and the vector of sectoral technologies $(C(t), \boldsymbol{N}(t))$. Moreover, we can also write the aggregate allocation of employment in the R&D sector as a function of

[11] As we show in Appendix 2 this relation also holds in a Schumpeterian version of the model.

the control and state variables $(C(t), \boldsymbol{N}(t))$, as follows

$$Z(t) = H - L(t) = H \cdot \left[1 - \frac{\zeta}{1+\zeta}\left(\sum_{i=1}^{I} \Xi_i \left(N_i(t)^{-\frac{1}{\zeta}} C(t)^{\epsilon_i}\right)^{1-\sigma}\right)^{\frac{1}{1-\sigma}}\right]. \tag{27}$$

Now, assuming $Z_i > 0$, let us combine the free entry condition (22) and the Bellman equation (21) to find

$$\begin{aligned} r(t) &= -\dot{s}_i(t) + \frac{1}{1+\zeta}\frac{P_i(t)\, Y_i(t)}{N_i(t)\, V_i(t)}, \\ &= -\dot{s}_i(t) + \frac{1}{\zeta A(t)}\frac{L(t)}{H}\frac{\Omega_i(t)}{\Lambda_i(t)}, \end{aligned} \tag{28}$$

where in the first equality, we have substituted for $\Pi_i(t)$ from Equation (11), and in the second equality we have used Equations (18) and (24). Equation (28) is an investment arbitrage condition: it states that the rate of return to investment in innovation across all sectors have to be equated with the interest rate $r(t)$. This equation is one of the two key conditions, along with the free entry condition (22), that together characterize the sectoral allocation of innovation in our model.

To make further progress, we need to assume a specific form for the spillover functions S_i to link the rate of growth of R&D productivity $\dot{s}_i$ to the rates of sectoral productivity growth. We assume the definition 1 and Equation (15) for the spillover function. With these definitions

$$\begin{aligned} A(t) &= \frac{1}{H}\sum_i \eta_i \left(\frac{N_i(t)}{S(t)}\right)^{1-\delta}, \\ \Lambda_i(t) &= \frac{\eta_i N_i(t)^{1-\delta}}{\sum_{i'} \eta_{i'} N_{i'}(t)^{1-\delta}}, \end{aligned} \tag{29}$$

where $S(t)$ is defined according to Equation (16). The following lemma characterizes sectoral R&D investment, rates of productivity growth, and the interest rate along the equilibrium path.

Lemma 1. *Anywhere along the equilibrium path where the interest rate is positive $r(t) > 0$ and there is R&D investment in all sectors, that is, $Z_i(t) > 0$ for all i, the allocations of R&D employment and the rates of sectoral growth are given by*

$$Z_i(t) = Z(t) \cdot \Lambda_i(t) + \frac{1}{\delta\zeta} L(t) \cdot [\Omega_i(t) - \Lambda_i(t)], \tag{30}$$

$$\dot{n}_i(t) = \frac{1}{A(t)}\left(\frac{Z(t)}{H} + \frac{1}{\delta\zeta}\frac{L(t)}{H} \cdot \left[\frac{\Omega_i(t)}{\Lambda_i(t)} - 1\right]\right). \tag{31}$$

and the interest rate is given by

$$r(t) = \frac{1}{A(t)} \left\{ \frac{1}{\zeta\delta} \left[1 - (1-\delta) \overline{\overline{\left(\frac{\Omega_i}{\Lambda_i} \right)}} \right] \frac{L(t)}{H} - \frac{1}{A(t)} \frac{Z(t)}{H} \right\}, \tag{32}$$

where $\overline{\overline{\left(\frac{\Omega_i}{\Lambda_i} \right)}}$ *is defined as the average of the relative output-to-assets sectoral shares, weighted by the shares of sectoral knowledge in the general-purpose economy-wide stock of knowledge*

$$\overline{\overline{\left(\frac{\Omega_i}{\Lambda_i} \right)}} \equiv \frac{\sum_i \vartheta_i^{1-\varsigma} N_i^{\varsigma} (\Omega_i / \Lambda_i)}{\sum_i \vartheta_i^{1-\varsigma} N_i^{\varsigma}}.$$

Lemma 1 provides an intuitive characterization for the allocation of innovation across sectors. Equation (30) decomposes the allocation R&D employment (inputs) into two components: a first component that allocates total R&D employment proportionally to the shares of sectors in total knowledge assets, and a second component that pulls R&D toward sectors whose market shares exceed their shares in assets. The pull-force of market size sharply manifests itself in the latter component, which is a direct consequence of the investment arbitrage condition (28). Since the returns to investment in sectoral innovation has to be equalized, and since the flow of monopoly profits depend directly on the market shares of sectors, innovation investments responds one-to-one to the changes in market size.

Equation (31) characterizes the implications for the rate of innovation and technical growth across sectors. To the extent that investments are allocated proportionally to the sectoral shares in knowledge assets, the implied rates of technical growth are the same across sectors (first term). However, the rates of technical growth across sectors vary to the extent that sectoral market shares deviate from their shares in knowledge assets. This implies potentially complex dynamics for the rates of innovation investments across sectors, depending on the evolution of the relative sectoral shares in output and assets. In addition, the proof of the lemma in Appendix A shows that

$$\dot{n}_i(t) \propto \left[\frac{1}{1+\zeta} \frac{P_i(t) Y_i(t)}{Z_i(t)} - \delta \right]^{-1},$$

suggesting a one-to-one relationship between sectoral R&D intensity ($\frac{Z_i}{P_i Y_i}$) and the relative rates of technical growth across sectors.

We can now put the different results of this section together to characterize the aggregate dynamics of the economy. If the economy starts at a technological state $\boldsymbol{N}(0) = (N_1(0), \cdots, N_I(0))$. Assuming that the equilibrium allocation remains interior everywhere along the path, we can write the evolution of the entire economy as the following dynamical system:

$$\begin{aligned}\frac{\dot{C}(t)}{C} &= \mathcal{F}(C(t), \boldsymbol{N}(t)),\\ \frac{\dot{N}_i(t)}{N_i(t)} &= \mathcal{G}_i(C(t), \boldsymbol{N}(t)), \qquad \text{for } 1 \le i \le I,\end{aligned}$$

where $\mathcal{F}$ is given by the Euler equation (7) and the expression for the interest rate Equation (32) and $P_i(t) = N_i(t)^{-1/\zeta}$, and $\mathcal{G}_i$ is given by Equations (31), in which the expressions for the aggregate employment in R&D and production $Z(t)$ and $L(t)$ are given by Equation (27).

3.6. Constant Growth Path

In this section, we focus attention to a class of equilibrium allocations that involve asymptotically constant rates of growth of consumption and sectoral technologies. Such equilibria closely parallel the balanced growth paths commonly studied in single sector growth models.

Definition. Constant Growth Path (CGP): An equilibrium path is CGP if along the allocation path aggregate consumption $C(t)$ and sectoral technologies asymptotically grow at constant rates. That is, if there exist constant nonnegative values $(g^*, \gamma_1, \cdots, \gamma_I)$ such that the following limits exist

$$\begin{aligned}\lim_{t\to\infty} \dot{c}(t) &= g^*,\\ \lim_{t\to\infty} \dot{n}_i(t) &= \gamma_i g^*, \qquad \text{for } 1 \le i \le I.\end{aligned}$$

Correspondingly, let us define the asymptotic levels of real per-capita consumption and states of sectoral technologies as

$$\begin{aligned}C^* &\equiv \lim_{t\to\infty} C(t)\, e^{-g^* t},\\ N_i^* &\equiv \lim_{t\to\infty} N_i(t)\, e^{-\gamma_i g^* t}.\end{aligned}$$

We will now study how the demand side and the innovation technology side, each impose a distinct set of constraints on the allocation of employment and expenditure, and on the rates of technical growth across sectors along any CGP. We discuss these constraints on the rates of growth through two lemmas. Lemma 2 presents the constraints imposed by the demand side and characterizes the set of industries with nonnegligible asymptotic shares in production markets. Lemma 3 presents the constraints imposed by the innovation technology and characterizes the set of industries with nonnegligible asymptotic shares in asset markets. Each lemma is followed by a corollary that discusses the constraints on the asymptotic *levels* in the allocations of sectoral technical states and per-capita consumption. Finally, Corollary 4 combines the two lemmas and shows that the two sets in fact have to coincide.

Lemma 2. *(Demand Side Constraints on CGPs) Along any CGP, the distribution of of sectoral consumption expenditure, employment, and output in our economy converges to a stationary* $\{\Omega_i^*\}_i$*, and total production employment converges to a constant, that is*

$$\lim_{t\to\infty} \Omega_i(t) = \Omega_i^*, \tag{33}$$

$$\lim_{t\to\infty} L(t) = L^* > 0. \tag{34}$$

Moreover, the asymptotic growth rates of technologies in different sectors (normalized by the growth rate of real consumption) satisfy

$$\zeta = \min_i \left\{ \frac{\gamma_i}{\epsilon_i} \right\} = \frac{\overline{\gamma}^*}{\overline{\epsilon}^*}, \tag{35}$$

where $\overline{\gamma}^*$ *and* $\overline{\epsilon}^*$ *denote average (normalized) rates of technical growth and income elasticity parameters of different sectors under distribution* $\{\Omega_i^*\}_i$*. Let* $\mathcal{I}^*$ *denote the set of industries that achieve the minimum in Equation (35). The production shares of different sectors asymptotically fall at the rate*

$$\lim_{t\to\infty} \dot{\omega}_i(t) = (1-\sigma)\left(\epsilon_i - \frac{\gamma_i}{\zeta}\right) g^* \leq 0, \tag{36}$$

with the inequality being strict for $i \neq \mathcal{I}^*$.

Proof. See Appendix A. □

Lemma 2 characterizes the set $\mathcal{I}^*$ of industries that asymptotically constitute a non-negligible share of consumption expenditure and production employment, for a given γ of relative rates of sectoral technical growth. The set $\mathcal{I}^*$ is comprised of sectors with the lowest ratio of technical growth to income elasticity parameter, as in Equation (35). Asymptotically, the shares of all other sectors in output converges to zero.[12] As with the model of Ngai and Pissarides (2007), when $\epsilon_i = 1$ for all sectors i and the preferences are homothetic, $\mathcal{I}^*$ includes only the sectors with the slowest rate of technical growth. However, in our setting the combination of supply and demand (income elasticity) forces together determine the asymptotic sectoral composition of the economy.

Another key implication of Lemma 2 is that, asymptotically, a constant and finite share $\frac{L^*}{H}$ of labor is employed in production and the remainder of labor is employed in the R&D sector. From Equation (34) and market clearing conditions (19) and (18), total

[12]This result also holds along an equilibrium path for the economy described in the model of (Comin et al., 2015), in which the rates of sectoral technical growth are exogenous.

consumption expenditure $E(t)$ and total R&D employment $Z(t)$ converge to constant values E^* and Z^* asymptotically. The Corollary below summarizes the constraints on the asymptotic levels implied by Lemma 2.

Corollary 2. *The asymptotic shares of consumption expenditure, employment, and output in sector i is given by*

$$\Omega_i^* = \begin{cases} \left(\frac{(C^*)^{\epsilon_i}}{E^*\left(N_i^*\right)^{1/\zeta}}\right)^{1-\sigma}, & i \in \mathcal{I}^*, \\ 0, & i \notin \mathcal{I}^*, \end{cases} \tag{37}$$

where the total consumption expenditure of households and the total production employment are asymptotically given by

$$E^* = \left(1+\frac{1}{\zeta}\right)\frac{L^*}{H} = \left[\sum_{i\in\mathcal{I}^*}\Xi_i\left(\frac{(C^*)^{\epsilon_i}}{\left(N_i^*\right)^{1/\zeta}}\right)^{1-\sigma}\right]^{\frac{1}{1-\sigma}} \in \left(0, 1+\frac{1}{\zeta}\right), \tag{38}$$

as functions of $(C^*, \boldsymbol{N}^*)$.

While Lemma 2 characterizes the sectoral composition of the production side of employment based on the demand-side forces, Lemma 3 below characterizes the sectoral composition of the R&D side based on the technological push forces. But before introducing the constraints imposed by the technological innovation side, let us introduce a further assumption regarding the nature of inter-industrial innovation spillovers.

Assumption 2. *Consider a time path $[\boldsymbol{N}(\cdot)]_{t=0}^{\infty}$ for the sectoral stocks of technological knowledge such that $N_i(t) \to N_i^* e^{\gamma_i g^* t}$ for all i. We assume that the technological spillover functions $\boldsymbol{S}$ has the property that along any such path, the following limits exist*

$$\Sigma_{ij}^*(g^*, \gamma) \equiv \lim_{t\to\infty}\left.\frac{\partial s_i}{\partial n_j}\right|_{\boldsymbol{n}=\boldsymbol{n}(t)}, \qquad \textit{for all } i,j \in \{1,\cdots,I\}.$$

It is straightforward to check that the class of intersectoral innovation spillover functions introduced in Example 1 satisfy the condition stated in Assumption 2. This assumption implies that we can define the asymptotic rate of growth of innovation spillovers to sector i at $\gamma_i^S g^*$. Along a CGP, we have that

$$\gamma_i^S \equiv \lim_{t\to\infty}\frac{\dot{s}_i(t)}{g^*} = \sum_j \Sigma_{ij}^*\gamma_j.$$

Lemma 3. *(Innovation Side Constraints on CGPs) Along any CGP, our economy converges to a stationary distribution $\{\Lambda_i^*\}_i$ of sectoral shares of corporate assets. First, define the asymp-*

totic rate of growth of innovation spillovers to sector i at $\gamma_i^S g^*$. Along a CGP, we have that

$$\gamma_i^S \geq \gamma_i, \tag{39}$$

for all i. Let $\mathcal{I}^{\dagger}$ the set of sectors that satisfy the expression (39) with equality. The shares of sector i in R&D employment and total corporate assets fall at a rate

$$\lim_{t\to\infty} \dot{z}_i(t) = \lim_{t\to\infty} \dot{\lambda}_i(t) = \left(\gamma_i - \gamma_i^S\right) g^* \leq 0,$$

with the inequality being strict for $i \neq \mathcal{I}^{\dagger}$ (note that $\lambda \equiv \log \Lambda$).

Furthermore, the vector of asymptotic rates of technological growth for different sectors has the following relationship with the income elasticities of sectoral products

$$\gamma_i^S - \gamma_i = (1-\sigma)\left(\frac{\gamma_i}{\zeta} - \epsilon_i\right). \tag{40}$$

Proof. See Appendix A. □

Lemma 3 establishes conditions for a sector to asymptotically constitute a nonnegligible share of R&D employment and corporate assets in the economy. This result is the R&D parallel to Lemma 2, which provided the same analysis for production employment. The key result is fairly intuitive: the R&D resources required to sustain a constant rate of technical growth in a sector crucially depends on the rate of growth of productivity of R&D workers and, in turn, the spillovers to that sector. If spillovers grow faster than the state of technology in a given sector, asymptotical R&D investment in that sector falls over time. Because of the free entry condition, the value of corporate assets in that sector inherit this shrinkage and the share of the sector in total value of firms also falls over time.

The lemma further connects the relative asymptotic rates of technical growth, R&D productivity growth, and income elasticity across different sectors in Equation (40). This result summarizes one of the key insights of the model: it shows how sectoral rates of technical growth will be determined by an interaction of the technology push forces, as captured by γ_i^S, and the demand-pull ones, as captured by demand elasticity parameters ϵ_i. Below, we will see how this condition helps characterize the asymptotic rates of productivity growth across sectors in the context of the specifications of intersectoral innovation spillover functions defined in Example 2.

In parallel to Corollary 2, which characterized the asymptotic composition of production employment and consumption in the economy, Corollary 3 below characterizes the asymptotic composition of R&D employment and corporate assets.

Corollary 3. *The asymptotic shares of different sectors in corporate assets across are given by*

$$\Lambda_i^* \equiv \lim_{t\to\infty} \Lambda_i(t) = \frac{1}{\gamma_i g^* A^*}\frac{Z_i^*}{H} = \begin{cases} 0, & i \notin \mathcal{I}^\dagger, \\ \frac{N_i^*}{S_i^* \cdot H \cdot A^*}, & i \in \mathcal{I}^\dagger, \end{cases}$$

where $S_i^* \equiv \lim_{t\to\infty} S_i(t)\, e^{-\gamma_i g^* t}$ *and we further have*

$$Z^* = H - L^* = g^* A^* H \left(\sum_{i\in\mathcal{I}^\dagger} \Lambda_i^* \gamma_i \right). \tag{41}$$

In light of the results of Lemmas 2 and 3, we can now combine the two sets of constraints imposed by the demand side and the innovation sides of the model for any CGP. Corollary 4 below states the key implication for the rates of technological growth across sectors.

Corollary 4. *Along any CGP, the set of sectors with asymptotically nonnegligible shares of production and R&D employment are identical, that is,* $\mathcal{I}^* = \mathcal{I}^\dagger$*. For all such sectors* $i \in \mathcal{I}^*$*, we have*

$$\gamma_i = \gamma_i^S = \zeta \epsilon_i.$$

Proof. Consider a sector i in $\mathcal{I}^\dagger$. From (39), we have that $\gamma_i^S = \gamma_i$ and then from Equation (40) we find $\gamma_i = \zeta\epsilon_i$, which from (36) implies that $i \in \mathcal{I}^*$. The reverse argument is identical. □

Corollary 4 is a powerful result that holds for any intersectoral innovation spillover functions that are compatible with a constant growth path. It states that the asymptotic rate of technological growth in sectors that constitute significant shares of economic activity, in production and R&D, is increasing in the income elasticity of the goods produced by those sectors. In other words, *technological growth is faster in sectors with higher income elasticities.* This result rests on two main forces: the free entry condition and nonhomotheticity. To gain a better intuition about this result, let us combine the free entry condition with the R&D investment arbitrage condition to find the following condition for any technologically growing sector:

$$1 = \underbrace{S_i(t)}_{\text{Spillovers}} \times \underbrace{\frac{1}{N_i(t)}}_{\text{Price Effect}} \times \underbrace{\Omega_i(t)}_{\text{Market Size Effect}} \times \frac{E^*}{r_i^*},$$

where r_i^* is a constant denoting the sector-specific discount rate. Furthermore, we know that the market size effect implies

$$\Omega_i(t) \propto N_i(t)^{-\frac{1-\sigma}{\zeta}} \times C(t)^{(1-\sigma)\epsilon_i}.$$

The argument behind Corollary 4 is now transparent. Sectors that require nonnegligible R&D employment are those for which S_i and N_i grow at the same rate, which then implies that the market size should be asymptotically constant. In order for the market size to be asymptotically constant we need $\gamma_i = \zeta\epsilon_i$.

If we are further willing to make functional form assumptions about the intersectoral innovation spillover functions, we can further characterize which specific sectors belong to $\mathcal{I}^*$, what happens to other sectors, and what the overall rate of consumption growth in this economy is.

Proposition 2. *The asymptotic rate of technological growth for each sector i along any SGP is given by*

$$\gamma_i = \zeta\left(\varrho\epsilon_i + (1-\varrho)\,\epsilon_{max}\right), \qquad \varrho \equiv \frac{1-\sigma}{\zeta(1-\delta)+1-\sigma} \in (0,1). \tag{42}$$

Assume that there is a unique $i^* \in \mathcal{I}$, such that $\epsilon_{i^*} = \epsilon_{max} > \epsilon_i$ for all $i \neq i^* \in \mathcal{I}$. We have that $\mathcal{I}^* = \mathcal{I}^\dagger = \{i^*\}$, and the total value of assets asymptotically converges to

$$H \cdot A^* = \eta_{i^*}\vartheta_{i^*}^{(1-\delta)\left(1-\frac{1}{\zeta}\right)}.$$

If the discount rate ρ satisfies

$$\frac{1}{\zeta A^*}\left(1 - \frac{\theta - 1 + \zeta\,\epsilon_{max}}{1+(\zeta+1)\,\epsilon_{max}}\right) < \rho < \frac{1}{\zeta A^*}, \tag{43}$$

the CGP exists and is unique. In this CGP, the asymptotic real interest rate r^* is given by

$$r^* = \frac{1}{\zeta A^*} - \frac{(1+\zeta)\,\epsilon_{max}}{\theta - 1 + \zeta\,\epsilon_{max}}\left(\frac{1}{\zeta A^*} - \rho\right).$$

The asymptotic rate of growth in aggregate consumption is given by

$$g^* = \frac{1/\zeta A^* - \rho}{\theta - 1 + \zeta\,\epsilon_{max}},$$

and the asymptotic share of employment in production by

$$\frac{L^*}{H} = \frac{\rho\zeta A^*\epsilon_{max} + \theta + (1+\zeta)\,\epsilon_{max} - 1}{\theta - 1 + \zeta\,\epsilon_{max}}.$$

Proof. See Appendix A. □

The proposition characterizes the long-run rates of technical growth across different sectors. Equation (42) suggests that, asymptotically, relative rates of technical growth are linear in the income elasticity parameters. This result allows us to explain the empirical patterns uncovered in Section 2. Let us assume that our proxy for innovation outputs, i.e., the number of patents, is a linear function of the number of new varieties, that is, $Patents_i(t) = k_{1,i} \cdot \dot{N}_i(t)$, and that the R&D expenditure is a linear function of the wages paid to R&D workers in the model, that is, $RDX_i(t) = k_{2,i} \cdot Z_i(t)$, for two sector-level constants. Asymptotically, along a CGP we have that $\dot{n}_i(t) = \dot{N}_i(t) / N_i(t) \to \gamma_i g^*$, which implies

$$PatentGrowth_i(t) = \frac{\ddot{N}_i(t)}{\dot{N}_i(t)} \to \gamma_i g^* = \zeta g^* \left(\varrho \epsilon_i + (1-\varrho) \epsilon_{max} \right).$$

Moreover, from Equation (14), we find

$$\begin{aligned} RDXGrowth(t) = \frac{\dot{Z}_i(t)}{Z_i(t)} &= \frac{\dot{N}_i(t)}{N_i(t)} - \frac{\dot{S}_i(t)}{S_i(t)} + \frac{\ddot{n}_i(t)}{\dot{n}_i(t)}, \\ &\to \left(\gamma_i - \gamma_i^S \right) g^* = \zeta \varrho (1-\delta) g^* (\epsilon_i - \epsilon_{max}). \end{aligned}$$

The two results above provides theoretical explanations for the correlations between the growth rates of our innovation proxies and our measures of income elasticity.

4. Conclusion

In this paper, we construct the first theory that endogenously determines the direction of innovation across sectors that produce goods with robustly heterogenous income elasticities. Our theory provides a framework to study the general equilibrium interactions between demand-pull and technology-push forces in determining the equilibrium rates of innovation and productivity growth across industries. We provide simple and intuitive characterizations for the evolution of R&D intensity across sectors. In the model, the asymptotic rates of innovation and productivity growth are pinned down by the income elasticities of sectoral outputs. We show that the rates of growth of R&D expenditure and patents in the US show sizable correlations with the income elasticities of their outputs, estimated using micro-level household consumption surveys.

References

Acemoglu, Daron, "Directed Technical Change," *Review of Economic Studies*, 2002, *69* (4), 781–809. 3, 6

— , "Equilibrium bias of technology," *Econometrica*, 2007, *75* (5), 1371–1409. 4, 6

— , *Introduction to Modern Economic Growth*, Princeton, NJ: Princeton University Press, 2009. 31

— **and Joshua Linn**, "Market size in innovation: theory and evidence from the pharmaceutical industry," *Quarterly Journal of Economics*, 2004, *119* (3), 1049–1090. 6

Aguiar, Mark and M Bils, "Has Consumption Inequality Mirrored Income Inequality?," *American Economic Review*, 2015, *105* (9), 2725–2756. 2, 3, 7, 8, 9, 10

Baumol, William J, "Macroeconomics of unbalanced growth: the anatomy of urban crisis," *American Economic Review*, 1967, *57* (3), 415–426. 2

Boppart, Timo, "Structural change and the Kaldor facts in a growth model with relative price effects and non-Gorman preferences," *Econometrica*, 2014, *82* (6), 2167–2196. 2

— **and Franziska J Weiss**, "Non-homothetic preferences and industry directed technical change," 2013, (June), 1–45. 6

Budish, Eric, Benjamin N Roin, and Heidi Williams, "Do firms underinvest in long-term research? Evidence from cancer clinical trials.," *The American economic review*, 2015, *105* (7), 2044–2085. 6

Caron, Justin, Thibault Fally, and James R Markusen, "Internationl trade puzzles: a solution linking production and preferences," *Quarterly Journal of Economics*, 2014, *129* (3), 1501–1552. 6

Cerda, Rodrigo A., "Endogenous innovations in the pharmaceutical industry," *Journal of Evolutionary Economics*, 2007, *17* (4), 473–515. 6

Clark, Gregory, "Winter is coming: Robert gordon and the future of economic growth," *American Economic Review*, 2016, *106* (5), 68–71. 2

Cohen, Wesley M., "Fifty years of empirical studies of innovative activity and performance," in "Handbook of the Economics of Innovation," 1 ed., Vol. 1, Elsevier B.V., 2010, pp. 129–213. 5

Cohen, Wesley M, Richard C Levin, and David C Mowery, "Firm size and R&D intensity: a re-examination," *Journal of Industrial Economics*, 1987, *34* (2), 217–223. 6

Comin, Diego, Danial Lashkari, and Marti Mestieri, "Structural change with long-run income and price effects," 2015. 2, 4, 6, 10, 23, 31

Gordon, Robert J, *The rise and fall of American growth: The US standard of living since the civil war*, Princeton University Press, 2016. 2

Griliches, Zvi and Jacob Schmookler, "Inventing and maximizing," *American Economic Review*, 1963, *57* (3), 415–426. 5

Grossman, Gene M and Elhanan Helpman, "Trade, innovation, and growth," *American Economic Review*, feb 1990, *151* (3712), 867–8. 5

Hanoch, Giora, "Production and Demand Models with Direct or Indirect Implicit Additivity," *Econometrica*, 1975, *43* (3), 395–419. 4, 6, 10
Innovation pessimism: Has the ideas machine broken down?

Innovation pessimism: Has the ideas machine broken down?, *The Economist, jan 2013. 2*

Jaravel, Xavier, *"The unequal gains from product innovations: evidence from the US retail sector," 2017. 6*

Ngai, L Rachel and Christopher A Pissarides, *"Structural Model of Growth Change in a Multisector," American Economic Review, 2007, 97 (1), 429–443. 23*

Pakes, Ariel and Mark Schankerman, *"An exploration into the determinants of research intensity," in Zvi Griliches, ed., R&D, Patents, and Productivity, Vol. I, University of Chicago Press, 1984, pp. 209–232. 5*

Romer, Paul M, *"Increasing Returns and Long-Run Growth," Journal of Political Economy, 1986, 94 (5), 1002–1037. 5*

Romer, Paul M., *"Endogenous Technological Change," Journal of Political Economy, 1990, 98 (S5), S71. 5*

Sato, Ryuzo, *"The Most General Class of CES Functions," Econometrica, 1975, 43 (5), 999–1003. 6, 10*

Scherer, Frederic M, *"Demand-pull and technological invention: Schmookler revisited," Journal of Industrial Economics, 1982, 34 (2), 225–237. 5*

Schmookler, Jacob, *Invention and Economic Growth, Harvard University Press, 1966. 2, 5*

Young, Alwyn, *"The African Growth Miracle," Journal of Political Economy, 2012, 120 (4), 696–739. 10*

_ , *"Inequality, the urban-rural gap, and migration," Quarterly Journal of Economics, 2013, 128 (4), 1727–1785. 10*

A. Proofs

Proposition. *Proposition 1.*

Proof. The derivations for the intratemporal allocation as given by Equation (3) with the price index as defined by Equation (4) are straightforward (see Comin et al., 2015). We only present the proof on the intertemporal component of the solution.

For a given path of real interest rate $[r(t)]_{t=0}^{\infty}$ and sectoral good prices $[\boldsymbol{P}(t)]_{t=0}^{\infty}$, the current-value Hamiltonian for the consumer problem (1) may be written as

$$\hat{\mathcal{H}} \equiv \frac{C(t)^{1-\theta}-1}{1-\theta} + \lambda(t)\left[1 + r(t)\,A(t) - \mathcal{E}\left(C(t);\boldsymbol{P}(t)\right)\right],$$

where we have defined the expenditure function $\mathcal{E} \equiv \mathcal{P}\left(C(t);\boldsymbol{P}(t)\right) \times C(t)$, where the price-index function is defined by Equation (4). Let us start with the necessary conditions. The FOCs for the Hamiltonian are as follows:

$$\frac{\partial \hat{\mathcal{H}}}{\partial C} = 0 \quad \Rightarrow C^{-\theta} - \lambda \frac{\partial \mathcal{E}}{\partial C} = 0, \tag{44}$$

$$\frac{\partial \hat{\mathcal{H}}}{\partial A} = \rho\lambda - \dot{\lambda} \quad \Rightarrow -\frac{\dot{\lambda}}{\lambda} = r - \rho. \tag{45}$$

In addition, we impose that the solution satisfy the transversality condition:

$$\lim_{t\to\infty} e^{-\rho t}\lambda(t)\,A(t) = 0. \tag{46}$$

Equations (44) and (45) together with the law of evolution of assets (5) and the transversality equation (6) characterize paths of per capita real aggregate consumption and asset holdings $[C(\cdot), A(\cdot)]$, and costate $\lambda(\cdot)$ that satisfy necessary conditions for optimality.

Next, we show the conditions that ensure the solution above indeed corresponds to the unique solution to the household utility maximization problem. A standard argument (using (45) and the No-Ponzi constraint) shows that *for all feasible pairs* $[C(\cdot), A(\cdot)]$, we have that $\lim_{t\to\infty} \exp(-\rho t)\,\lambda(t)\,A(t) \geq 0$.[13] Therefore, we can establish that the pair characterized by Equations (44), (45), and (46) indeed correspond to the optimum if the Hamiltonian is concave in C. Furthermore, since the Hamiltonian is separable in (C, A) and linear A, strict concavity in C implies the uniqueness of the optimum for the household problem.

Combining Equations (44), (45), and (46) with the definition of the expenditure function $\mathcal{E} = \mathcal{P}\,C$ and the price index (4) gives the Euler equation (7) and the transversality

[13]For a discussion of necessary and sufficient conditions for the optimality of a solution to similar continuous-time dynamic programming problems, see Acemoglu (2009).

condition (8). It remains for us to find conditions that ensure the strict concavity of $\mathcal{E}$ in C to ensure the sufficiency of the conditions above and uniqueness of the solution.

The second order condition for C is

$$\begin{aligned} o &> -\theta C^{-(\theta+1)} - \lambda \frac{\partial^2 \mathcal{E}}{\partial C^2}, \\ &= -C^{-(\theta+1)} \left(\theta + \eta_{\mathcal{E}_C}\right), \end{aligned}$$

where $\eta_{\mathcal{E}_C} \equiv \frac{C \frac{\partial^2 \mathcal{E}}{\partial C^2}}{\frac{\partial \mathcal{E}}{\partial C}}$ denotes the elasticity of marginal expenditure with respect to real consumption C. In the equality above, we have substituted for $\lambda = \left(C^{\theta} \partial \mathcal{E} / \partial C\right)^{-1}$ from Equation (44). The second order condition therefore implies that a sufficient condition for the strict concavity of the Hamiltonian to be $\eta_{\mathcal{E}_C} > -\theta$.

Using Equation (4), we can compute the elasticity to find

$$\eta_{\mathcal{E}_C} = \bar{\epsilon}(t) \left[1 + (1-\sigma) \, Var\left(\frac{\epsilon_i}{\bar{\epsilon}(t)}, t\right)\right] - 1.$$

It then follows that conditions $\sigma \in (0,1]$ and $\epsilon_i > 1 - \theta$ for all i are sufficient to ensure the strict concavity of the Hamiltonian in C. □

Lemma. *See Lemma 1.*

Proof. We can substitute for the rate of growth of R&D productivity in sector i as

$$\dot{s}_i(t) = \delta \cdot \dot{n}_i(t) + (1-\delta) \cdot \dot{s}(t).$$

Substituting this expression in Equation (28), the rate of growth of productivity in sector i should satisfy

$$\dot{n}_i(t) = \frac{1}{\delta}\left(-\left[r(t) + (1-\delta)\, \dot{s}(t)\right] + \frac{1}{\zeta A(t)} \frac{L(t)}{H} \frac{\Omega_i(t)}{\Lambda_i(t)}\right). \tag{47}$$

Once again, using the free entry condition, we can write the rate of sectoral productivity growth as

$$\dot{n}_i(t) = \frac{\dot{N}_i(t)}{N_i(t)} = \frac{Z_i(t)}{N_i(t)\, V_i(t)} = \frac{1}{A(t)} \frac{Z_i(t)}{H} \frac{1}{\Lambda_i(t)}, \tag{48}$$

where we have combined Equations (12) and (22). Using this result, multiplying both sides of Equation (47) by $A(t)\, \Lambda_i(t)$ and summing over i we can find the expression inside the brackets in Equation (47) to be given by □

$$A(t) \cdot \left[r(t) + (1-\delta) \cdot \dot{s}(t)\right] = -\delta \frac{Z(t)}{H} + \frac{1}{\zeta} \frac{L(t)}{H}. \tag{49}$$

Substituting this expression back in Equation (47) and using Equation (48) we find that, if $Z_i(t) > 0$ for all i, the allocations of R&D employment and the rates of sectoral growth are given by Equations (30) and (31).

To find the expression for the interest rate $r(t)$, we first compute the rate of growth of the general-purpose stock of knowledge,

$$\begin{aligned}\dot{s} &= \frac{\sum_i \vartheta_i^{1-\varsigma} N_i^{\varsigma} \dot{n}_i}{\sum_i \vartheta_i^{1-\varsigma} N_i^{\varsigma}}, \\ &= \frac{1}{A}\frac{Z}{H} + \frac{1}{\delta\zeta}\frac{L}{H}\left[\sum_i \vartheta_i^{1-\varsigma}\eta_i \left(\frac{N_i}{S}\right)^{\varsigma+\delta-1}\Omega_i - \frac{1}{A}\right],\end{aligned}$$

where in the second expression, we have substituted from Equation (31) and have used the fact that $\Lambda_i \equiv \frac{\eta_i}{A}(N_i/S)^{1-\delta}$ and $S^{\varsigma} \equiv \sum_i \vartheta_i N_i^{\varsigma}$. Substituting this expression in Equation (49), we find the expression for the interest rate to be given by Equation (32).

Finally, we can rewrite the rates

$$\dot{n}_i(t) = \frac{1}{\delta}\left[-r(t) - (1-\delta)\dot{s}(t) + \frac{1}{1+\zeta}\frac{P_i(t)Y_i(t)}{Z_i(t)}\dot{n}_i(t)\right]$$

of technical growth as

$$\dot{n}_i(t) = \frac{r(t) + (1-\delta)\dot{s}(t)}{\frac{1}{1+\zeta}\frac{P_i(t)Y_i(t)}{Z_i(t)} - \delta},$$

suggesting a one-to-one relationship between sectoral R&D intensity $(\frac{Z_i}{P_iY_i})$ and the sectoral rate of technological growth. Note that this expression suggests the following constraint on the R&D intensity of sectors, satisfied everywhere along an equilibrium path

$$\frac{Z_i(t)}{P_i(t)Y_i(t)} < \frac{1}{\delta(1+\zeta)}.$$

Lemma. *See Lemma 2.*

Proof. Along a CGP as characterized by Definition (3.6), the growth in the relative shares of sectors i and j can be found from Equation (3) to be

$$\lim_{t\to\infty} \dot{\omega}_i(t) - \dot{\omega}_j(t) \equiv (1-\sigma)g^*\left[(\epsilon_i - \epsilon_j) - \frac{\gamma_i - \gamma_j}{\zeta}\right],$$

which is a constant. Therefore, relative shares asymptotically evolve in a monotonic fashion; their time derivatives maintaining their signs. Since shares are nonnegative numbers that sum to one, they belong to a compact set and therefore have to converge to constant shares $\{\Omega_i^*\}_{i=1}^I$.

Under the asymptotic distribution $\{\Omega_i^*\}_{i=1}^I$, we can define $\overline{\gamma}^*$ and $\overline{\epsilon}^*$ to be average expenditure-weighted technical growth rates and income elasticity parameters across sectors. From Equation (9), we find that the growth rate of consumption expenditure is given by

$$\lim_{t\to\infty} \dot{e}(t) = \left(\overline{\epsilon}^* - \frac{1}{\zeta}\overline{\gamma}^*\right) g^*,$$

which, again, suggests that $E(t)$ either asymptotically grows or falls at a constant rate. From the market clearing we know that $E(t) = \frac{1+\zeta}{\zeta}\frac{L(t)}{H} \leq \frac{1+\zeta}{\zeta}$ and therefore $E(t)$ also belongs to a closed and bounded set. Therefore, $\lim_{t\to\infty} E(t) = E^* > 0$, where the strict positivity follows from the transversality condition (8). This implies that the production employment also converges to a constant $L^* > 0$. Furthermore, the growth of $E(t)$ has to asymptotically be zero, that is,

$$\zeta = \frac{\overline{\gamma}^*}{\overline{\epsilon}^*}. \tag{50}$$

Since all shares are converge to constants, we have that

$$\begin{aligned} \lim_{t\to\infty} \dot{\omega}_i(t) &= (1-\sigma) g^* \left((\epsilon_i - \overline{\epsilon}^*) - \frac{\gamma_i - \overline{\gamma}^*}{\zeta} \right), \\ &= (1-\sigma) g^* \left(\epsilon_i - \frac{\gamma_i}{\zeta} \right) \leq 0, \end{aligned}$$

where we have used Equation (50) in the second equality . This suggests that $\zeta\epsilon_i \leq \gamma_i$ and therefore $\zeta \leq \gamma_i/\epsilon_i$ for all i, implying the result

$$\zeta = \min_i \left\{ \frac{\gamma_i}{\epsilon_i} \right\}.$$

Finally, Equations (37) and (38) follow from the relations above and Equations (3) and (4).

□

Lemma. *See Lemma 3.*

Proof. First, from $\dot{n}_i(t) = S_i(t) Z_i(t) / N_i(t) \longrightarrow \gamma_i g^*$ and from the previous lemma, we know that $\gamma_i \geq \zeta\epsilon_i > 0$. We then asymptotically find

$$\gamma_i^S - \gamma_i + \lim_{t\to\infty} \dot{z}_i(t) = 0.$$

Since the R&D employment Z_i is bounded by Z^*, its rate of growth above cannot be positive. Therefore, we have that $Z_i(t)$'s converge to constants and we have

$$\gamma_i^S \geq \gamma_i, \tag{51}$$

for all i. For a sector $i \in \mathcal{I}^\dagger$ for which the asymptotic share of R&D employment is not zero, we have $\gamma_i^S = \gamma_i$ and its size of R&D employment is given by

$$Z_i^* = \gamma_i g^* \times \lim_{t\to\infty} \left(\frac{N_i(t)}{S_i(t)} \right)$$

□

Proof. We can directly employ the free entry condition to study the properties of the CGP. First, notice that

$$\begin{aligned}
\lim_{t'\to\infty} \frac{\Pi_i(t'+t)}{\Pi_i(t')} &= \lim_{t'\to\infty} \left(\frac{N_i(t')}{N_i(t'+t)} \right) \left(\frac{\Omega_i(t'+t)}{\Omega_i(t')} \right) \frac{Y(t'+t)}{Y(t')}, \\
&= \exp \left[\left(\underbrace{-\gamma_i}_{\text{technology}} + \underbrace{(1-\sigma)\left(\epsilon_i - \frac{\gamma_i}{\zeta}\right)}_{\text{market size}} \right) g^* t \right], \\
&= e^{-\left[\left(1+\frac{1-\sigma}{\zeta}\right)\gamma_i - (1-\sigma)\epsilon_i\right] g^* t}, \qquad (52)
\end{aligned}$$

where in the second line, we have used the fact that $\lim_{t\to\infty} Y(t) = HE^*$ and have substituted for the asymptotic rate of growth of sectoral output shares from Equation (36). Note that since $\Omega_i(t)$ is a share variable, we have that $\lim \dot{\omega}_i \leq 0$ with equality being satisfied at least by one sector. Now, we find the asymptotic value of owning a firm in sector i to be

$$\begin{aligned}
\lim_{t\to\infty} \frac{V_i(t)}{\Pi_i(t)} &= \left(\lim_{t\to\infty} \int_0^\infty e^{-\left(\frac{1}{t'}\int_0^{t'} r(t+t')\right)t'} \left(\frac{\Pi_i(t+t')}{\Pi_i(t)} \right) dt', \right) \\
&= \int_0^\infty e^{-\left[r^* + g^*\left(\gamma_i - (1-\sigma)\left(\epsilon_i - \frac{\gamma_i}{\zeta}\right)\right)\right]t'} dt', \\
&= \frac{1}{r^* + g^*\left(\gamma_i - (1-\sigma)\left(\epsilon_i - \frac{\gamma_i}{\zeta}\right)\right)},
\end{aligned}$$

where in the first equality we have used the definition of $V_i(t)$ from Equation (20), and in the second equality, we have used Equation (52) and the fact that the Euler equation and Lemma 2 imply asymptotically constant the real interest rate r^*.

Now, we can use the free entry condition, either in its direct form (22), or through Equation (21) to find

$$\begin{aligned}
r^* &= \lim_{t\to\infty} -\dot{s}_i(t) + \frac{\Pi_i(t)}{V_i(t)}, \\
&= -\gamma_i^S + r^* + g^*\left(\gamma_i - (1-\sigma)\left(\epsilon_i - \frac{\gamma_i}{\zeta}\right)\right), \qquad (53)
\end{aligned}$$

which implies Equation (40). □

Proposition. *See Proposition 3.*

Proof. First, it is easy to see that $\gamma_{max} = \zeta\epsilon_{max}$. Note that from the definition (1) and for $S(t)$ defined for $\varsigma > 0$, we have $\lim \dot{s} = \gamma_{max}$ and we find

$$\gamma_i^S - \gamma_i = \delta\gamma_i + (1-\delta)\,\gamma_{max} - \gamma_i = (1-\delta)\,(\gamma_{max} - \gamma_i)\,.$$

This means that for $i \in \mathcal{I}^\dagger$, we need to have $\gamma_i = \gamma_{max} = \zeta\epsilon_i$. Now, consider the sector j with $\epsilon_j = \epsilon_{max}$ and, to find a contradiction, assume that $j \notin \mathcal{I}^\dagger$. This implies that $j \notin \mathcal{I}^*$ and therefore $\gamma_j > \zeta\epsilon_{max}$. This is a contradiction, because it implies that $\gamma_j > \gamma_{max}$. Therefore, we find

$$\mathcal{I}^* = \mathcal{I}^\dagger = \{i \,|\, \epsilon_i = \epsilon_{max}\}\,.$$

From Equation (40), we then have

$$(1-\delta)\,(\zeta\epsilon_{max} - \gamma_i) = (1-\sigma)\left(\frac{\gamma_i}{\zeta} - \epsilon_i\right),$$

implying

$$\begin{aligned}\gamma_i &= \left(1-\delta+\frac{1-\sigma}{\zeta}\right)^{-1}\left[(1-\delta)\,\zeta\epsilon_{max} + (1-\sigma)\,\epsilon_i\right],\\ &= \zeta\left[\frac{1-\sigma}{\zeta\,(1-\delta)+1-\sigma}\epsilon_i + \frac{\zeta\,(1-\delta)}{\zeta\,(1-\delta)+1-\sigma}\epsilon_{max}\right].\end{aligned}$$

The remainder of the proof follows as a special case of the proof of the Proposition 3 in Section B. □

B. Generalized Spillover Function

In this section, we examine the properties of our model for a more general class of spillover functions.

Definition 2. Consider spillover functions $\boldsymbol{S}$ defined according to

$$S_i(\boldsymbol{N}) \equiv \frac{1}{\eta_i}\left[\delta_i^{1-\psi_i}N_i^{\psi_i} + (1-\delta_i)^{1-\psi_i}\,\tilde{S}_i(\boldsymbol{N})^{\psi_i}\right]^{\frac{1}{\psi_i}}, \tag{54}$$

$$\tilde{S}_i(\boldsymbol{N}) \equiv \left(\sum_j \vartheta_{ij}^{1-\varsigma_i}N_j^{\varsigma_i}\right)^{\frac{1}{\varsigma_i}}, \tag{55}$$

for all i, where $\delta \in (0,1)$, $\vartheta_{ij} > 0$ for all $i,j \in \{1,\cdots,I\}$, and $\sum_j \vartheta_{ij} = 1$ for all i. We *rule out* the set of model parameters that for any sector i satisfy $\psi_i \le 0$ and $\varsigma_i \le 0$.

According to Equations (54) and (55), innovation spillovers to each sector have two components: the spillovers from the same sector, the $\delta_i^{1-\psi_i} N_i^{\psi_i}$ term, and the spillovers from all other sectors, the term involving $\tilde{S}_i$. The degree of substitutability between the two types of spillovers is determined by ψ_i. If $\psi_i > 0$, N_i and $\tilde{S}_i$ are gross substitutes and function S_i is log-submodular in the two arguments. If $\psi_i < 0$, N_i and $\tilde{S}_i$ are gross complements and function S_i is log-supermodular in the two arguments. Similarly, parameter ς_i for sector i determines the degree of substitutability among technologies of different sectors in their innovation spillovers to sector i. We will see below that the two parameters (ψ_i, ς_i) play an important role in the characterization of the asymptotic rate of technical growth for sector i. The case of $\psi_i \to 0$, when $\delta_i \equiv \delta$, $\varsigma_i \equiv \varsigma$, and $\vartheta_{ij} \equiv \vartheta_j$ for all $i$and j corresponds to the case of Definition 1, with the stock of general-purpose technologies S defined correspondingly. We rule out the case where both N_t and $\tilde{S}_i$ are gross substitutes and $\tilde{S}_i$ is log-submodular in its arguments, since we show below that this case does not allow for the existence of a CGP.

The following proposition characterizes the long-run behavior of the economy for this class of spillover functions.

Proposition 3. *Consider an economy with the intersectoral innovation spillover function introduced in Example 2 and a set of income elasticity parameters that consists of a set of distinct values, that is, $\varepsilon_i \neq \varepsilon_{i'}$ for any pair of $i \neq i'$.*

Consider then a potential CGP as one characterized by a set of parameters

$$\left(\mathcal{I}^* \subset \mathcal{I}, \{\Omega_i^*, \Lambda_i^*\}_{i \in \mathcal{I}^*}, g^*, L^*, \gamma, \boldsymbol{N}^*\right).$$

First, the set of sectors that asymptotically constitute a nonvanishing share of economic activity $\mathcal{I}^*$ consists of

(1) Any sector i with $\varsigma_i > 0$ and $\psi_i < 0$, or $\varsigma_i < 0$ and $\psi_i > 0$,

(2) Any sector i with $\varsigma_i > 0$ and $\psi_i \geq 0$ if $\epsilon_i \geq \epsilon_{i'}$ for all i'.

For all of these sectors, the rate of technological growth is asymptotically given by $\gamma_i = \zeta \epsilon_i$. Any other sector has a vanishing share of employment in production and output, and a vanishing share in total assets. The asymptotic rate of technological growth for such a sector i is given by

$$\gamma_i = \zeta\left(\varrho_i \epsilon_i + (1 - \varrho_i)\,\epsilon_{max}\right), \qquad \varrho_i \equiv \begin{cases} \frac{1-\sigma}{\zeta+1-\sigma}, & \psi_i > 0, \\ \frac{1-\sigma}{\zeta(1-\delta_i)+1-\sigma}, & \psi_i \to 0. \end{cases}$$

Denote by $\langle \epsilon_i \rangle^*$ and $Var \langle \epsilon_i \rangle^*$ the average and the variance of income elasticity parameters under the sectoral distribution implied by the asymptotic sectoral shares of corporate assets $\{\Lambda_i^*\}_{i \in \mathcal{I}^*}$.

If the discount rate ρ satisfies

$$\frac{1}{\zeta A^*}\left(1-\frac{\theta+\langle\epsilon_i\rangle^*-1+(\zeta+1)\langle\epsilon_i\rangle^*}{1+(\zeta+1)\langle\epsilon_i\rangle^*}\right)<\rho<\frac{1}{\zeta A^*}, \tag{56}$$

the CGP exists and is unique. In this CGP, the asymptotic real interest rate r^* and the consumption expenditure weighted income elasticity parameters $\overline{\epsilon}^*$ will be found by the unique solution to the system of equations

$$\begin{aligned}\overline{\epsilon}^* &= \langle\epsilon_i\rangle^*+\frac{Var\langle\epsilon_i\rangle^*}{r^*+\zeta\langle\epsilon_i\rangle^*},\\ r^* &= \frac{1}{\zeta A^*}-\frac{(1+\zeta)\langle\epsilon_i\rangle^*}{\overline{\epsilon}^*+\theta-1+(1+\zeta)\langle\epsilon_i\rangle^*}\left(\frac{1}{\zeta A^*}-\rho\right).\end{aligned}$$

The asymptotic rate of growth in aggregate consumption is given by

$$g^*=\frac{1/\zeta A^*-\rho}{\theta+\overline{\epsilon}^*-1+(\zeta+1)\langle\epsilon_i\rangle^*},$$

and the asymptotic share of employment in production by

$$\frac{L^*}{H}=\frac{\rho\zeta A^*\langle\epsilon_i\rangle^*+\theta+\overline{\epsilon}^*-1+\zeta\langle\epsilon_i\rangle^*}{\theta+\overline{\epsilon}^*-1+(\zeta+1)\langle\epsilon_i\rangle^*}.$$

Finally, $\left(\Omega_i,N_i^*\right)_{i\in\mathcal{I}^*}$ are given by Equations (25) and (26) and

$$\Omega_i^*=\frac{\zeta}{L^*}\left(r^*+\zeta g^*\epsilon_i\right)\Lambda_i^*,\qquad \textit{for all } i\in\mathcal{I}^*.$$

Proof. Let γ_{min} and γ_{max} denote the minimum and the maximum in the set of all relative rates of technical growth $\{\gamma_i\}_{i=1}^I$, respectively, and sets $\mathcal{I}_{min}$ and $\mathcal{I}_{max}$ to include the sectors that achieve the correspondingly values. Let us further define $\gamma_i^S\equiv\lim_{t\to\infty}\frac{\dot{s}_i(t)}{g^*}=\sum_j\Sigma_{ij}^*\gamma_i$. We can now rewrite Equation (53) as

$$\left(1+\frac{1-\sigma}{\zeta}\right)\gamma_i-\gamma_i^S=(1-\sigma)\,\epsilon_i,\qquad \text{for all } i.$$

From Equation (39) we know that $\gamma_i\le\gamma_i^S$ and therefore $\gamma_i\ge\zeta\epsilon_i$.

Now, we will group sectors based on the degree of substitutability parameters (ψ_i,ς_i) in their respective intersectoral innovation spillover functions.

1. *Sectors with $\psi_i,\varsigma_i<0$:* In this case, S_i asymptotically grows at the rate dictated by the slowest (technologically) growing sectors $\mathcal{I}_{min}$. Therefore, for all such sectors we have $\gamma_i^S=\gamma_{min}$. However, in order to satisfy condition (39), we need to ensure that $\gamma_i=\zeta\epsilon_i=\gamma_{min}$. But since $\gamma_i\ge\zeta\epsilon_i$ for all i, this could be the case

only if $\epsilon_i = \epsilon_{min}$. We can further compute the asymptotic corporate assets held in sector i as

$$\begin{aligned}
\Lambda_i^* &= \lim_{t\to\infty} \frac{N_i(t)}{S_i(t)}, \\
&= \lim_{t\to\infty} \eta_i \left(\delta_i^{1-\psi_i} + (1-\delta_i)^{1-\psi_i} \left(\sum_{j\neq i} \vartheta_{ij}^{1-\varsigma_i} \left(\frac{N_j(t)}{N_i(t)} \right)^{\varsigma_i} \right)^{\frac{\psi_i}{\varsigma_i}} \right)^{-\frac{1}{\psi_i}}, \\
&= \eta_i \left(\delta_i^{1-\psi_i} + (1-\delta_i)^{1-\psi_i} \left(\sum_{j\neq i:\, \epsilon_j=\epsilon_{min}} \vartheta_{ij}^{1-\varsigma_i} \left(\frac{N_j^*}{N_i^*} \right)^{\varsigma_i} \right)^{\frac{\psi_i}{\varsigma_i}} \right)^{-\frac{1}{\psi_i}}.
\end{aligned}$$

2. *Sectors with $\psi_i < 0$ and $\varsigma_i > 0$:* In this case S_i asymptotically grows at a rate dictated by the minimum of the rate of growths of N_i and $\tilde{S}_i$. The spillover term $\tilde{S}_i$ in turn asymptotically grows at the rate dictated by the fastest (technologically) growing sectors γ_{max} (note that condition stated in the proposition rules out the case where all sectors other than i technologically grow at strictly lower rates). Therefore, we always have that $\gamma_i^S = \gamma_i = \zeta\epsilon_i$.

$$\begin{aligned}
\Lambda_i^* &= \lim_{t\to\infty} \frac{N_i(t)}{S_i(t)}, \\
&= \eta_i \left(\delta_i^{1-\psi_i} + (1-\delta_i)^{1-\psi_i}\, \mathbb{I}\{\epsilon_i = \epsilon_{max}\} \left(\sum_{j\neq i:\, \epsilon_j=\epsilon_i} \vartheta_{ij}^{1-\varsigma_i} \left(\frac{N_j^*}{N_i^*} \right)^{\varsigma_i} \right)^{\frac{\psi_i}{\varsigma_i}} \right)^{-\frac{1}{\psi_i}},
\end{aligned}$$

where $\mathbb{I}\{\epsilon_i = \epsilon_{max}\} = 1$ only if $\epsilon_i = \epsilon_{max}$.

3. *Sectors with $\psi_i > 0$ and $\varsigma_i < 0$:* In this case S_i asymptotically grows at a rate dictated by the maximum of the rates of growth of N_i and $\tilde{S}_i$. However, the rate of growth of $\tilde{S}_i$ is now determined by the slowest (technologically) growing sector other than i. Therefore, once again we will have $\gamma_i^S = \gamma_i = \zeta\epsilon_i$, unless if $\epsilon_{i'} > \epsilon_i$ for all $i' \neq i$. We will have

$$\begin{aligned}
\Lambda_i^* &= \lim_{t\to\infty} \frac{N_i(t)}{S_i(t)}, \\
&= \eta_i \left(\delta_i^{1-\psi_i} + (1-\delta_i)^{1-\psi_i}\, \mathbb{I}\{\epsilon_i = \epsilon_{min}\} \left(\sum_{j\neq i:\, \epsilon_j=\epsilon_i} \vartheta_{ij}^{1-\varsigma_i} \left(\frac{N_j^*}{N_i^*} \right)^{\varsigma_i} \right)^{\frac{\psi_i}{\varsigma_i}} \right)^{-\frac{1}{\psi_i}}.
\end{aligned}$$

If if $\epsilon_{i'} > \epsilon_i$ for all $i' \neq i$, then we have that $\gamma_i^S = \gamma_{min}^{-i} \equiv \min\{\gamma_{i'}\}_{i' \in \mathcal{I} \setminus \{i\}}$.

$$\gamma_i = \zeta\epsilon_i \left(1 + \frac{\zeta}{\zeta + 1 - \sigma}\left(\frac{\gamma_{min}^{-i}}{\zeta\epsilon_i} - 1\right)\right) > \zeta\epsilon_i,$$

since $\zeta\epsilon_i = \gamma_i = \gamma_{min} < \gamma_{min}^{-i}$.

4. *Sectors with $\psi_i > 0$ and $\varsigma_i > 0$:* In this case S_i asymptotically grows at the rate determined by the fastest (technologically) growing sectors $\mathcal{I}_{max}$, and we have $\gamma_i^S = \gamma_{max}$ for all i. This first suggests that $\gamma_{max} = \zeta\epsilon_{max}$ where ϵ_{max} is the sector with the largest income elasticity. For this sector

$$\begin{aligned} \Lambda_i^* &= \lim_{t\to\infty} \frac{N_i(t)}{S_i(t)}, \\ &= \eta_i \left(\delta_i^{1-\psi_i} + (1-\delta_i)^{1-\psi_i} \mathbb{I}\{\epsilon_i = \epsilon_{max}\} \left(\sum_{j\neq i:\, \epsilon_j = \epsilon_i} \vartheta_{ij}^{1-\varsigma_i} \left(\frac{N_j^*}{N_i^*}\right)^{\varsigma_i}\right)^{\frac{\psi_i}{\varsigma_i}}\right)^{-\frac{1}{\psi_i}}. \end{aligned}$$

For all other sectors follows from Equation (53) that

$$\gamma_i = \zeta\epsilon_i \left(1 + \frac{\zeta}{\zeta + 1 - \sigma}\left(\frac{\epsilon_{max}}{\epsilon_i} - 1\right)\right).$$

5. *Sectors with $\psi_i \to 0$:* In this case

$$S_i = \frac{1}{\eta_i} N_i^{\delta_i} \tilde{S}_i^{1-\delta_i},$$

and $\dot{s}_i = \delta_i \dot{n}_i + (1-\delta_i)\dot{\tilde{s}}_i$. If $\varsigma_i < 0$, again, we can only have $\gamma_i = \zeta\epsilon_i$ with $\epsilon_i = \epsilon_{min}$. If $\varsigma_i > 0$, we have

$$\gamma_i = \zeta\epsilon_i \left(1 + \frac{\zeta(1-\delta_i)}{\zeta(1-\delta_i) + 1 - \sigma}\left(\frac{\epsilon_{max}}{\epsilon_i} - 1\right)\right).$$

In this case, if ϵ_i is the unique maximum in the set $\{\epsilon_i\}_{i\in\mathcal{I}}$, we do not have a CGP.

Note that in cases 1-3 and the case 4 when $\epsilon_i = \epsilon_{max}$, we have that $\gamma_i^S = \gamma_i$ and therefore $i \in \mathcal{I}^\dagger$ as defined by Lemma 3. Only sectors with $\gamma_i > \zeta\epsilon_i$, that is, sectors with $\psi_i \geq 0$ and $\varsigma_i \geq 0$ are not in $\mathcal{I}^\dagger$ and therefore asymptotically vanish.

From the Euler Equation (7), we find that asymptotically

$$g^* = \frac{r^* - \rho}{\theta - 1 + \bar{\epsilon}^*\left[1 + (1-\sigma)\, Var^*\left(\frac{\epsilon_i}{\bar{\epsilon}^*}\right)\right] - \frac{1}{\zeta}\bar{\gamma}^*\left[1 + (1-\sigma)\, Cov^*\left(\frac{\epsilon_i}{\bar{\epsilon}^*}, \frac{\gamma_i}{\bar{\gamma}^*}\right)\right]}.$$

Let us first compute θ^* defined such that $r^* = g^*\theta^* + \rho$, that is

$$\theta^* \equiv \theta - 1 + \overline{\epsilon}^* \left[1 + (1-\sigma)\, Var^*\left(\frac{\epsilon_i}{\overline{\epsilon}^*}\right)\right] - \frac{1}{\zeta}\overline{\gamma}^* \left[1 + (1-\sigma)\, Cov^*\left(\frac{\epsilon_i}{\overline{\epsilon}^*}, \frac{\gamma_i}{\overline{\gamma}^*}\right)\right].$$

For all sectors i that are included in the moments that appear in θ^*, we have $\gamma_i = \zeta\epsilon_i$. Therefore, the covariance term is $Cov^*\left(\frac{\epsilon_i}{\overline{\epsilon}^*}, \frac{\epsilon_i}{\overline{\epsilon}^*}\right) = Var(\frac{\epsilon_i}{\overline{\epsilon}^*}) - 1$.

$$\begin{aligned}\theta^* &= \theta + \overline{\epsilon}^* - 1 + \left(\overline{\epsilon}^* - \frac{1}{\zeta}\overline{\gamma}^*\right)\left[1 + (1-\sigma)\, Var^*\left(\frac{\epsilon_i}{\overline{\epsilon}^*}\right)\right], \\ &= \theta + \overline{\epsilon}^* - 1,\end{aligned}$$

where in the second equality we have used Equation (35).

We will now pin down the rate of growth of aggregate real consumption g^*. Substituting from $\gamma_i = \zeta\epsilon_i$ for $i \in \mathcal{I}^*$ in Equation (41) we find:

$$\frac{1}{\zeta}(H - L^*) = g^* A^* \sum_{i\in\mathcal{I}^*} \epsilon_i \Lambda_i^*. \tag{57}$$

Then, taking the asymptotic limit of Equation (28), we find

$$\left(r^* + \gamma_i^S g^*\right)\Lambda_i^* = \Omega_i^* \frac{L^*}{\zeta A^*}. \tag{58}$$

Summing over $i \in \mathcal{I}^*$ for which $\gamma_i^S = \gamma_i = \zeta\epsilon_i$ and substituting for $r^* = \theta^* g^* + \rho$, we reach

$$\rho\left(\sum_{i\in\mathcal{I}^*}\Lambda_i^*\right) + g^*\left[\theta^*\left(\sum_{i\in\mathcal{I}^*}\Lambda_i^*\right) + \zeta\left(\sum_{i\in\mathcal{I}^*}\epsilon_i\Lambda_i^*\right)\right] = \frac{L^*}{\zeta A^*}.$$

Summing this expression from Equation (57), we find

$$\frac{H}{\zeta A^*} = \rho\left(\sum_{i\in\mathcal{I}^*}\Lambda_i^*\right) + g^*\left[\theta^*\left(\sum_{i\in\mathcal{I}^*}\Lambda_i^*\right) + (\zeta+1)\left(\sum_{i\in\mathcal{I}^*}\epsilon_i\Lambda_i^*\right)\right],$$

which pins down the economy-wide rate of growth as

$$g^* = \frac{1/\zeta A^* - \rho}{\theta^* + (\zeta+1)\langle\epsilon_i\rangle^*},$$

where $A^* \equiv \frac{1}{H}\sum_{i\in\mathcal{I}^*}\Lambda_i^*$ is the asymptotic value of assets owned by the households and $\langle\epsilon_i\rangle^* \equiv \frac{\sum_{i\in\mathcal{I}^*}\epsilon_i\Lambda_i^*}{\sum_{i\in\mathcal{I}^*}\Lambda_i^*}$ is the average of income elasticity parameters as weighted by the sector's asymptotic shares of corporate assets. Similarly, we can derive the expression for the asymptotic share of employment in the production sector

$$\frac{L^*}{H} = \frac{\rho\zeta A^* + \theta^* / \langle \epsilon_i \rangle^* + \zeta}{\theta^* / \langle \epsilon_i \rangle^* + 1 + \zeta}.$$

Finally, for the real interest rate we find

$$\begin{aligned} r^* &= \rho + \frac{\theta^* / \langle \epsilon_i \rangle^*}{\theta^* / \langle \epsilon_i \rangle^* + 1 + \zeta} \left(\frac{1}{\zeta A^*} - \rho \right), \\ &= \left(\frac{\theta^* / \langle \epsilon_i \rangle^*}{\theta^* / \langle \epsilon_i \rangle^* + 1 + \zeta} \right) \frac{1}{\zeta A^*} + \left(\frac{1 + \zeta}{\theta^* / \langle \epsilon_i \rangle^* + 1 + \zeta} \right) \rho. \end{aligned}$$

The positivity of g^* and the transversality condition $g^* < r^*$ together imply the following constraints on the discount rate ρ

$$\frac{1}{\zeta A^*} \left(1 - \frac{\theta^* + (\zeta + 1) \langle \epsilon_i \rangle^*}{1 + (\zeta + 1) \langle \epsilon_i \rangle^*} \right) < \rho < \frac{1}{\zeta A^*}.$$

Since, as we see below $\bar{\epsilon}^* \geq \langle \epsilon \rangle^*$, Equation (43) is stronger than the condition above as a restriction on ρ. Therefore, if condition (43) is satisfied the CGP is unique.

Let us now calculate $\bar{\epsilon}^*$. Substituting from Equation (58) for Ω_i^* we find

$$\begin{aligned} \bar{\epsilon}^* &= \frac{\sum_{i \in \mathcal{I}^*} (r^* + \zeta \epsilon_i) \Lambda_i^* \epsilon_i}{\sum_{i \in \mathcal{I}^*} (\rho + \theta^* g^* + \zeta \epsilon_i) \Lambda_i^*}, \\ &= \frac{r^* \left(\sum_{i \in \mathcal{I}^*} \Lambda_i^* \epsilon_i \right) + \zeta \left(\sum_{i \in \mathcal{I}^*} \Lambda_i^* \epsilon_i^2 \right)}{r^* \left(\sum_{i \in \mathcal{I}^*} \Lambda_i^* \right) + \zeta \left(\sum_{i \in \mathcal{I}^*} \Lambda_i^* \epsilon_i \right)}, \\ &= \langle \epsilon_i \rangle^* \frac{r^* + \zeta \frac{\langle \epsilon_i^2 \rangle^*}{\langle \epsilon_i \rangle^*}}{r^* + \zeta \langle \epsilon_i \rangle^*}, \\ &= \langle \epsilon_i \rangle^* \left(1 + \frac{\frac{Var \langle \epsilon_i \rangle^*}{\langle \epsilon_i \rangle^*}}{r^* + \zeta \langle \epsilon_i \rangle^*} \right), \end{aligned}$$

where we have defined the variance of the income elasticity parameters under the distribution implied by sectoral shares in corporate assets $Var \langle \epsilon_i \rangle^* \equiv \frac{\sum_{i \in \mathcal{I}^*} \left(\epsilon_i - \langle \epsilon_i \rangle^* \right)^2 \Lambda_i^*}{\sum_{i \in \mathcal{I}^*} \Lambda_i^*}$.

Combining these equations and letting $\varepsilon \equiv \frac{\bar{\epsilon}^*}{\langle \epsilon_i \rangle^*} - 1$ we find a system of two equations

$$\begin{aligned} \varepsilon &= \frac{Var \langle \epsilon_i \rangle^* / \langle \epsilon_i \rangle^*}{r^* + \zeta \langle \epsilon_i \rangle^*}, \\ r^* &= \frac{1}{\zeta A^*} - \frac{1 + \zeta}{\varepsilon + \frac{\theta + \langle \epsilon_i \rangle^* - 1}{\langle \epsilon_i \rangle^*} + 1 + \zeta} \left(\frac{1}{\zeta A^*} - \rho \right), \end{aligned}$$

that together determine (ε, r^*). If condition (43) is satisfied, this system of equations has a unique solution. The quadratic equation always has a solution. The first equation defines r^* as a decreasing function of ε that goes from ∞ to 0 as ε goes from 0 to the maximum of $Var\,\langle\epsilon_i\rangle^* / \zeta \left(\langle\epsilon_i\rangle^*\right)^2$. The second equation defines r^* as an increasing function of ε that goes from a $\frac{1}{\zeta A^*} - \frac{1+\zeta}{\frac{\theta+\langle\epsilon_i\rangle^*-1}{\langle\epsilon_i\rangle^*}+1+\zeta}\left(\frac{1}{\zeta A^*} - \rho\right)$ to $\frac{1}{\zeta A^*}$ as ε goes from 0 to ∞.[14] This completes the characterization of the CGP. □

C. Schumpeterian Model

Consider the following production function for the final good producers in each sector to replace the production function in Equation (10)

$$Y_i = \left(\int_0^1 Q_{iv}^{\frac{1}{\zeta+1}} X_{iv}^{\frac{\zeta}{\zeta+1}} dv\right)^{\frac{\zeta+1}{\zeta}},$$

where each sector has a unit interval of varieties v, each with a variety-specific quality Q_{iv}. The demand for the intermediate good v in sector i is given by

$$X_{iv} = Q_{iv}\left(\frac{P_i}{P_{iv}}\right)^{1+\zeta} Y_i,$$

which again suggests $P_{iv} = 1$ for all varieties in all sectors at all times. The resulting price index for goods in sector i is given by $P_i = Q_i^{-\frac{1}{\zeta}}$, where we have defined Q_i as the average quality of products in sector i, that is, $Q_i \equiv \int_0^1 Q_{iv} dv$. Note that this parallels the result in the expanding varieties model where Q_i, average sectoral quality, has replaced the number of intermediate goods N_i. Similarly, the profits for a given producer is given by

$$\Pi_{iv} = \frac{1}{\zeta} X_{iv} = \frac{1}{\zeta}\frac{Q_{iv}}{Q_i}\Omega_i Y,$$

which again parallels Equation (11).

Potential entrants invest in raising the quality of the available intermediate goods through investing in R&D, which targets a specific variety. When a potential entrant hires Z_{iv} R&D workers to improve on the quality of variety v in sector i with current quality Q_{iv}, it succeeds at a flow rate

$$\frac{S_i(\boldsymbol{Q})}{\eta_i}\frac{Z_{iv}}{Q_{iv}},$$

in which case the quality of the variety improves to $Q_{iv}\Gamma_i$ and the entrant takes over the current monopolist. We assume that $\Gamma_i \geq \left(\frac{\zeta+1}{\zeta}\right)^{\zeta}$ so that the monopolistic pricing

[14] Simple algebra shows that this system indeed can be reduced to a quadratic equation.

of the current incumbent does not allow the previous incumbent to enter the market. The Bellman equation for the net present value of the monopoly rights on variety v in sector i is given by $rV_{iv} - \dot{V}_{iv} = \Pi_{iv} - \frac{S_i}{\eta_i}\frac{Z_{iv}}{Q_{iv}}V_{iv}$. Due to constant returns to scale, the value function will be linear in quality, that is $V_{iv} = V_i Q_{iv}$ for some V_i. Similary, the rate of creative destruction is going to be the same for all intermediate goods within a sector. Define $Z_i \equiv Q_i \frac{Z_{iv}}{Q_{iv}}$ such that this rate for sector i is $\frac{Z_i}{Q_i}$ and the total number of R&D workers in sector i $\int_0^1 Z_i \frac{Q_{iv}}{Q_i} dv = Z_i$. Since Q_{iv} is a jump process, its derivative is zero and we have $\left(r + \frac{S_i}{\eta_i}\frac{Z_i}{Q_i}\right) V_i - \dot{V}_i = \frac{1}{\zeta}\frac{\Omega_i Y}{Q_i}$ where Z_{iv} is the number of R&D workers hired for any intermediate variety in sector i. The free entry condition is now given by

$$1 \geq \frac{S_i}{\eta_i / \Gamma_i} V_i.$$

The rate of growth of quality in sector i is $\dot{q}_i = \frac{\Gamma_i - 1}{\eta_i}\frac{S_i}{Q_i} Z_i$. [15] Therefore, we can rewrite the free entry condition as

$$\dot{q}_i = \left(1 - \frac{1}{\Gamma_i}\right)\frac{Z_i}{Q_i V_i}.$$

In Equation (23), we instead have $H \cdot A = \sum_i \int V_{iv} dv = \sum_i Q_i V_i$.

[15] To see this, note that

$$\begin{aligned} Q_i\left(t + dt\right) &= \left(\int_0^1 \left(\Gamma_i - 1\right) Q_{iv} \times \frac{S_i}{\eta_i}\frac{Z_i}{Q_i} dv\right) dt, \\ &= \frac{\Gamma_i - 1}{\eta_i} S_i Z_i dt, \end{aligned}$$

where in the second line we have used the fact that R&D employment is the same for all varieties within the same sector.