Economic Geography and Rational Expectations

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Abstract

Eventhough the consequences of the rational expectation assumption have been somewhat explored in the literature concerning two-country models, we are not aware of any attempt to explore the role that rational behavior may have in a continuous spatial economy. Our model builds on Krugman (1996)'s economic geography model. However, here, workers are assumed to be forward looking and to have perfect foresight abilities. Our result reemphasizes the role of the local market structure on the convergence process: like in Krugman (1996), scale economies at the local level and free mobility of workers contribute to spatial divergence. However, unlike in the corresponding myopic case studied by Mossay (2003), the size of agglomerations increases with the taste for variety and the proportion of the manufacturing population, and decreases with transport costs. The role of rational adjustments with respect to myopic adjustments is to distort the relationship between the amplification factor and the wavelength.

Keywords: agglomeration, trade, scale economies, monopolistic competition, migration, rational expectations

J.E.L. Classification: F12, R23, J61, R12, R13, D91, D84, C61

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1 Introduction

Since long economists have been interested in the study of the distribution of economic activities across the geographical space. The building of the European Union and the several *regional policy issues* which come along, have undoubtely contributed to boost interest in the field. As an illustration of this is the flourishing development of works in the so-called *new economic geography* - surveyed lately by Ottaviano and Puga (1998) and by Fujita and Thisse (1996) - as well as in the *regional labor market literature* - see Blanchard and Katz (1992), Fatas (1998) and Hojvat (1998) -.

Recent works by Krugman (1991a, 1993, 1996) have been particularly useful for understanding how increasing returns and labor mobility affect regional convergence. The basic framework is the following. There are two sectors: manufacturing and agriculture. Manufacturing employs mobile workers and agriculture immobile peasants. Consumers buy the manufacturing varieties on monopolistically competitive regional markets and the agricultural good on a competitive national market. Scale returns at the firm level contributes to regional divergence. This is because the more workers in a region, the more varieties in that region, and the higher their utility as they value variety, see Fujita *et al.* (1999). In turns this triggers additional inflows of workers in the region. On the other hand, the immobility of peasants contributes to regional convergence because firms locate close enough to the local markets they supply, so as to avoid prohibitive transport costst when supplying the immobile peasants in the unagglomerated areas.

In the case of a two-region model, conditions leading to convergence or divergence are related to the relative importance of increasing returns, transport costs and the labor proportion in the total population as shown by Krugman (1991a). In a multi-location version of the same model, numerical simulations suggest that multiple agglomerations systematically emerge and are roughly evenly spaced across the landscape, see Krugman (1993). In a continuous location version of his model, Krugman (1996) showed that the economy always displays regional divergence. The continuous spatial approach used by Krugman is crucial in that it allows to determine the size of emerging agglomerations which undoubtedly is one the main relevant spatial features of an economy. In his work, Krugman characterized the shape of the emerging agglomerations by performing numerical computations of the preferred wavelength.

The work of Mossay (2003) differs from Krugman (1996)'s work in one respect only: the type of spatial adjustment. While migration is assumed to be local in Mossay (2003), it is assumed to be global in Krugman (1996). Under global spatial adjustments, the size of the emerging agglomeration is determined in terms of the taste for variety, the proportion of the manufacturing population and the transport cost. In contrast with this, under local spatial adjustments, the spatial modes of which the amplification factors are the highest, turn out to be small spatial scale modes. Specifically, the amplification factor is the highest for the spatial mode with infinite frequency. Though the spatial adjustments differ in these two continuous spatial models, both spatial economies always diverge. It results that scale economies at the firm level and free mobility (either local or global) of workers contribute to spatial divergence. This means that the spatial divergence result holds, regardless of the spatial foresight ability of agents.

Works such as Krugman (1996) or Fujita *and al.* (1999) show how to derive central features of a global economy such as what a flow depends on, or *endogenous spatial scales*, ... They also make clear the need to develop appropriate methods to deal with the Partial Differential Equations (PDEs) which govern the evolution of a spatial economy. These studies are undoubtedly a step towards the comprehension of the functioning of a global economy. Yet, in this literature, myopic behavior is assumed on behalf of agents. Migration is based on current available returns. A consequence is that migration flows are positively correlated with spatial return differentials. However, in reality, agents are interested not only in current available returns but also in the returns they expect in the future, see Krugman (1991b). The role of expectations turns out to be crucial. It has been shown in two-country models that expectations can give rise to self-fulfilling prophetics: when discounting is low enough, that is when future matters, the steady-state of the economy is determined by the expectations that agents make, see Krugman (1991b), Ottaviano (1999), or Ottaviano and Thisse (2000).

Though the *temporal role of the rational expectation assumption* has been somewhat explored in the literature concerning two-country models, we are not aware of any attempt to explore the *spatial role that rational behavior* may have. So as to fill up this gap, we build a model of rational workers who choose what to consume over time, as well as which spatial itinerary to follow.

Our results are the following. We show that, like in Krugman (1996) and Mossay (2003), spatial divergence always occur. This reemphasizes the role of the local market structure on the convergence process: *scale economies* *at the local level and free mobility contribute to spatial divergence.* However, unlike in the corresponding myopic case studied by Mossay (2003), *the size of agglomerations increases with the taste for variety and the proportion of the manufacturing population, and decreases with transport costs.* The *role of rational adjustments with respect to myopic adjustments is thus to distort the relationship between the amplification factor and the wavelength.*

In much of the existing economic literature, geography is summarized by two or a finite number of countries, ie. Krugman (1991a, 1993). In the framework developed here, a *field approach* - or equivalently, a *continuous spatial approach* - is used to describe spatial flows. The need to rely on a spatio-temporal description of the economy has been re-emphasized lately in the new economic geography by Krugman (1996), Fujita *and al.* (1999), and in the regional labor market literature by Quah (1996). As well described by Quah, regions are geographical units spread out on a two-dimensional map. Their economic performance can be represented as a distribution - a mathematical surface - over their physical geography. In this context, the field approach consists in describing the evolution of this distribution over time. While Krugman relies on the field approach to describe the emergence of agglomerations, Quah uses it to describe the distribution dynamics of European regions. Here, the field approach will allow us to describe not only cross-political-border flows as it is done in discrete models, but also internal flows of people. More importantly, we will derive how these flows are generated in a continuous spatial economy. Empirical work supports the idea that contiguous geographical interactions play an important role in explaining local variations, see Quah (1996). So as to incorporate that aspect into our model, migration is assumed to be local. This implies that migration flows are dependent on contiguous return differentials.

In this paper, trade modelling is built on the continuous version of Krugman's model (1996). Along a one dimensional geographical space, there is a monopolistically competitive manufacturing sector employing mobile workers and a perfectly competitive agricultural sector employing immobile peasants. Flows of goods take place on the international scene. Chamberlinian imperfect competitive economies are allowed to trade and product differentiation makes trade desirable. Specifically, product differentiation leads to gains from trade even when economies have identical consumptions tastes, production technology, and factor endowments.

We believe that there is a sharp distinction between consumption and migration decisions. While adjusting consumption is costless, migration is costly. So as to capture this distinction, *prices on markets will form instantaneously* as in the Walrasian tradition and *migration is modeled as an adjustment process*. Here, the migration decision is an investment decision, so that agents choose to migrate up to the point where the marginal cost of moving is equated to the marginal present value of moving, see Sjaastad (1962). In this sense, it constrasts with existing continuous spatial models, where the migration decision is based on current returns only.

When looking at economic geography problems, economic theorists and geographers generally adopt two different approaches. The economic theorist wants to make a description of the spatial economy which is consistent with the *rational choice* of individual agents. So as to provide microeconomic foundations to economic geography models, economists rely on an *agent-description* of the economy. This leads to a *temporal description of the behavior of agents*. The model of Townsend and Wallace (1982) is very illustrative of the agent-description used in general equilibrium models when addressing spatial issues. In our context, this corresponds to knowing how the consumption of a worker evolves over time and how his migration costs are balanced over time. On the other hand, when describing a spatial economy, the crucial point for geographers is to describe it within its *geographic scope*. When describing what happens in a location, geographers rely on a *space-description* of the economy. This leads to a *temporal description of locations*. The model of Tobler (1981) relies on a field approach to describe how geographic mobility may affect the distribution of people over space. This is very illustrative of the spatial-decription used in geography. In our context, this corresponds to describing how migration flows in a given location evolve over time, and how the consumption of workers in a given location evolves. Thus, according to the microeconomic or geographic feature we want to focus on, one approach a priori seems more appropriate than the other. In the field models mentioned above [Krugman (1996) or Mossay (2003)], the role of the local market structure on spatial convergence is studied by relying on a space-description of the economy. In these works, *the space-description is obtained in a straightforward manner because of the myopic behavior* -lack of foresight- *assumption*. Agents relocate in terms of the current state of the spatial economy only. In contrats with this existing literature, *it turns out that both the agent- and the space- descriptions are needed to analyze the role of expectations* on spatial convergence. So as to provide microeconomic foundations, i.e. to state the consumption-migration problem at the individual level, we rely on the agent-description of the economy, referred to as the *Lagrangian approach*. Because the spatial convergence issue is clearly a geographic feature of the spatial economy, the spatial-description of the economy, referred to as the *Eulerian approach*, seems more appropriate when dealing with convergence.[1]Even though the formalisms used in these two approaches differ, they should be two ways to describe the same physical spatial economy. What follows is a heuristic argument supporting this equivalence. Suppose that the agent-description of the economy is known, or equivalently that the temporal behavior of all workers in the economy in terms of their consumption and their location at any time, is known. You can then easily imagine what happens in a given location. To do so, perform the following operation. At any time, count the number of workers who are in that location. You can then describe the temporal flows in that location, how the number of workers varies over time, and, since the consumption for each worker is known, what consumption is made in that location over time. Repeat this operation for all locations, and you get the spatial-description of the economy.

To study the convergence/divergence process in a continuous space, we need an adequate tool to analyze the spatial stability of the uniform long-run equilibrium. We will apply the normal mode method which a standard linear stability method in the hydrodynamic stability literature, see Drazin and Reid (1991). The general idea is to find the conditions under which a small spatial perturbation is stable or not. To do so, the spatial perturbation is decomposed as a sum of elementary periodic perturbations. We then study whether each of the elementary periodic perturbation grows or is damped over time. If at least one of these elementary perturbations grows over time, then the long-run equilibrium is unstable. This technique has been applied by Krugman and Venables (1996) to study a spatial model of international specialization, and by Krugman (1996) to perform numerical computations of the preferred wavelength of emerging agglomerations, that is the wave-length of the dominant unstable perturbation. The corresponding discrete technique has also been used by Papageorgiou and Smith (1983). Their purpose was to find the conditions under which some spatial externality may lead the spatial uniform equilibrium to be unstable. In this paper we will use the normal mode method to determine how agglomerations emerge from local instability of a uniform long run equilibrium.

[1]The Lagrangian and Eulerian terminology is borrowed from the physics of continuum media. In that literature, they refer respectively to a particle, and a space-description.

The rest of this paper is organized as follows. Section 2 lays out the economic environment. We describe the short-run equilibrium of the spatial economy in section 3. The worker's intertemporal decision problem is described in section 4. We describe the temporal behavior of a worker across locations, in terms of given spatio-temporal prices. In section 5 we discuss the spatial description of the economy and introduce the Eulerian formalism to describe the temporal evolution of a location, in terms of given spatio-temporal prices. In section 6 we define a market equilibrium of the spatial economy. In section 7 we define long-run equilibria and study spatial divergence away from a long-run equilibrium. Finally, section 8 summarizes the main results.

2 The Economic Environment

In this section, the economic environment is described. We consider a spatial economy with a continuum of locations $s \in]-\infty, +\infty[$. Time is denoted by $t \in]t_0, \infty[$. There are two types of consumers: mobile workers and immobile peasants. The densities of workers and peasants in location s at time t are denoted respectively by $L(s,t)$ and A. There are two sectors in the economy : the manufacturing sector, which exhibits increasing returns, and agriculture, which has constant returns.

Assumption 1 (Preference) *Consumers have a Cobb-Douglas utility function.*

A consumer (either a worker or a peasant) at location s and time t enjoys a Cobb-Douglas utility from two types of goods

$$U(s,t) = C_M^{\mu}(s,t) C_A^{1-\mu}(s,t) \tag{1}$$

where μ is the share of manufactured goods in expenditure, C_A the consumption of the agricultural good, and C_M the consumption of the manufactured aggregate which is defined by

$$C_M(s,t) = \left[\int_{-\infty}^{+\infty} \left(\int_0^{n(z,t)} c_i(z,s,t)^{\frac{\sigma-1}{\sigma}} di \right) dz \right]^{\frac{\sigma}{\sigma-1}}$$

where $n(z,t)$ is the density of manufactured varieties available at location z at t, $c_i(z,s,t)$ is the consumption of variety i produced at z, and $\sigma > 1$ is the elasticity of substitution among manufactured varieties.

In the model below, our intention is to focus on how location and consumption decisions are taken simultaneously. We will follow the approach widely used in the economic geography literature by disregarding issues related to intertemporal trading, see Krugman (1991a) or Ottaviano (1998), and in related domains, see Matsuyama (1991) or Krugman (1991b).

Assumption 2 (Intertemporal Trade) *There is no intertemporal trade.*

Equivalently, this means that agents spend all their current income on current consumption.

Farming Activity

Farming is an activity that takes place under constant returns to scale

$$Q_A(s,t) = A(s,t)$$

where $A(s,t)$ is the density of peasants needed in location s at time t to produce Q_A units of the agricultural good in s at t. The profit-maximizing behavior leads to

$$W^a = p_A$$

where p_A is the price of the agricultural good.

Manufacturing Activity

Manufacturing variety i involves a fixed cost and a constant marginal cost. Economies of scale are thus realized at the firm level

$$L_i(s,t) = \alpha + \beta Q_{M,i}(s,t)$$

where $L_i(s,t)$ is the amount of labor used in location s at time t to produce $Q_{M,i}$ units of variety i in s at t.

Transportation costs affect manufactured goods and take the Samuelson iceberg form. More precisely, when the amount Z of some variety is shipped from location z to location x, then the amount X of that variety which is effectively available at location x is given by

$$X = Z \exp\left[-\tau \left|x - z\right|\right]$$

where τ is the tranport cost per unit of distance, and $|x - z|$ the distance between locations z and x.

Assumption 3 (Monopolistic Competition)

We assume that there is a large number of manufacturing firms. Each of them produces a single variety, and faces a demand with a constant elasticity σ. The optimal pricing behavior of any firm at location s and time t is therefore to set the price $p_i(s,t)$ of variety i at a fixed markup over marginal cost

$$p_i(s,t) = \frac{\sigma}{\sigma - 1}\beta W(s,t) \tag{2}$$

where $W(s,t)$ is the worker wage rate prevailing at location s at time t.

Firms are free to enter into the manufacturing sector so that their profits are driven to zero. Consequently, their output is given by

$$Q_{M,i}(s,t) = \frac{\alpha}{\beta}(\sigma - 1) \tag{3}$$

Since all varieties are produced at the same scale, the density $n(s,t)$ of manufactured goods produced at each location is proportional to the density $L(s,t)$ of workers at that location,

$$L(s,t) = \int_0^{n(s,t)} L_i(s,t)di = \alpha\sigma n(s,t) \tag{4}$$

This relationship is crucial. When some workers move to a new location, they no longer produce the same mix of products but other differentiated products. As a result, varieties produced in one location are different from those produced in any other location. Since consumers are characterized by a preference for variety, they will buy from all locations so that *trade of varieties between any location pair will arise.*

In addition to enjoying consumption of goods, each worker can travel along the real line s as time evolves. We believe that there is a sharp distinction in costs in adjusting consumption and location over time. While *adjusting consumption is costless, migration is costly.* This distinction makes the migration decision quite distinct from the consumption decision. So as to capture this distinction, *prices on markets will form instantaneously* as in the Walrasian tradition and *migration will be modelled as an adjustment process.* Furthermore, we suppose that when dealing with the migration decision, workers consider only *local migration* opportunities, as is the case in Hotelling (1921) or Mossay (2003). By local migration we mean that workers are more likely to migrate to nearby neighbhorhoods than to more distant locations.

Assumption 4 (Migration) Migration is local.

Equivalently, this means that, in our model, agents cannot have instantaneous access to a location which is at a finite distance from where they are, unlike in Krugman (1996) or Krugman and Venables (1995). The local migration behavior is consistent with empirical findings according to which the intensity of migration flows declines with the increasing distance between origin and destination, see Ravenstein (1885), Shaw (1975), or Wheeler *et al.* (1998). By making the local migration assumption we tend to focus on what we believe to be the most important part of migration flows.

Even though long range and international migrations play an important role in explaining the growth of large cities, the local aspect of migration is found back in many migratory patterns in world history. A first illustratory case is the westward movement of Anglo-Saxons and other Europeans in the United States during the 19th century. They gradually moved away from the East coast towards the West coast. *This movement was a slow and continuous westward process.*[2] Another case where the local aspect is found, is the rural migratory patterns in France during the 18th and 19th centuries, see Ariès (1979). *Peasants migrated over short distances* and gave rise to numerous small-size cities (called "bourgs" in French) spread all over France. In view of the westward migration in the US and the early phase of the rural exode in France, assumption 4 does not appear completely unrealistic.

Let us denote the location of a worker at time t by $s(t)$. An implication of assumption 4 is that the spatial itinerary $s(t)$ followed by each worker in the spatial economy, is continuous. The time derivative ds/dt can be interpretated as the travelling speed of a worker. The travelling speed assesses how fast a worker adjusts his itinerary over time, and therefore can constitute a natural measure for migration costs. In our analysis, costs are assumed to be a quadratic function of the travel speed. This corresponds to the migration cost assumption usually made in the adjustment dynamics literature, see Mussa (1978), Krugman (1991b), Baldwin and Venables (1994).

Assumption 5 (Migration Costs) *Migration costs are a quadratic function of the travel speed.*

[2] A field-approach was proposed by Hotelling (1921) to describe these migration flows. Migration was assimilated to a pure diffusion process. However, no microeconomic foundation was proposed by Hotelling.

As pointed out in the adjustment dynamics literature, concavity of the cost function facilitates the existence of a solution to the optimization problem. In that literature, adjustment takes place between two countries and aggregate migration costs are a quadratic function of the aggregate migration flows between the two countries. The microeconomic foundations which are consistent with this aggregate modelling, are the following. When moving from one country to the other, a migrant incurs a cost which depends on the intensity of migration flows and imposes a negative externality on the other migrants by congesting the migration process. So the larger the number of migrants, the larger the migration cost. Furthermore, sufficient heterogeneity among migrants is needed so as to explain why migrants are not moving all at the same time from one country to the other. In contrast with this approach, *we do require neither the presence of externality nor of heterogeneity in the migration process.* In our model we follow the standard microeconomic approach. We formulate the decision problem at the individual level, and the homogeneity of workers, is assumed. The total intertemporal migration cost for a worker following the itinerary $s(t)$, is thus proportional to $\int_0^T [ds/dt]^2 \exp(-rt)dt$, where r denotes the discounting factor..

In the present model, workers are interested not only in current returns but also in the returns they expect in the future. Although the role of expectations has been examined in economic geography two-country models [e.g., Ottaviano (1999), Ottaviano and Thisse (2000), Mussa (1978), or Baldwin and Venables (1994)], the forward-looking approach has never been dealt with in a continuous spatial setting, at least to our knowledge. So far, only myopic behavior has been studied in models involving continuous spatial economies, see Sonnenschein (1981, 1982), Mossay (2001, Chapters 2 and 3), Krugman (1996) or Krugman and Venables (1995). Here, in our model, we endow workers with forward-looking and rational expectation abilities.

Assumption 6 (Expectations) Workers are forward-looking and have rational expectations.

In our context, since there is no uncertainty, the rational expectation assumption corresponds to perfect foresight.

3 Short-Run Equilibrium

Before moving to the worker's intertemporal decision problem, and thereafter to the dynamic equilibrium, let us make precise what we mean by a short-run equilibrium of the spatial economy ($\mathbb{E}$). Following Krugman (1996), we suppose that workers are immobile so that the spatial distribution $L(s,t)$ is fixed.

Total income Y at location s and time t is given by

$$Y(s,t) = Ap^A + L(s,t)W(s,t) \tag{5}$$

where A is the constant density of peasants, $L(s,t)$ the density of workers, and p^A the price of the agricultural good.

Workers are not interested in nominal wages but rather in utility levels. In order to consume at x, one unit of variety i produced at location s, $\exp[\tau\,|x-s|]$ units must be shipped so that the delivery price is $p_i(s,t)$ $\exp[\tau\,|x-s|]$. The price index of manufactured aggregate to consumers at location x, $\Theta(x,t)$, is obtained by computing the minimum cost of purchasing one unit of the manufactured aggregate $C_M(x,t)$

$$\Theta(x,t) = \left[\int_{-\infty}^{+\infty}\int_0^{n(z,t)} p_i(s,t)^{-(\sigma-1)}\exp[-\tau(\sigma-1)\,|s-x|]ds\right]^{-\frac{1}{(\sigma-1)}}$$

Using the pricing rule (2) and relation (4), $\Theta(x,t)$ may be rewritten as

$$\begin{aligned}\Theta(x,t) &= \beta\sigma/(\sigma-1)\,(\alpha\sigma)^{1/(\sigma-1)}\\ &\left[\int_{-\infty}^{+\infty} L(s,t)W(s,t)^{-(\sigma-1)}\exp[-\tau(\sigma-1)\,|s-x|]ds\right]^{-\frac{1}{(\sigma-1)}}\end{aligned} \tag{6}$$

The consumption of variety $i\in[0,n(s,t)]$ produced at s may be expressed for workers and peasants located at x as follows

$$\begin{aligned} c_i^w(s,x,t) &= \mu W(x,t)p_i(s,t)^{-\sigma}\exp[-\tau(\sigma-1)\,|s-x|]\Theta(x,t)^{\sigma-1}\\ c_i^a(s,x,t) &= \mu p^A p_i(s,t)^{-\sigma}\exp[-\tau(\sigma-1)\,|s-x|]\Theta(x,t)^{\sigma-1}\end{aligned}$$

The total demand for variety i produced at s is obtained by integrating the demand for that variety of all the consumers along the real line,

$$
\begin{aligned}
Q^D_{M,i}(s,t) &= \int_{-\infty}^{+\infty} [L(x,t)c_i^w(s,x,t) + Ac_i^a(s,x,t)]dx \\
&= \int_{-\infty}^{+\infty} \mu(L(x,t)W(x,t) + Ap^A)p_i(s,t)^{-\sigma} \\
&\quad \exp[-\tau(\sigma-1)\left|s-x\right|]\Theta(x,t)^{\sigma-1}dx
\end{aligned}
$$

By the total income expression (5), we get

$$
Q^D_{M,i}(s,t) = \int_{-\infty}^{+\infty} \mu Y(x,t)p_i(s,t)^{-\sigma}\exp[-\tau(\sigma-1)\left|s-x\right|\Theta(x,t)^{\sigma-1}dx \quad (7)
$$

The market-clearing condition for variety i produced at s is obtained by equating the demand $Q^D_{M,i}$ (7) and the supply $Q_{M,i}(s,t)$ (3) of that variety,

$$
p_i(s,t) = [\mu\frac{\beta}{\alpha(\sigma-1)}\int_{-\infty}^{+\infty} Y(x,t)\Theta(x,t)^{\sigma-1}\exp[-\tau(\sigma-1)\left|s-x\right|]dx]^{1/\sigma}
$$

Because of the pricing rule (2),

$$
\begin{aligned}
W(s,t) &= (\frac{\sigma-1}{\beta\sigma})[\mu\frac{\beta}{\alpha(\sigma-1)}]^{1/\sigma} \\
&\quad [\int_{-\infty}^{+\infty} Y(x,t)\Theta(x,t)^{\sigma-1}\exp[-\tau(\sigma-1)\left|s-x\right|]dx]^{1/\sigma} \qquad (8)
\end{aligned}
$$

The manufacturing wage $W(s,t)$ is the wage prevailing at location s and time t such that firms at s break even given the income levels $Y(x,t)$, price indices $\Theta(x,t)$ in all locations and the transportation cost technology.

The indirect utility $\Omega(s,t)$ of a worker located at s is then obtained through (1) by

$$
\begin{aligned}
\Omega(s,t) &= U(\Theta(s,t),W(s,t)) \\
&= C_M^{\mu}(\Theta(s,t),W(s,t))C_A^{1-\mu}(\Theta(s,t),W(x,t)) \\
&= (\mu W(s,t)/\Theta(s,t))^{\mu}[(1-\mu)W(s,t)/p^A]^{1-\mu} \\
&= \mu^{\mu}(1-\mu)^{1-\mu}(p^A)^{-(1-\mu)}\Theta^{-\mu}(s,t)W(s,t) \qquad (9)
\end{aligned}
$$

Definition 1 *A short-run equilibrium at location s and time t, is defined, taking $L(s,t)$ as given, by equations (5), (6), (8), (9).*

4 The Worker's Decision Problem

In this section we introduce the worker's decision problem. We want to study how consumption and migration decisions are simultaneously taken in terms of given intertemporal prices - that is in terms of a spatial distribution of prices which evolves over time -. Therefore we will follow a worker along the itinerary he chooses and describe his behavior. By doing so we introduce the Lagrangian description.

Consider a worker initially located in s_0 at time t_0. Since he cares about the consumption $C_M(t)$, $C_A(t)$ he gets at time $t \in [t_0, T]$, he is willing to choose the local markets $s(t)$ where to trade. From time t_0 on, because prices can vary across locations, he may find it advantageous to move across locations.

The problem that a worker initially located in s_0 faces, is to choose an optimal stream of consumption $C_M(t)$, $C_A(t)$ and to select an optimal spatial itinerary $s(t)$, so as to maximize his intertemporal well-being subjected to his budget constraint

$$\max_{s(t), C_M(t), C_A(t)} \int_{t_0}^{\infty} \left[U(C_M(t), C_A(t)) - \frac{1}{2k} \dot{s}^2 (t) \right] \exp(-rt) \, dt$$

$$\begin{aligned} \text{st.} \quad \Theta(s(t), t) C_M(t) + p_A(s(t), t) C_A(t) &= W(t) \ , \forall t \in [t_0, \infty[\\ s(t_0) &= s_0 \end{aligned} \qquad (\mathbb{P})$$

where k measures the migration cost incurred by a worker and $U = C_M^{\mu}(t) C_A^{1-\mu}(t)$.

Definition 2 (Lagrangian Description) *The Lagrangian description of the spatial economy is an agent description. It describes the behavior of a given agent as time evolves.*

So as to see how we can use the Lagrangian approach to solve the problem for all workers. Consider the situation at time t_0. Workers are distributed along the real line according to the initial distribution $L_0(s)$. Then consider a location s. By solving problem $(\mathbb{P})$ with $s_0 = s$, one obtains the behavior of workers who were initially located in s. By doing so, we can describe the itinerary path and the demand function of these workers, moving initially from s. By applying the same procedure to workers initially located in other initial locations, we can get describe the behavior of any worker over time. Note that to identify a worker, we need to know his initial location s_0. Our reasoning, so far, has relied on an agent-description. It has aimed at describing what happens to *a given worker* identified by his initial location, as time evolves.

In what follows, we may decompose the consumer's problem ($\mathbb{P}$) into two sub-problems ($\mathbb{P}_1$) and ($\mathbb{P}_2$): a pure consumption problem and a pure migration problem. While the pure consumption problem concerns the choice of an optimal consumption in a given local market, the pure migration problem concerns the choice of an optimal spatial itinerary to follow, conditionally on the fact that local consumption is optimal. The reason for which the decomposition is possible, is the no saving assumption.

The Pure Consumption Problem

This problem may be stated as following. What is the optimal consumption a worker should choose when he is in a given local market at a given date ? To address this question, consider a worker located in the local market $s(t)$ at time t. He makes his consumption decision $C_M(t)$, $C_A(t)$ taking the price $\mathbf{\Theta}(s(t), t)$ which prevails in the local market he is in, as given. Hence, his optimal local consumption $C_M(t)$, $C_A(t)$ at time t is determined as follows

$$\max_{C_M(t), C_A(t)} U(C_M(t), C_A(t))$$

$$\text{st.} \qquad \Theta(s(t), t))C_M(t) + p_A C_A(t) = W(t) \tag{$\mathbb{P}_1$}$$

The optimal demand functions are written

$$\begin{aligned} C_M(t) &= C_M(\Theta(s(t), t)) \\ C_A(t) &= C_A(\Theta(s(t), t)) \end{aligned} \tag{10}$$

The Pure Migration Problem

This problem may be stated as following. What is the optimal spatial itinerary a worker should choose once local consumption is supposed to be optimal ? To address this question, suppose that local consumption is optimal - that is, determined by (10). By selecting a trajectory along the real line s, a worker goes through some local markets, enjoying optimal local consumption, and incurring migration costs associated with how fast he moves along his trajectory. Hence the clear dependence of his intertemporal well being upon the itinerary he chooses. Consequently, a worker will select his spatial itinerary $s(t)$ so as to

$$\max_{s(t)} \int_{t_0}^{T} \left\{ U(C_M \left[\Theta(s(t),t\right], C_A \left[\Theta(s(t),t)\right]) - \frac{1}{2k}\, \dot{s}^2\, (t) \right\} \exp(-rt)dt$$

$$\text{st.} \qquad s(t_0) = s_0 \tag{$\mathbb{P}_2$}$$

Timing of the consumption and migration decisions

The decomposition of problem ($\mathbb{P}$) into the two sub-problems ($\mathbb{P}_1$) and ($\mathbb{P}_2$) emphasizes the distinction between consumption and migration decisions. These decisions can be considered as being made in a sequential order. In a local market, adjusting consumption is costless. This makes the consumption decision the "easiest" one to make. Because of this, the choice of consumption $C_M(t)$, $C_A(t)$ is the first an agent makes by solving ($\mathbb{P}_1$). In contrast, migration is costly, and requires balancing migration costs $\dot{s}\,(t)^2$ in an optimal way over time. This is why the choice of the optimal itinerary $s(t)$ is made thereafter by solving ($\mathbb{P}_2$).

Remark The pure migration problem ($\mathbb{P}_2$) for a worker may be rewritten as

$$\max_{s(t)} \int_{t_0}^{T} L(t, s, \dot{s})dt$$

$$\text{with } s(t_0) = s_0 \tag{11}$$

where we have defined the Lagrangian function by $L(t, s, \dot{s}) \equiv \exp(-rt)[U\,[s,t] - 1/(2k)\left(\dot{s}\right)^2]$. The migration decision ($\mathbb{P}_2$) can be identified as a problem in the calculus of variations where the end-point is variable. In problem ($\mathbb{P}_2$), the Lagrangian function L is non-autonomous but separable in s, $\dot{s}$. For economic problems of the same canonical form as (11), see for instance the Ramsey neoclassical growth model in Ekeland (1986), or Tabuchi's optimal spatial planning (1986).

4.1 Necessary Conditions

In this section we derive necessary conditions the spatial itinerary should meet so as to solve program ($\mathbb{P}_2$). Examples where these necessary conditions are also sufficient, and thus lead to an optimal itinerary, are largely discussed in Mossay (2001, Chapter 4). The existence issue as well as the derivation of the Euler-Lagrange conditions and other regularity considerations can also be found in Mossay (2001, Chapter 4).

What we would like to determine is how a worker chooses his spatial itinerary, conditionally on optimal local demand in any location at any time. Or equivalently, we are interested in how, over time, a worker balances the benefits of moving to higher utility level locations and the associated migration costs. Does he choose a monotonic spatial itinerary or is he about to go back and forth. As migration is motivated by higher utility levels, the worker's utility level is about to be "relatively" high at the end of his lifetime. One could think a priori that a worker should incur the migration costs during the corresponding life period.

4.1.1 First Order Condition

The worker's decision concerning migration is to select an itinerary $s(.)$ which is a function of time. The integrand depends on the itinerary $s(.)$ and the displacement speed $V(.) \equiv ds/dt$. The first order condition (FOC) associated with $(\mathbb{P}_2)$ corresponds to the Euler-Lagrange equation

$$\frac{d}{dt}\left[-\frac{1}{k}\frac{ds}{dt}\right] - \frac{dU}{ds} + \frac{r}{k}\frac{ds}{dt} = 0$$

which may be rewritten as

$$\frac{d^2}{dt^2}s = -k\frac{dU}{ds^h} + r\frac{ds}{dt}$$

$$\ddot{s}\,(t) = -k\partial_s U + r\,\dot{s}\,(t) \tag{12}$$

Equation (12) is a second-order ordinary differential equation (ODE). Typically, two additional conditions are required to determine a solution.

When a worker decides to migrate, he is ready to incur some migration costs because he expects to benefit from higher utility levels in the future. Therefore there is no reason for a worker to decide to incur migrations costs when $t \to \infty$. This is stated in the following additional necessary condition.

4.1.2 Terminal Condition

The free terminal condition associated with $(\mathbb{P}_2)$ is

$$\lim_{t\to\infty} V(t)\exp(-rt) = 0 \tag{13}$$

Equation (13) tells us that at the end of his lifetime, a worker does not want to migrate anymore, since migrating is no longer beneficial.

4.1.3 Economic Interpretation

At any time during his lifetime, a worker selects his displacement speed $V(t)$ so that migration is carried out to the point where its marginal cost is equated to its marginal value

$$\begin{array}{c}\text{Marginal Cost}\\ \text{of Migration}\end{array} = \begin{array}{c}\text{Marginal Value}\\ \text{of Migration}\end{array}, \, t \in [t_0, T]$$

Proof. Integrate the Euler-Lagrange equation (12) from t to T, and use the free terminal condition (13). ■

The time derivatives $\dot{s}$ and $\ddot{s}$ are respectively the displacement speed and the acceleration of the agent. When there is no discounting ($r = 0$), equation (12) means that *a worker decelerates in the direction of the spatial gradient of utility* $\nabla_s U$. This result is strongly in contradiction with myopic migration behavior where the migration speed $V(.)$ is proportional to the spatial gradient of utility $\nabla_s U$, see Sonnenschein (1981, 1982), or Mossay (2001, Chapters 2 and 3).

Many examples of the intertemporal problem ($\mathbb{P}$, $\mathbb{P}_1$, $\mathbb{P}_2$) and their closed-form solutions are studied in Mossay (2001, Chapter 4).

5 From an Agent Description to a Location Description

So far, we have described what happens to workers who are initially at a given location s_0. We have described the optimal itinerary $s(t)$ they follow over time, and the optimal demand function C_M,C_A they choose in terms of the time-evolving price distribution $\Theta(s,t)$. We know where these workers are located at any time, and what their demand function in each local market they go through, is. To solve the entire problem, we need to tell what happens to each worker in the economy at any time. To do so, two approaches are possible. The first one is an agent-description, referred to as the Lagrangian approach. It tells you how the behavior of *a given agent* evolves over time. The second one is a space-decription, referred to as the Eulerian approach.

It tells you what happens in *a given location* as time evolves. We will see that a link exists between the two approaches.[3]

The Lagrangian description is the one which first enters one's mind, in the sense that it describes explicitely what decisions a given agent is about to make over time. This approach is particularly appropriate when one wants to focus on the agent dynamics. For instance, patterns of consumption or migration during the lifetime are within this scope. Beside these agent-based issues, other characteristics of the economy are of great interest, namely the spatial features of an economy. Among these, the spatial convergence issue and the emergence of agglomerations are of major importance. The following related recurrent questions must be put. Will the economy converge toward a long-run equilibrium? If so, will the convergence process be monotonic or cyclical? When this happens, is the long-run equilibrium uniform or are agglomerations about to emerge. Furthermore, in this latter case, what can be said about the agglomerations (ie. about their size). To deal with these questions, we need to assess how all the individual actions about migration interact through the market mechanism to give rise to a spatial order, or eventually to an apparent complexity. In order to address these issues, it seems natural to rely on a space-description of the economy. To do so, we will formulate our problem in terms of spatial distributions describing the workers $L(s,t)$, their migration speeds $V(s,t)$, their consumptions C_M, C_A and the price system $\Theta(s,t)$. This will lead to the Eulerian description of the economy.

Definition 3 (Eulerian Description) *The Eulerian description of the economy is a spatial description. It describes what happens in a given location as time evolves.*

To make this clear, suppose that we are interested in describing what is going on, in location s. In this case, we may want to tell how the number of workers changes over time in that location, or how the optimal demand of workers in s vary over time in terms of $\Theta(s,t)$. Let us observe how this description differs from the Lagrangian one. In the Eulerian description, we do not describe the evolution of agents during their lifetime, but what happens in *a given location*. To further illustrate this, imagine that we describe the

[3]The Lagrangian and Eulerian terminology is borrowed from the physics of continuum media. In that literature, they refer respectively to a particle-description, and a space-description.

time-evolution of the consumption of workers in s. By doing so, we actually refer to consumption patterns of different workers. This is because the worker population in s changes as time evolves.

5.1 The Lagrangian and the Eulerian Approaches: the Link

From Lagrange to Euler

Indeed, both the Lagrangian and the Eulerian approaches should be equivalent. Suppose the problem is solved in the Lagrangian formalism. Then the time-behavior of each worker is known. So as to deduce the Eulerian description of the spatial economy, we need to describe what happens in *a given location* as time evolves. According to the Lagrangian approach, we know at time t where each worker identified by its inital location s_0, decides to locate. We can deduce from this how many of them are actually in location s, by summing up all workers in s at t. By doing so, we can describe how the number of workers varies in location s. What we have got is precisely the Eulerian description of the evolution of workers in s.

Lagrangian and Eulerian Variations

The migration speed of a worker, who is located in s at time t, is denoted $V(s,t)$. There are two ways to describe time-variations of V. The most natural way is probably to tell how the migration speed of this given worker varies over time: this is the Lagrangian derivative $dV/dt(s,t)$. Another way, is to tell how the migration speed for workers located in s evolves over time: this is the Eulerian derivative $\partial_t V(s,t)$. More generally, we will use the Lagrangian and the Eulerian derivatives to refer to time-variations occuring respectively during the lifetime of a worker and in a given location.

The link between these two derivatives is formally[4]

$$d_t(.) = \partial_t(.) + V\partial_s(.) \tag{14}$$

The meaning of this relationship is that agents' holdings are transported by agents' flows.

Remember the *Lagrangian form* of the Euler-Lagrange equation (12)

$$\frac{d}{dt}V(t) = -k\frac{d}{ds}U + rV(t) \tag{15}$$

[4]See a general reference in fluid dynamics (i.e., Batchelor (2000)).

By using the relationship (14) in equation (15), we can obtain the *Eulerian form* of the Euler-Lagrange equation

$$\partial_t V(s,t) + V(s,t)\partial_s V(s,t) = -k\partial_s U\left[\Theta(s,t)\right] + rV(s,t) \tag{16}$$

Comparison of the Lagrangian and Eulerian forms

Consider a worker. Equation (15) tells how this worker adapts his travelling speed over time as a response to the spatial gradient of indirect utility in the location he is in. This corresponds clearly to the Lagrangian description of the spatial economy. In contrast, consider a location s. The first left-hand side term $\partial_t V$ of equation (16) tells how the travelling speed of workers coming to that location, evolves over time. This term is given by the spatial gradient of indirect utility in that location, and the transport term $V\partial_s V$ which reflects that, even a time-invariant field $V(s)$ can involve individual motion of workers. This corresponds clearly to the Eulerian description of the spatial economy.

5.2 Evolution Law of Workers

In this section, we want to describe how the spatial density $L(s,t)$ of workers in s evolves over time in response to migration flows determined so far in (16). We show that, once the distribution of migration speed $V(s,t)$ is known at time t, we can deduce how the density L changes at time t.

The evolution law for the worker distribution L will be shown below to be

$$\partial_t L + \partial_s(LV) = 0 \tag{17}$$

where V^h is given by (16).

<u>Interpretation of the evolution law</u>

Equation (17) may be rewritten as

$$\partial_t L = -\partial_s \Phi \tag{18}$$

where $\Phi(s,t)$ is defined by LV, thereby representing the flow of workers through location s at time t. For a reason which will appear clear below, equation (18) is called the conservative form corresponding to equation (17).

So as to interpret equation (17), consider a region $\Gamma = [s_1, s_2]$, and denote its overall worker population at time t by $\mathcal{L}(t)$. By integrating (18) spatially

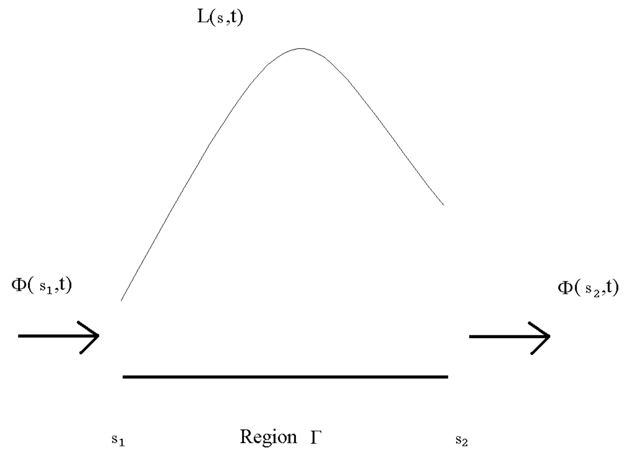

Figure 1: Conservation Law of Labor

over Γ, we have

$$\frac{d}{dt}\mathcal{L}(t) = -(\Phi(s_2, t) - \Phi(s_1, t)) \tag{19}$$

Equation (19) is the *conservation law* of the worker population. It has the following meaning.

Conservation Law Equation (19) asserts that the rate of increase of the worker population in region Γ is equal to the flow of workers into Γ through its borders s_1 and s_2; see Figure **??**.

To be even more illustrative, approximate (19) for small time variations, and get the net increase of the worker population in region Γ during time span $[t, t + dt]$ in terms of flows of workers through borders s_1 and s_2

$$\begin{aligned} \mathcal{L}(t + dt) - \mathcal{L}(t) &\simeq -(\Phi(s_2, t) - \Phi(s_1, t))dt \\ &= (V(s_1, t)L(s_1, t) - V(s_2, t)L(s_2, t))dt \end{aligned}$$

6 Market Equilibrium

In this section, we precise what we mean by a market equilibrium in our context. So as to avoid the difficulties concerning the optimality of extremals, the time horizon under study should be small enough in order for the Euler-Lagrange conditions to be also sufficient, see Mossay (2001, Chapter 4). Indeed this will be the case in section 7 where the time interval under study will turn out to be very small due to the local nature of the performed stability analysis.

Consider a time-sequence of spatial distributions of workers $L(s,t)$, migration speed $V(s,t)$, income $Y(s,t)$, manufacturing price index $\Theta(s,t)$, worker's wage $W(s,t)$ and indirect utility level $\Omega(s,t)$, that is

$$\{L(s,t), V(s,t), Y(s,t), \Theta(s,t), W(s,t), \Omega(s,t)\}_{t\in[t_0,\infty[, s\in]-\infty,+\infty[}$$

This sequence constitutes a market equilibrium if the following conditions are satisfied

Maximization

• the demand for workers and peasants are optimal in each location s, at any time t

$$\begin{aligned} C_M(s,t) &= C_M(\Theta(s,t), W(s,t)); C_A(s,t) = C_A(\Theta(s,t), W(s,t)) \quad , \quad \forall s,t \\ C_M^a(s,t) &= C_M(\Theta(s,t)); C_A^a(s,t) = C_A(\Theta(s,t)) \quad , \quad \forall s,t \end{aligned}$$

• the migration speed for workers is optimal in each location s at any time t

$$\partial_t V + V\partial_s V = -k\partial_s \Omega + rV \quad , \quad \forall s,t$$

• the migration speed for workers satisfies the terminal condition in each location s when t tends to ∞

$$\lim_{t\to\infty} V(s,t)\exp(-rt) = 0 \quad , \quad \forall s$$

Evolution Law of Workers

• the density of workers in each location s at any time t satisfies

$$\partial_t L + \partial_s (VL) = 0 \quad , \quad \forall s,t$$

Market Clearing

• the income, manufaturing price index, wage and indirect utility satisfy the equilibrium conditions in any location s at any time t

$$
\begin{aligned}
Y(s,t) &= Ap_A + L(s,t)W(s,t) \\
\Theta(s,t) &= \beta\frac{\sigma}{\sigma-1}(\alpha\sigma)^{\frac{1}{\sigma-1}}\left[\int_{-\infty}^{+\infty} L(z,t)W(z,t)^{-(\sigma-1)}\exp\left[-\tau(\sigma-1)\left|z-s\right|\right]dz\right]^{-\frac{1}{\sigma-1}} \\
W(s,t) &= \frac{\sigma-1}{\beta\sigma}\left[\frac{\mu\beta}{\alpha(\sigma-1)}\right]^{\frac{1}{\sigma}}\left[\int_{-\infty}^{+\infty} Y(z,t)\Theta(z,t)^{\sigma-1}\exp\left[-\tau(\sigma-1)\left|z-s\right|\right]dz\right]^{\frac{1}{\sigma}} \\
\Omega(s,t) &= \mu^{\mu}(1-\mu)^{1-\mu}p_A^{-(1-\mu)}\Theta^{-\mu}(s,t)W(s,t)
\end{aligned}
$$

6.1 Evolution Equation

Let us make the following change of variables

$$Y(s,t) = \overline{Y}y(s,t); \Theta(s,t) = \overline{\Theta}\theta(s,t); W(s,t) = \overline{W}w(s,t); \Omega = \overline{\Omega}\omega(s,t); L(s,t) = \mu l(s,t);$$

where $\overline{Y}$, $\overline{\Theta}$, $\overline{W}$, $\overline{\Omega}$ are given by

$$
\begin{aligned}
p_A &= \overline{W}; A = 1-\mu \\
\overline{Y} &= Ap_A + \overline{LW} \\
\overline{\Theta} &= \beta\frac{\sigma}{\sigma-1}(\alpha\sigma)^{\frac{1}{\sigma-1}}\overline{L}^{-\frac{1}{\sigma-1}}\overline{W}\left[\frac{2}{\tau(\sigma-1)}\right]^{-\frac{1}{\sigma-1}} \\
\overline{W} &= \frac{\sigma-1}{\beta\sigma}\left[\frac{\mu\beta}{\alpha(\sigma-1)}\right]^{\frac{1}{\sigma}}\overline{Y}^{\frac{1}{\sigma}}\overline{\Theta}^{\frac{\sigma-1}{\sigma}}\left[\frac{2}{\tau(\sigma-1)}\right]^{-\frac{1}{\sigma-1}} \\
\overline{\Omega} &= \mu^{\mu}(1-\mu)^{1-\mu}p_A^{-(1-\mu)}\overline{\Theta}^{-\mu}(s,t)\overline{W}(s,t)
\end{aligned}
$$

We then can rewrite the partial differential equations (PDE) that distributions $l(s,t)$, $V(s,t)$, $y(s,t)$, $\theta(s,t)$, $w(s,t)$ and $\omega(s,t)$ should satisfy for being a market equilibrium

$$
\begin{aligned}
y(s,t) &= (1-\mu) + \mu l(s,t)w(s,t) \\
\theta(s,t) &= \left[\frac{\tau(\sigma-1)}{2}\int_{-\infty}^{+\infty} l(z,t)w(z,t)^{-(\sigma-1)}\exp(-\tau(\sigma-1)\left|z-s\right|)dz)\right]^{-\frac{1}{\sigma-1}}
\end{aligned}
$$

$$
\begin{aligned}
w(s,t) &= \left[\frac{\tau(\sigma-1)}{2}\int_{-\infty}^{+\infty} y(z,t)\theta(z,t)^{(\sigma-1)}\exp(-\tau(\sigma-1)\left|z-s\right|)dz)\right]^{\frac{1}{\sigma}} \\
\omega(s,t) &= \theta^{-\mu}(s,t)w(s,t)
\end{aligned}
$$

$$
\begin{aligned}
\partial_t l + \partial_s(Vl) &= 0 \\
\partial_t V + V\partial_s V &= -K\partial_s\Omega + rV \\
\lim_{t\to\infty} V(s,t)\exp(-rt) &= 0
\end{aligned} \tag{20}
$$

where K has been defined as

$$K \equiv k\overline{\Omega}$$

7 Behavior around a Uniform Long-Run Equilibrium

We now define long-run equilibria of the spatial economy $(\mathbb{E})$.

Definition 4 *A uniform long-run equilibrium (ULRE) is a set of constants* $\{l, V, y, \theta, w, \omega\}$ *which satisfies the system (20).*

In this section we analyze the spatial stability of a long-run equilibrium. The idea is to find the conditions under which a small spatial pertubation is stable or not. We will restrict our attention to perturbations which are in some sense close to the long-run equilibrium. This allows us to focus on the linearized equations of the system. In order to study the time evolution of the spatial perturbation, we decompose it as a sum of elementary periodic perturbations. The reason for doing so is that an arbitrary perturbation may be expressed as a linear combination of periodic perturbations according to Fourier decomposition. For the sake of simplicity, periodic perturbations may be viewed as $\sin(\nu s)$. High (low) values of ν correspond to high (low) frequency perturbations. Later on, we need to introduce more general periodic perturbations called *normal modes*. We then study whether each of these elementary periodic perturbations grows or is damped over time. If at least one of the elementary periodic perturbations is unstable, that is growing over time, then the long-run equilibrium is unstable. More details concerning the linear spatial stability analysis may be found in a general reference in the hydrodynamic stability literature, see Drazin and Reid (1991). Our analysis will allow us to examine how agglomerations emerge as well as the role that expectations may have on the nature of agglomerations.

7.1 Perturbation Linearized Equations

In order to perform the linearization of equations (20), we decompose the variables into their steady state value and their corresponding deviation. The steady state values 1 for variables $l(s,t)$, $y(s,t)$, $\theta(s,t)$, $w(s,t)$, $\omega(s,t)$ and 0 for variable $V(s,t)$. Let us denote the corresponding deviations by $l'(s,t)$, $y'(s,t)$, $\theta'(s,t)$, $w'(s,t)$, $\omega'(s,t)$ and write

$$\begin{aligned} y(s,t) &= 1 + y'(s,t); \theta(s,t) = 1 + \theta'(s,t); \\ w(s,t) &= 1 + w'(s,t); \omega(s,t) = 1 + \omega'(s,t); \\ l(s,t) &= 1 + l'(s,t); V(s,t) = 0 + V'(s,t) \end{aligned} \tag{21}$$

The perturbation equations are then obtained by the substitution of (21) in the PDE system (20). Neglecting second-order terms such as $l'w'$ leads to the perturbation linearized equations

$$\begin{aligned} y'(s,t) &= \mu\left[l'(s,t) + w'(s,t)\right] \\ \theta'(s,t) &= -\frac{\tau}{2}\int_{-\infty}^{+\infty}\left[l'(z,t) + (1-\sigma)w'(z,t)\right]\exp(-\tau(\sigma-1)\left|z-s\right|)dz \\ w'(s,t) &= \frac{\tau(\sigma-1)}{2\sigma}\int_{-\infty}^{+\infty}\left[y(z,t) + (\sigma-1)\theta(z,t)\right]\exp(-\tau(\sigma-1)\left|z-s\right|)dz \\ \omega'(s,t) &= w(s,t) - \mu\theta(s,t) \end{aligned}$$

$$\begin{aligned} \partial_t l' + \partial_s(V') &= 0 \\ \partial_t V' + K\partial_s \Omega' &= rV' \end{aligned} \tag{22}$$

7.2 Normal Mode Stability Analysis

The main idea in what follows is to study how periodic perturbations evolve over time. For simplicity, you may think periodic perturbations as being like $\sin(\nu s)$. High (low) values of ν correspond to high (low) frequency pertubations. However, we need a more general approach in order to deal with our problem.

Definition 5 *A spatial mode is determined by its frequency ν, and is defined as* $\exp[I\nu s]$, *with* $I^2 = -1$.

As suggested in the case of sinusoidal functions, high (low) frequency modes have a low (large) spatial scale.

Our analysis consists in determining whether the spatial modes have a damped or explosive behavior. A priori, there should be no reason for spatial modes to have the same time behavior.

Since equations (22) are linear, we may construct a general solution to these equations by an appropriate linear combination of the normal modes (elementary solutions which constitutes a complete set). As these equations are linear, this suggests that one looks for solutions where all perturbations are proportional to $\exp[\delta t + I\nu s]$.

So perturbations are assumed of the following type

$$\begin{bmatrix} y'(s,t) \\ \theta'(s,t) \\ w'(s,t) \\ \omega'(s,t) \\ l'(s,t) \\ V'(s,t) \end{bmatrix} = \exp\left[\delta t + I\nu x\right] \begin{bmatrix} y'_0 \\ \theta'_0 \\ w'_0 \\ \omega'_0 \\ l'_0 \\ V'_0 \end{bmatrix} \tag{23}$$

where for instance, y'_0 is the constant amplitude of the density perturbation associated with the variable $y'(s,t)$; and similarly for θ'_0, w'_0, ω'_0, l'_0, V'_0. Replacing (23) in (22) yields

$$\begin{aligned} y_0 &= \mu(l_0 + w_0) && (24) \\ \theta_0 &= -\frac{1}{\sigma - 1} h l_0 + h w_0 && (25) \\ w_0 &= \frac{1}{\sigma} h y_0 + \frac{\sigma - 1}{\sigma} h \theta_0 && (26) \\ \omega_0 &= w_0 - \mu\theta_0 && (27) \\ 0 &= \delta l_0 + I\nu v_0 && (28) \\ r v_0 &= \delta v_0 + K I \nu \omega_0 && (29) \end{aligned}$$

where h measures the spatial scale of the spatial mode ν and is given by

$$h = [1 + \nu^2/(\tau(\sigma - 1))^2]^{-1} \tag{30}$$

Replacing (24) and (25) into (26), we get $w_0 = w_0(l_0)$. Next $\theta_0(l_0)$ and $\omega_0(l_0)$ are obtained by using (25) and (27) respectively. By replacing $\omega_0(l_0)$

in (29), and then $v_0(l_0)$ in (28), we get

$$[a\delta^2 + ra\delta + b]l_0 = 0 \tag{31}$$

where $a = ((\sigma - 1)h^2 + \mu h - \sigma)$, $b = K\tau^2(\sigma - 1)(1-h)(\mu(1-2\sigma) + h(\sigma(\mu^2 + 1) - 1))$.

Ignoring the trivial solution $y_0 = \theta_0 = w_0 = \omega_0 = l_0 = 0$, it follows that a solution to (24), (25), (26), (27), (28), (29) only exists if the coefficient of l_0 in (31) is zero, that is, when

$$\delta(h) = \frac{r \pm \sqrt{f(\sigma, \mu)}}{2} \tag{32}$$

where

$$f(\sigma, \mu) = r^2 - \frac{4K\tau^2(\sigma - 1)(1 - h)(\mu(1 - 2\sigma) + h(\sigma(\mu^2 + 1) - 1))}{(\sigma - 1)h^2 + \mu h - \sigma}$$

The above condition gives the possible values for δ and h. By integrating over the normal modes, one can get the most general solution, i.e. $y(s,t) = y \int \exp[\delta(\nu)t + Ik\nu]d\nu$.

7.3 Spatial Divergence

Definition 6 *A spatial mode defined by h is unstable if* $\mathrm{Re}\,\delta(h) > 0$.

Definition 7 *A long-run equilibrium is unstable if there exists a spatial mode $h \in]0,1]$ such that* $\mathrm{Re}\,\delta(h) > 0$.

In what follows, spatial divergence always occur so that we are interested in the shape the emerging agglomerations may have. So as to do so, we compute the preferred wavelength, that is the spatial mode of which the amplification factor is the highest.

Definition 8 *The preferred wavelength h_{pre} characterizing the emerging agglomeration is the spatial mode of which the amplification factor is the highest.*

Defined as such, h_{cr} is the spatial scale of the emerging agglomeration, and therefore constitutes a measure of the size of agglomerations.

Proposition 1 (Preferred Wavelength) *If the taste for variety and the proportion of the manufacturing population are not too high ($g(\sigma, \mu) < 1$), then the preferred wavelength is given by $h_{pre} \longrightarrow \mu(2\sigma - 1)/(\sigma\mu^2 + \sigma - 1)$ where the function $g(\sigma, \mu)$ is defined by*

$$g(\sigma, \mu) = \frac{\mu - \sigma - 3\mu\sigma + \sigma^2 + 2\mu\sigma^2 + \mu^2\sigma^2 - \sqrt{(1-\mu)^3(1+\mu)^2(\sigma-1)^2\sigma)}}{1 + (-2 - 2\mu - \mu^2 + \mu^3)\sigma + (1+\mu)^2\sigma^2}$$

Conversely, if the taste for variety and the proportion of the manufacturing population are relatively high ($g(\sigma, \mu) > 1$), then the preferred wavelength is given by $h_{pre} \longrightarrow 1$.

Proof.

Remember that the interval under study for h is $]0, 1]$. Here we are interested in the spatial modes h which satisfy the terminal condition as given in (20), that is in modes h such that $\delta(h) \leq r$. In order to do so, we will determine when the curve $\delta(h)$ crosses the horizontal line r. Like in other continuous spatial models [ie. Krugman (1996), Fujita *and al.* (1999)], there is no reason why these modes should behave in the same way over time: some modes may be more amplified over time than others. The spatial mode of which the amplification factor is the highest corresponds to the preferred wevelength, see Krugman (1996), or Fujita *and al.* (1999). First note that $\delta(0) < r$, $\delta'(0) \geq 0$ and $\delta(1) = r$. This means that when the curve $\delta(h)$ does not cross the horizontal line r at $h < 1$, all spatial modes satisfy the terminal condition, and $h_{pre} \longrightarrow 1$. See Figure 2. On the other hand, when the curve $\delta(h)$ crosses the horizontal line r at $h < 1$, some modes are not admissible. It turns out that the curve $\delta(h)$ can cross the horizontal line r at most once for $h < 1$. This actually happens when $\delta'(h)$ gets to zero at some $h < 1$, that is when $g(\sigma, \mu) < 1$. The preferred wavelength is then given by $h_{pre} \longrightarrow \mu(2\sigma - 1)/(\sigma\mu^2 + \sigma - 1)$ for which $\delta(h_{pre}) = r$. See Figure 3. ■

The result of Proposition 1 is an extension of the works of Krugman (1996) and Mossay (2003) to the case where workers have rational expectations. When increasing returns at equilibrium are relatively low and the proportion of the manufacturing population is relatively low, agglomerations emergence and the preferred wavelength is given by $h_{pre} \longrightarrow \mu(2\sigma - 1)/(\sigma\mu^2 + \sigma - 1)$. This arises when $\sigma >> 1$ and μ is small. In this case, $g(\sigma, \mu)$ is smaller than 1, see Figure 4. On the other hand, when increasing returns at equilibrium

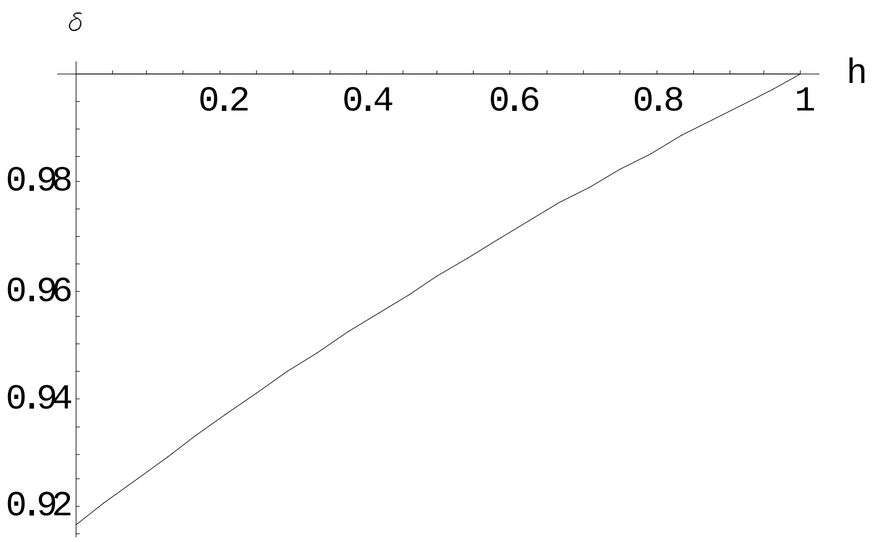

Figure 2: Relationship between the amplication factor and the wavelength when $g(\sigma, \mu) > 1$.

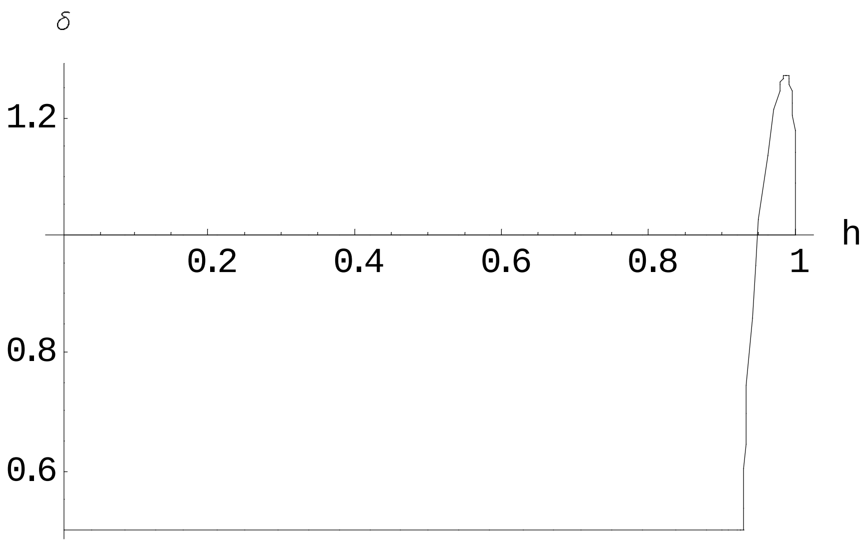

Figure 3: Relationship between the amplication factor and the wavelength when $g(\sigma, \mu) < 1$.

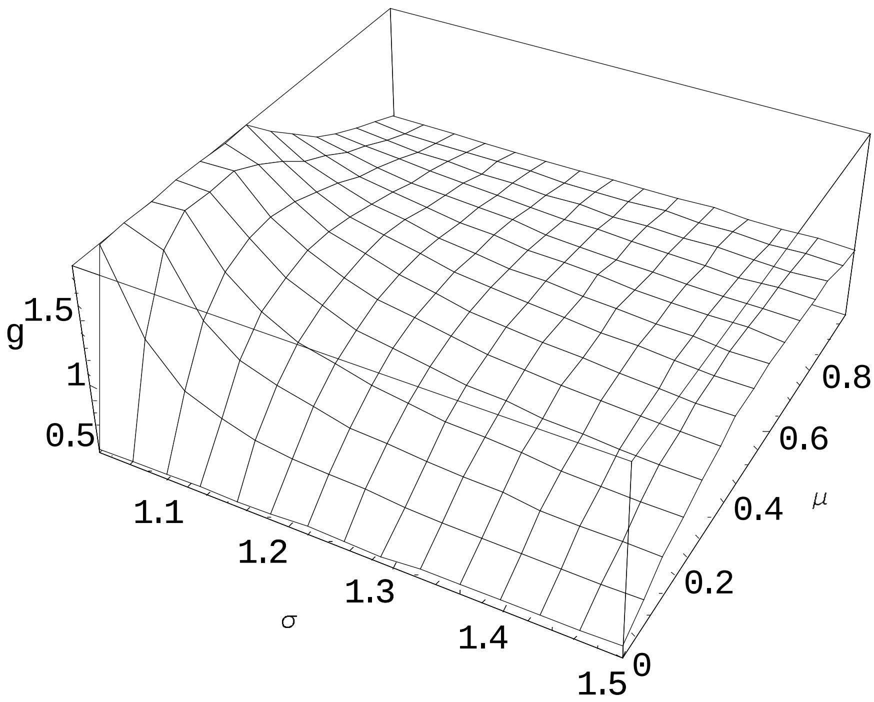

Figure 4: Function g in terms of parameters σ and μ

are relatively high and the manufacturing population relatively high, the preferred wavelength tends to infinity ($h_{pre} \longrightarrow 1$). In this second case, $g(\sigma, \mu)$ is higher than 1, see Figure 4. See Table 1 for some numerical computations of the preferred wavelength.

$\sigma\backslash\mu$	0.1	0.2	0.3	0.4
1.1	1	1	1	1
1.2	0, 66	1	1	1
1.3	0, 43	0, 79	1	1
1.4	0, 36	0, 66	0, 89	1
1.5	0, 32	0, 60	0, 81	0, 96

Table 1: Preferred Wavelength in terms of σ and μ

While workers are assumed to be myopic in Mossay (2003) and Krugman (1996), here, they are assumed to have rational expectations. Eventhough expectations differ in these three works, agglomerations emerge in the three cases: *scale economies at the firm level and free mobility of workers contribute to the emergence of agglomerations.* This shows *the predominant role of the local market structure in determining the emergence of agglomerations.* We also note that unlike in the corresponding two-region model of Krugman (1991), divergence always occur in continuous spatial models, see also Krugman (1996), Mossay (2003). In other words, in a continuous spatial setting, iceberg transport costs can never balance the agglomeration force due to scale economies. This is because when space is continuous there is always a location toward which transporting the manufacturing good can be made as cheap as wanted.

Since the local market structure drives the divergence result, *what is then the role of expectations in the emergence of agglomerations.* To isolate the role expectations may have on the spatial economy, we compare our results with the ones obtained in Mossay (2003). This will make the comparison sensible since that work differs from this work in one respect only: the expectation formation. In Mossay (2003), where workers are myopic, we showed that h_{pre} is always equal to 0, meaning that the emerging agglomerations, which have the higher amplification factor, have very small size. In contrast, here, the preferred wavelength has a finite measure provided that $g(\sigma, \mu) < 1$. *This shows how the type of expectation may distort the relationship between the amplification factor and the wavelength, and thus affect the preferred wavelength.*

We now see how the preferred wavelength varies with σ and μ in the following proposition.

Proposition 2 (Size of Agglomerations) *Provided that the taste for variety and the proportion of the manufacturing population are not too high ($g(\sigma, \mu) < 1$), the size of agglomerations increases with the taste for variety (inversely related to σ) and the proportion of the manufacturing population, and decreases with transport costs.*

Proof.
As $g(\sigma, \mu) < 1$, the preferred wavelength is given by $h_{pre} \longrightarrow \mu(2\sigma - 1)/(\sigma\mu^2 + \sigma - 1)$ as of Proposition 1. We have $\partial_\sigma h_{pre} = \mu(\mu^2 - 1)/(\sigma - 1 + \mu^2\sigma)^2 < 0$. Therefore, the higher the taste for variety (the lower σ) , the larger the size h_{cr} of agglomerations. Also $\partial_\mu h_{pre} = (1 - 2\sigma)\ (1 + \sigma(\mu^2 - 1))/(\sigma - 1 + \sigma\mu^2)^2 > 0$. This is because it can be shown that $1 + \sigma(\mu^2 - 1) < 0$ when $g(\sigma, \mu) < 1$. Therefore, the larger the proportion μ of the manufacturing population, the larger the size h_{cr} of agglomerations. Finally, since h is always positively related to transport costs (see how h has been rescaled from k in equation (30), the higher the transport costs, the smaller the agglomerations. $\blacksquare$

Proposition 2 makes sense. If the taste for variety is high (σ is low), then increasing returns realized at equilibrium are high, and agglomerations are large but not numerous (h_{pre} is large). Moreover the higher the proportion μ of the manufacturing population, the larger the agglomerations because then there are less peasants to supply in the unagglomerated areas. On the other hand, when transport costs are high, it is then sensible for firms to locate closer to local markets in order to avoid prohibitive transport costs when supplying immobile peasants in the unagglomerated areas. Finally, when the taste for variety and the proportion of the manufacturing population are relatively high, $h_{pre} \longrightarrow 1$ as given by Proposition 1, and space plays no role on the spatial divergence process: the spatial mode of which the amplification factor is the highest, is the uniform spatial mode.

8 Conclusion

We have modeled migration as part of a rational decision over a continuous set of locations. Therefore, it contrasts with all previous continuous economic geography models [ie. Krugman (1996), Fujita and al. (1999), Mossay

(2003)]. On the consumer's side, we derived the first order conditions (FOCs) characterizing an optimal spatial itinerary.

Some concepts were introduced to analyze the structure of the partial differential equations (PDEs) which govern the evolution of the economy over space and time. Among those, the normal mode analysis allowed us to study the linear spatial stability of the uniform long-run equilibrium. As a result of the analysis, like in Krugman (1996) and Mossay (2003), *spatial divergence always takes place.* Spatial divergence is thus obtained under the same conditions as the ones needed when adjustments are myopic, see Mossay (2003). This shows how predominant is the role of the local market structure in the convergence process: *scale economies and free mobility of workers contribute to spatial divergence, regardless of the temporal foresight ability of agents.* Furthermore, unlike in the corresponding myopic case studied by Mossay (2003), *the size of agglomerations increases with the taste for variety* (inversely related to σ) *and the proportion of the manufacturing population, and decreases with transport costs.* This actually happens provided that the taste for variety and the proportion of the manufacturing population are not too high. This shows that *the spatial role of expectations is to distort the relationship between the amplification factor and the wavelength, and thus to affect the preferred wavelength.*

In addition, continuous spatial economic geography models [Krugman (1996) or Mossay (2003), as well as this paper] contrast qualitatively with corresponding two-country models [Krugman (1991), Ottaviano (1999)]. In continuous models, iceberg transport costs can never balance the agglomeration force due to scale economies. The intuitive explanation is the following. When space is continuous there is always a location toward which transporting the manufacturing good can be made as cheap as wanted.

In Mossay (2001), we have identified four key elements which may affect the spatial and temporal evolution of an economy: *the local market structure, the type of spatial adjustment* (local/global), and *the type of expectations* (myopic/rational). Further attention should be devoted to these different aspects so as to identify their own or joined impacts on a spatial economy even more clearly. Eventough they have not been included in our discussion, other aspects deserve careful attention, namely intertemporal trading, asset markets, and uncertainty. This work should thus be seen as part of a very broad research program aiming at a better understanding of both spatial and temporal fluctuations.

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