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Option Pricing Using EGARCH Models

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by

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Abstract
Various empirical studies have shown that the time-varying volatility of asset returns can be described by GARCH (generalised autoregressive conditional heteroskedasticity) models. The corresponding GARCH option pricing model of Duan (1995) is capable of depicting the "smile-effect" which often can be found in option prices. In some derivative markets, however, the slope of the smile is not symmetrical. In this paper an option pricing model in the context of the EGARCH (Exponential GARCH) process will be developed. Extensive numerical analyses suggest that the EGARCH option pricing model is able to explain the different slopes of the smile curve.

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1 Introduction

In their seminal paper, Black and Scholes (1973) derived a valuation formula for European options which has been applied to many option markets.[1] The Black-Scholes model assumes that the returns of the underlying asset follow a normal distribution with constant volatility. Empirical evidence has shown, however, that the model is in conflict with at least four stylised facts. First, early studies by Mandelbrot (1963) and Fama (1965) have indicated that short-run returns in commodity and stock markets are not normally distributed but have fat tails and are peaked, i.e. they have leptokurtic distributions. However, for longer investment horizons of a month or more, the return distribution seems to converge to a normal distribution.[2] Second, more recent evidence has shown that the assumption of constant volatility is often strongly violated in financial markets.[3] Third, there is a tendency for changes in stock prices to be negatively correlated with changes in volatility.[4] This is often referred to as the "leverage effect".

The fourth type of conflicting evidence is related to systematic patterns in implied volatilities. When the Black-Scholes formula is inverted to compute implied volatilities from reported option prices, volatility estimates differ across exercise prices and time to maturity. In a plot of implied volatilities against strike prices often two distinct patterns can be observed: the "volatility smile" and the "volatility skew".[5] As time to maturity increases, these curves typically flatten out.[6] The volatility smile is associated with a U-shaped pattern of implied volatilities where at-the-money options have the smallest implied volatility. The volatility smile has been found in stock index options[7] in the period prior the '87 market crash and in currency options.[8] After the crash, however, skewed implied-volatility patterns were often observed. Studying post-crash S&P 500 options and futures options, Rubinstein (1994), Derman and

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1 The paper of Bates (1995) surveys the extensive empirical literature on stock options, options on stock indexes and options on currencies.

2 See Fama (1976), Chapter 1.

3 See, for example, Taylor (1986).

4 See, for example, Black (1976), Christie (1982), French, Schwert and Stambaugh (1987), Schwert (1989), Nelson (1991) and Gallant, Rossi and Tauchen (1992).

5 Dumas, Fleming and Whaley (1996) call it "sneer".

6 See Ghysels, Harvey and Renault (1995).

7 See Sheikh (1991) for S&P 100 index options and Rubinstein (1994) for S&P 500 index options.

8 See Shastri and Wethyavivorn (1987) and Taylor and Xu (1994a).

Kani (1994) and Taylor and Xu (1994b) showed that implied volatilities decrease monotonically as the exercise price rises relative to the index level.[9]

In recent years, much attention has been focused on modelling financial-market returns by processes other than simple Gaussian white noise and on extending option pricing models to incorporate moneyness effects. To capture the property of time-varying volatility, Engle (1982) introduced the ARCH (AutoRegressive Conditional Heteroskedasticity) model. Bollerslev's (1986) extension of this model, the GARCH (Generalised ARCH) model, has gained widespread acceptance in the literature and is often used for modelling stochastic volatility in financial time series.[10] Although GARCH models give adequate fits for exchange-rate dynamics, these models often fail to perform well in modelling the volatility of stock returns because GARCH models assume that there is a symmetric response between volatility and returns. Therefore, GARCH models are not able to capture the "leverage effect" of stock returns. Nelson (1991) introduced the EGARCH (Exponential GARCH) model in order to model asymmetric variance effects.

More recently, much attention has been directed at examining the implication of ARCH models for option prices.[11] The option pricing theory for ARCH models was first developed by Duan (1995) in an equilibrium setting and by Kallsen and Taqqu (1995) in an arbitrage-free continuous-time framework. In a simulation study Chaudhury and Wei (1995) compared Duan's GARCH option pricing model with the Black-Scholes model. They confirmed Duan's finding that the GARCH option pricing model produces the strongest price effects for out-of-the-money options with short maturity. Satchell and Timmermann (1993) and Amin and Ng (1993) found that the GARCH option pricing model produced significantly better fits to market prices than the Black-Scholes model. These studies have confirmed that the standard GARCH option pricing model can capture the "smile effect" of implied volatilities, however it is not compatible with the "volatility skew". The aim of this paper is, therefore, to extend previous research by studying the option price effects of EGARCH models.

It should be mentioned that parallel to the development and application of ARCH-type models in finance, a different strand of research emerged which extended the Black-Scholes model by introducing an additional stochastic differential equation for the volatility. Examples of bivariate diffusion models along these lines are Hull and White (1987), Scott (1987), Wiggins (1987), Johnson and Shanno (1987), Chesney and Scott (1989), Melino and Turnbull (1990), Stein and Stein (1991) and Heston (1993).

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9 See Rubinstein (1994) and Taylor and Xu (1994b).

10 Bera and Higgins (1993), Bollerslev, Chou and Kroner (1992) and Bollerslev, Engle and Nelson (1994) provide extensive reviews of applications of ARCH-type models for financial time series.

11 See, for example, Engle and Mustafa (1992) and Day and Lewis (1992).

The rest of this paper is organised as follows. Section 2 looks at empirical volatility patterns of options on the German stock index DAX. After a brief introduction to the EGARCH model in Section 3, in Section 4 of this paper the GARCH option pricing model of Duan (1995) is extended to the EGARCH option pricing model by using an EGARCH(1,1) stock return volatility specification. Section 5 explains the numerical procedure for calculating EGARCH option prices by Monte Carlo simulations. In Section 6, I present and discuss the option price effects for several parameter combinations. Section 7 concludes the paper.

2 Implied Volatility Patterns of DAX Options

The Volatility Smile and the Volatility Skew can also be observed in the implied volatilities of options on the German stock index DAX. In August 1991, the DAX option was introduced at the German Futures and Options Exchange (DTB). Since then it has developed into the most liquid option traded on the DTB.[12] The value of an option contract is the current index level multiplied by 10 German Marks (DM). Option prices are quoted in points where each point represents DM 10,- of contract value. The tick size is 0.1 points which corresponds to a tick value of DM 1,-. The DAX option is an European-style option.

To provide an illustration of volatility effects in the DAX option market, I derived implied volatilities for call options traded on 20 June, 1994 using the Black/Scholes formula.[13] Within this day I selected all transactions between 11:00 a.m. and 1:00 p.m.[14] Similar to Shimko (1993) and Oliver (1995), I fitted a quadratic regression for the relationship between moneyness m and implied volatilities σ_{imp} from the inverted Black-Scholes formula:

$$\sigma_{imp} = a \cdot m^2 + b \cdot m + c \qquad (1)$$

where m is defined as[15]

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[12] The trading volume of DAX options is greater than that of all 20 DTB-traded stock options together.

[13] This particular day was chosen because it was a day with very large transaction volume.

[14] The DAX option is traded from 9:30 a.m. to 4:00 p.m. Frankfurt time.

[15] If $m = 1$, the option is *at the money*. For $m > 1$ (respectively $m < 1$) the option is said to be *in the money* (respectively *out of the money*). I use here a slightly different definition of moneyness with respect to the usual one. Ghysels, Harvey and Renault (1995), who define moneyness as $m = \ln(S/(X \cdot e^{-r(T-t)}))$ point out: "Indeed, it is more common to call at the money / in the money / out of the money options, when S=X / S>X / S>X respectively. From an economic point of view, it is more appealing to compare S with the present value of the strike price X."

$$m = \frac{S}{X \cdot e^{-r(T-t)}} \tag{2}$$

where S denotes the current DAX index level, X the exercise price, r the risk free interest rate and $(T\text{-}t)$ the time to maturity.

The resulting regression curves for July 94 contracts (based on 655 transactions) and December 94 contracts (based on 66 transactions) are presented in Figure 1. The implied volatilities for short-term options show a slightly skewed smile effect, whereas the long-term options show the volatility skew. This result is in agreement with the findings of Taylor and Xu (1994b) for the S&P 500 futures options market.[16]

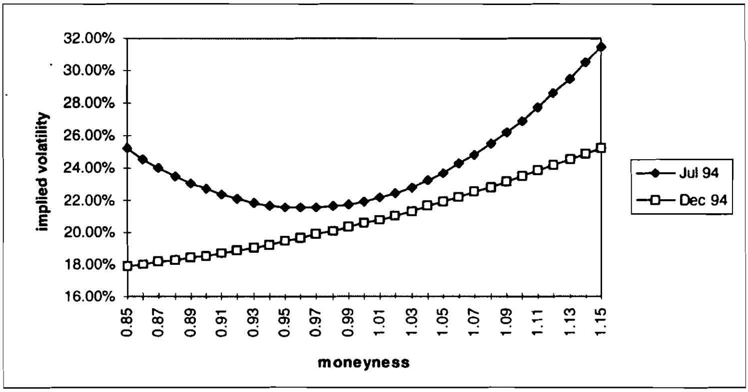

Figure 1: Implied Volatility Patterns for DAX Options traded on 20 June, 1994

3 From ARCH to EGARCH

Since the seminal paper of Engle (1982) a rich literature has emerged for the modelling of heteroskedasticity in financial time series. Engle (1982) introduced the ARCH(p) model in which the conditional variance σ_t^2 is a linear function of lagged squared innovations ε_t:

$$\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_p\varepsilon_{t-p}^2 \qquad \text{with } a_i > 0 \text{ for all } i \tag{3}$$

and

$$\varepsilon_t | \phi_{t-1} \sim N(0, \sigma_t^2) \ ,$$

[16] See Taylor and Xu (1994b), Figures 4 and 5.

where ϕ_t is the information set of all information up to and including time t. It should be noted that for ARCH models and their variants (see below) the *unconditional* distribution of ε_t is always leptokurtic.[17] Therefore, ARCH-type models are consistent with the distributional properties of short-run returns in financial markets. In applications of the ARCH(p) model, it often turned out that the required lag p was rather large.[18] In order to achieve a more parsimonious parametrisation, Bollerslev (1986) introduced the generalised ARCH(p,q) model (GARCH(p,q) for short):

$$\begin{aligned}&\sigma_t^2 = a_0 + a_1\varepsilon_{t-1}^2 + \cdots + a_p\varepsilon_{t-p}^2 + b_1\sigma_{t-1}^2 + \cdots + b_q\sigma_{t-q}^2 \\ &\text{with } a_i > 0 \text{ and } b_j > 0 \text{ for all } i \text{ and j.}\end{aligned} \tag{4}$$

In general, the value of p in (4) will be much smaller than the value of p in (3).

Important limitations of ARCH and GARCH models are the non-negativity constraints of the a_i's and b_j's which ensure positive conditional variances. Moreover, GARCH models assume that the impact of news on the conditional volatility depends only on the magnitude, but not on the sign, of the innovation. As mentioned above, empirical studies have shown that changes in stock prices are negatively correlated with changes in volatility. To overcome these drawbacks, Nelson (1991) introduced the exponential GARCH (EGARCH) model in which the logarithm of conditional variance is specified as:[19]

$$\ln\sigma_t^2 = a_0 + a_{1a}\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + a_{1b}\left(\frac{|\varepsilon_{t-1}|}{\sigma_{t-1}} - E\left[\frac{|\varepsilon_{t-1}|}{\sigma_{t-1}}\right]\right) + b_1\ln\sigma_{t-1}^2\ , \tag{5}$$

For $\varepsilon_t \sim N(0,\sigma_t^2)$ the standardised variable $\frac{\varepsilon_t}{\sigma_t}$ follows a standard normal distribution and consequently $E\left[\frac{|\varepsilon_t|}{\sigma_t}\right] = \sqrt{\frac{2}{\pi}}$.[20] The parameter a_{1a} captures the leverage effect. For "good news" $\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1}} > 0\right)$ the impact of the innovation ε_{t-1} is $(a_{1b} + a_{1a})\cdot\frac{\varepsilon_{t-1}}{\sigma_{t-1}}$ and for "bad news" $\left(\frac{\varepsilon_{t-1}}{\sigma_{t-1}} < 0\right)$ it is $(a_{1b} - a_{1a})\cdot\frac{\varepsilon_{t-1}}{\sigma_{t-1}}$. If $a_{1a} = 0$, $\ln\sigma_t^2$ responds symmetrically to $\frac{\varepsilon_{t-1}}{\sigma_{t-1}}$. To produce a leverage effect, a_{1a} must be negative. The fact that the EGARCH process is specified in terms of log-volatility implies that σ_t^2 is always positive and,

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[17] See Bollerslev, Engle and Nelson (1994).

[18] See, for example, Akgiray (1989), Fornari (1993), and Schmitt (1994).

[19] For convenience I restrict applications in this paper to the EGARCH(1,1) model which often proved to be adequate for the modelling of stock-price volatilities. See, for example, Heynen, Kemna and Vorst (1994).

[20] See Johnson and Kotz (1970), p. 49.

consequently, there are no restrictions on the sign of the model parameters. It can be shown that the stationary volatility of an EGARCH(1,1) process is[21]

$$\bar{\sigma}^2 = \exp\left[\frac{a_0 - a_{1b}\sqrt{\frac{2}{\pi}}}{1-b_1} + \frac{1}{2}\cdot\frac{\left(a_{1a}^2 + a_{1b}^2\right)}{1-b_1^2}\right]\cdot\prod_{m=0}^{\infty}\left[F_m\left(a_{1a},a_{1b},b_1\right) + F_m\left(-a_{1a},a_{1b},b_1\right)\right] \quad (6)$$

with

$$F_m = N\left[b_1^m \cdot \left(a_{1b} - a_{1a}\right)\right]\cdot\exp\left(b_1^{2m}\cdot a_{1a}\cdot a_{1b}\right)$$

where $N[.]$ is the cumulative standard normal distribution.

4 The EGARCH option pricing model

Following Duan's (1995) methodology for the GARCH model, I specify the EGARCH(1,1) model[22] for the stock price process S_t and the stock volatility as:

$$\ln\frac{S_t}{S_{t-1}} = r + \lambda\sigma_t - \frac{1}{2}\sigma_t^2 + \varepsilon_t\ , \quad (7)$$

$$\ln\sigma_t^2 = a_0 + a_{1a}\frac{\varepsilon_{t-1}}{\sigma_{t-1}} + a_{1b}\left(\frac{|\varepsilon_{t-1}|}{\sigma_{t-1}} - \sqrt{\frac{2}{\pi}}\right) + b_1\ln\sigma_{t-1}^2\ , \quad (8)$$

where r is the risk-free interest rate, λ is the risk premium of the stock, and a_0, a_{1a}, a_{1b} and b_1 are time-independent parameters. To ensure stationarity, b_1 is assumed to be less than one. ε_t has a normal distribution with a mean of zero and conditional variance σ_t^2 under probability measure P:

$$\varepsilon_t|\phi_{t-1} \sim N(0,\sigma_t^2)\ ,$$

Unlike in the bivariate diffusion models, the volatility for the next period of time, given the information set ϕ_t, is known with certainty in the EGARCH model. This allows to define the equilibrium price measure Q which is absolutely continuous with respect to the measure P. Therefore Q is said to satisfy the locally risk-neutral valua-

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[21] See Heynen, Kemna and Vorst (1994).

[22] Duan's (1995) GARCH option pricing model holds also for the general EGARCH(p,q) process. For simplicity the EGARCH(1,1) is used here.

tion relationship (LRNVR).[23] Under the pricing measure Q, the stock return process results in,

$$\ln \frac{S_t}{S_{t-1}} = r - \frac{1}{2}\sigma_t^2 + \xi_t \; , \tag{9}$$

$$\ln \sigma_t^2 = a_0 + a_{1a}\left(\frac{\xi_{t-1}}{\sigma_{t-1}} - \lambda\right) + a_{1b}\left(\left|\frac{\xi_{t-1}}{\sigma_{t-1}} - \lambda\right| - \sqrt{\frac{2}{\pi}}\right) + b_1 \ln \sigma_{t-1}^2 \; , \tag{10}$$

with

$$\xi_t | \phi_{t-1} \sim N(0, \sigma_t^2) \; .$$

Following Duan (1995), the terminal stock price at time T can be calculated as:

$$S_T = S_t \exp\left[(T-t)r - \frac{1}{2}\sum_{s=t+1}^{T}\sigma_s^2 + \sum_{s=t+1}^{T}\xi_s\right] \tag{11}$$

By taking the conditional expectation of the terminal payoff under the pricing measure Q and discounting at the risk-free interest rate, the value of an European call option with strike price X can be obtained:

$$C_t^{EGA} = e^{-(T-t)r} E^Q\left[\max\left(S_T - X, 0\right) | \phi_t\right] \tag{12}$$

The corresponding European put option value can be calculated by using the put-call parity. In this study I confine the price simulations to call options.

5 Design of the Monte Carlo Simulations

Since the distribution of the terminal asset price S_T cannot be derived analytically, Monte Carlo Simulations are used to compute the EGARCH option prices. The simulated EGARCH option price is

$$C_t^{EGA}(n) = e^{-(T-t)r} \cdot \frac{1}{n}\sum_{i=1}^{n}\left[\max\left(S_{T,i} - X, 0\right)\right] \tag{13}$$

where n=200,000 is the number of repetitions.

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[23] See Duan (1995), Definition 2.1.

To improve the efficiency of the EGARCH option price estimates, the antithetic variation technique[24] is employed. For any sample path of innovations ξ_i and its corresponding option price, the antithetic path $-\xi_i$ and its corresponding option price is calculated. The average of these two prices provides the output for one repetition. To further improve the efficiency of simulations, the control-variate method is employed.[25] The corresponding Monte Carlo Black-Scholes price C_t^{BS} is used as the control variable since for the Black-Scholes formula an analytical solution $\tilde{C}_t^{BS}$ exists:

$$\tilde{C}_t^{BS} = S_t \cdot N(d_1) - X \cdot e^{-r\cdot(T-t)} \cdot N(d_2) \tag{14}$$

with

$$d_1 = \frac{\ln\frac{S_t}{X} + \left(r + \frac{\sigma_{BS}^2}{2}\right)(T-t)}{\sigma_{BS}\sqrt{T-t}} \quad \text{and} \quad d_2 = d_1 - \sigma_{BS}\sqrt{T-t}.$$

The volatility estimate σ_{BS} for the Black-Scholes model is the stationary volatility $\overline{\sigma}$ of the EGARCH model.

The resulting revised EGARCH option price $\tilde{C}_t^{EGA}$ is then defined as

$$\tilde{C}_t^{EGA}(n) = C_t^{EGA}(n) - q(n) \cdot \left[C_t^{BS}(n) - \tilde{C}_t^{BS}\right] \tag{15}$$

where

$$q(n) = \frac{Cov\left(C_t^{EGA}(n), C_t^{BS}(n)\right)}{Var\left(C_t^{BS}(n)\right)}.$$

As is customary, I will refer to the *estimated* EGARCH price $\tilde{C}_t^{EGA}$ as the *EGARCH option price*.

To obtain parameter values for a_0, a_{1a}, a_{1b}, b_1 and λ, I estimated the EGARCH(1,1) models as specified in (7) and (8) for the daily return series of the German stock index DAX and 29 of the 30 incorporated stocks[26] for the period from 2 January,

[24] See Hammersley and Handscomb (1964), pp. 60-66.

[25] See Kleijnen and Rubinstein (1994).

[26] For VEBA only data starting in March 1987 is available.

1987 to 30 December, 1992. The minimum and maximum of the parameter estimates for the different stocks are shown in Table 1.

Table 1: Parameter estimates for the EGARCH(1,1) model

	a_0	a_{1a}	a_{1b}	b_1	λ
minimum	-1.6162	-0.1677	0.0151	0.7790	-0.2033
maximum	-0.0448	0.0098	0.3920	0.9904	0.2968

The leverage parameter a_{1a} took values between -0.1677 and 0.0098. The slightly positive value of 0.0098 (for Commerzbank) was the only positive value and the only non-significant estimate. All other parameter estimates for a_{1a} were negative and significant at the 1%-level. As expected, a_{1b} was always positive with a maximum value of 0.3920. For 29 out of 30 time series, λ was not significant at the 5% level and none was significant at the 1% level. Therefore, I assume in the simulations that the risk premium λ is constant at zero. For a_0 and b_1, I use the average parameter estimates of –0.70 and 0.92, respectively which closely match the estimates obtained for the DAX series. It was decided to vary only parameter values of a_{1a} and a_{1b} because a_0 has only an effect on the stationary volatility but not on the volatility dynamics and b_1 shows less variation in sample estimates than a_{1b}. For simplicity I assume that the risk-free interest rate is zero. This assumption also simplifies the definition of moneyness m.[27]

The simulations are based on the following parameter values:

a_0:	-0.70
a_{1a}:	0.00, -0.05, -0.10, -0.15
a_{1b}:	0.10, 0.20, 0.30, 0.40
b_1:	0.92
λ:	0
r:	0 [%]
$m = \frac{S}{X \cdot e^{-r(T-t)}}$:	0.80, 0.81, ... ,1.19, 1.20
T-t:	1 month, 2, 3, 4, 5, 6, 9 and 12 months

[27] In the literature, moneyness is either defined as S/X or F/X, where F denotes the forward price. For $r = 0$, both definitions are, of course, identical. I use here the definition (2), which is virtually the same as F/X.

Parameter estimates are based on daily time series and there are between 248 to 253 observations per year. To be consistent with the discrete nature of the stock-return series, I assume that a year has 252 (trading) days. Factors of $\sqrt{250}$ or $\sqrt{252}$ are also typically used in the annualisation of historical volatility from daily data.[28] Therefore, a month is defined to have $\frac{252}{12} = 21$ (trading) days. The variation of a_{1a}, a_{1b} and T-t gives 128 different combinations of the parameters. I use the stationary volatility of the particular EGARCH model, as calculated in (6), for the initial conditional variance σ_0 in the simulations. Table 2 shows the (annualised) stationary volatilities for the different parameter combinations.

As expected, stationary volatility increases with a_{1b} whereas the leverage parameter a_{1a} has only a minor influence.

Table 2: Annualised stationary volatilities for the EGARCH models

		a_{1b}			
		0.10	0.20	0.30	0.40
a_{1a}	0.00	20.10%	20.48%	21.12%	22.09%
	-0.05	20.17%	20.54%	21.18%	22.14%
	-0.10	20.38%	20.72%	21.34%	22.30%
	-0.15	20.74%	21.03%	21.62%	22.56%

6 Simulation Results

This section is organised in two subsections. Section 6.1 shows how the variation of the EGARCH parameters a_{1a} and a_{1b} and the parameters time to maturity and moneyness influence implied volatilities. In section 6.2 I use one specific parameter combination for a_{1a} and a_{1b} to study the absolute and relative deviations between EGARCH option prices and the corresponding Black-Scholes prices.

6.1 Implied Volatility Shapes of EGARCH Models

For a graphical comparison between the EGARCH and the Black-Scholes option prices, implied volatilities of the EGARCH option prices are derived. Using the Black-Scholes formula (14), the implied volatilities $\sigma_{EGA,imp}$ of EGARCH prices are calculated for the different values of moneyness $\left(\frac{S}{X}\right)$ and time to maturity (T-t) from

$$\tilde{C}^{EGA} = C^{BS}\left(\sigma_{EGA,imp}, \frac{S}{X}, r, T-t\right).$$

[28] For a discussion of annualisation factors see Hauck (1991), pp. 100-106.

The stationary volatilities $\overline{\sigma}$ of the EGARCH models are used as volatility estimates for Black-Scholes prices. To facilitate the comparison between models, implied volatility ratios (*IVR*) of EGARCH implied volatilities $\sigma_{EGA,imp}$ and the corresponding Black-Scholes volatilities σ_{BS} are calculated as

$$IVR = \frac{\sigma_{EGA,imp}}{\sigma_{BS}} = \frac{\sigma_{EGA,imp}}{\overline{\sigma}} \tag{16}$$

For ease of interpretation and to preserve space, a graphical display of results was chosen.

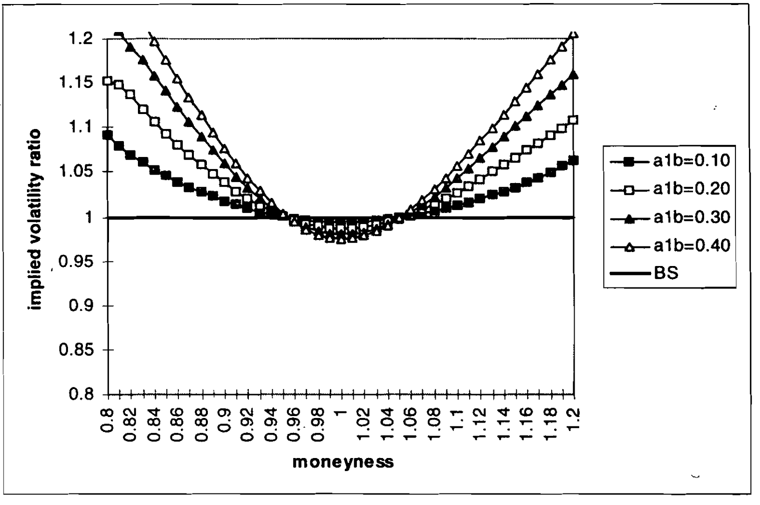

Figure 2: Implied volatility ratios: Time to maturity = 1 month, $a_{1a} = 0$

Figures 2 and 3 show the effects of varying the parameter a_{1b} which measures the impact of the size of innovations $\left(\frac{|\varepsilon_t|}{\sigma_t}\right)$ on volatility. In both figures, the time to maturity is fixed at 1 month. BS denotes the IVR for the Black-Scholes model, which is always one. Figure 2 displays a clear smile pattern of implied volatilities along moneyness. For at-the-money call options, implied volatility of the EGARCH model is systematically smaller than the corresponding Black-Scholes implied volatility but the implied volatility ratio is larger than 1 for both in-the-money options (*S/X*>1) and out-of-the-money options (*S/X*<1). The volatility smile pattern in Figure 2 very much resembles those obtained from stochastic volatility models and from GARCH models.[29] Figure 2 is based on simulations with $a_{1a} = 0$, i.e. without a leverage effect whereas the leverage parameter a_{1a} is set at -0.15 for the simulations underlying Figure 3. Varying a_{1b} again from 0.10 to 0.40 produces now quite different implied volatility patterns. For small values of a_{1b}, the skew effect from the leverage parameter a_{1a} is clearly dominant. The EGARCH model gives larger implied volatilities than the Black-Scholes model for in-the-money call options but smaller implied volatilities for out-of-the-money options.

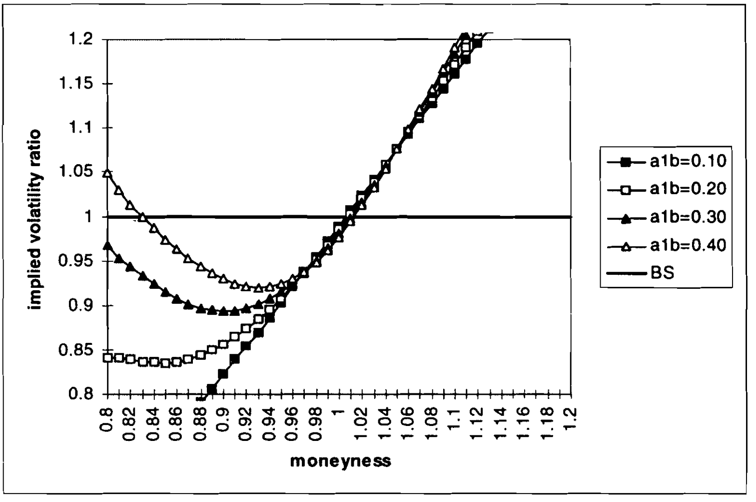

Figure 3: Implied volatility ratios: Time to maturity = 1 month, a_{1a} = -0.15

To better understand the volatility effect of leverage, recall that a negative parameter a_{1a} implies ceteris paribus that positive innovations are associated with decreases of volatility and vice versa. Out-of-the money options require large positive returns of the underlying to end up in the money at maturity. Since positive return innovations are associated with smaller volatility than negative returns, under leverage EGARCH out-of-the-money call prices should be smaller than Black-Scholes call prices. This explains the skew pattern in Figure 3. Note, however, that with an increase of a_{1b} relative to a_{1a}, the smile pattern gradually re-emerges.

Figure 4 shows the variation of leverage effects when a_{1a} varies but a_{1b} is constant at 0.20. The increasingly skewed pattern of implied volatility ratios is quite apparent when the leverage parameter a_{1a} is further decreased. The *IVR* pivots anti-clockwise

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[29] Kaehler (1994) shows that the volatility smile can be related to the leptokurtosis of the return distribution. The at-the-money effect is due to the peakedness of the distribution whereas the in-the-money and out-of-the-money effects are caused by the fat tails.

around a moneyness of 1. The economic intuition for the increase in implied volatility of in-the-money call options is as follows. Consider for a moment a put option which is out of the money, i.e. with $S>X$. The leverage effect implies that the spot price distribution at maturity has smaller skewness than the lognormal distribution which underlies the Black-Scholes model. Or, in other words, the probability that currently out-of-the-money puts will get into the money at maturity is higher for an asymmetric EGARCH process than for Gaussian white noise. It then follows from put-call parity that also call prices should increase in this scenario. Therefore, EGARCH implied volatilities exceed Black-Scholes implied volatilities for in-the-money call options.

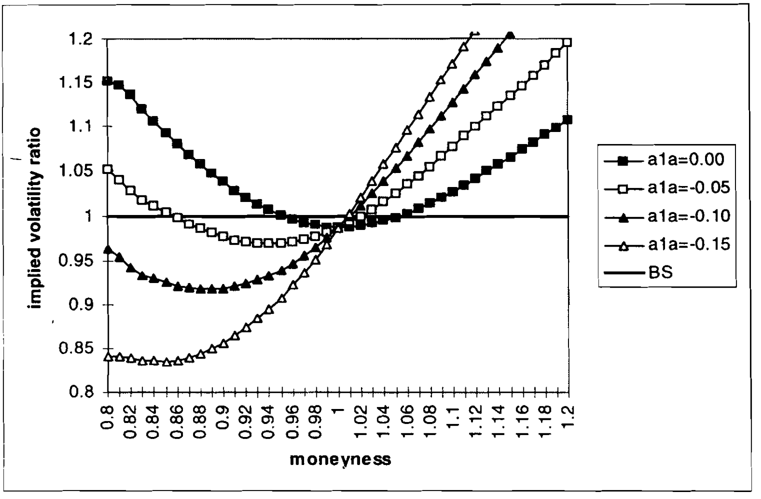

Figure 4: Implied volatility ratios: Time to maturity = 1 month, $a_{lb} = 0.20$

Turning to time-to-maturity effects in Figure 5, we see that smile effects decrease with an increase in the remaining life of an option. As noted in the Introduction, one of the stylised facts of financial markets is that return distributions converge to normality if the investment horizon is increased. It is, therefore, consistent with this stylised fact that EGARCH option prices should converge to Black-Scholes prices under temporal aggregation. The results of Figure 5 are based on EGARCH models without leverage effects and, therefore, without skew patterns.

Figure 6 displays time-to-maturity effects for asymmetric EGARCH models. With a_{la} fixed at -0.10 and a_{lb} at 0.20, one can again see that, in general, the implied-volatility effects are stronger for short-run options than for long-run options. However, notable exemptions are deep-out-of-the-money options with a moneyness of less than 0.9. The implied volatility ratio of 1-month calls with moneyness of 0.8 is 0.964 whereas the ratio decreases to 0.899 for 3-month call options and increases again to 0.957 for

12-months options. This interesting effect is due to the fact that, under temporal aggregation, the smile pattern disappears faster than the skew pattern. It should be noted that very similar maturity patterns were obtained for all other combinations of a_{la} and a_{lb} where $a_{la} < 0$.

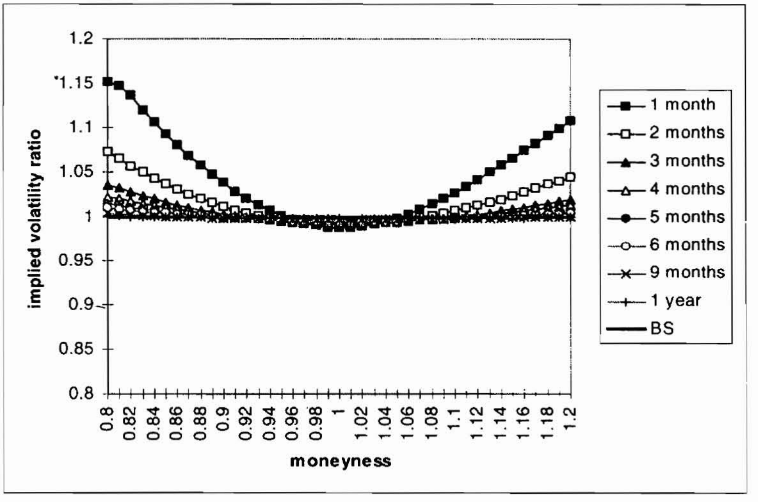

Figure 5: Implied volatility ratios: $a_{la} = 0.00$, $a_{lb} = 0.20$

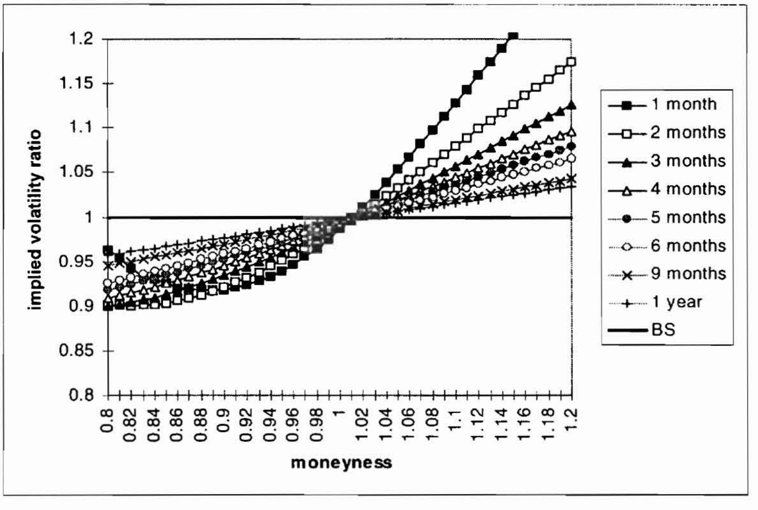

Figure 6: Implied volatility ratios: $a_{la} = -0.10$, $a_{lb} = 0.20$

All simulations of sample paths and option prices require starting values σ_0 of volatility. A natural choice is the stationary volatility of the corresponding EGARCH model as defined in (5). The results in the previous figures are all based on this as the starting value. It would be interesting to know whether simulation results are sensitive to the choice of starting values. Figures 7 and 8 explore this issue. Using a starting value of volatility of 15.72% which is smaller than the stationary volatility of 20.72% (based on $a_{1a} = -0.10$ and $a_{1b} = 0.20$, see Table 1), has the effect of shifting the implied volatility curves down (compare Figures 6 and 7). The economic rationale behind this shift is quite obvious. Black-Scholes prices are not affected since they are still based on the same stationary volatility but EGARCH prices would decrease due to the decrease in initial and subsequent volatilities.

Figure 8 illustrates the effects of using the relatively large value of 25.72% as a starting value. It should not be surprising that the large initial volatility shifts the volatility curves upwards (compare Figure 8 with Figure 6). Therefore, the choice of starting values determines the *position* of the IVR curves. It is interesting to note that the *shape* of the volatility curves is very little affected by variations of starting values.

Rubinstein (1985) and Sheikh (1991) examined the effect of the time to maturity on the the term structure of implied volatilities. They reported for at-the-money calls, the longer the time to maturity, the higher is the implied volatility of the option. But they reported also that over a different period a reversal occurs. Figures 7 and 8 show that with different initial volatilities the EGARCH option pricing model can explain these effects.

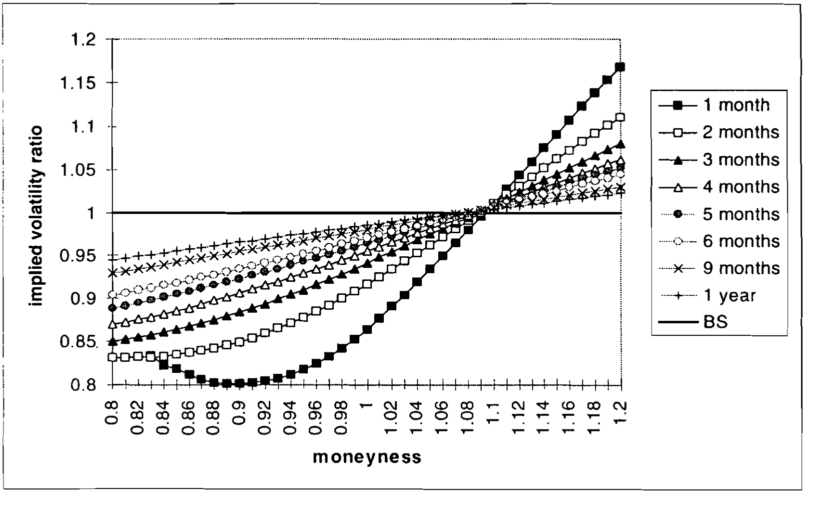

Figure 7: low initial volatility

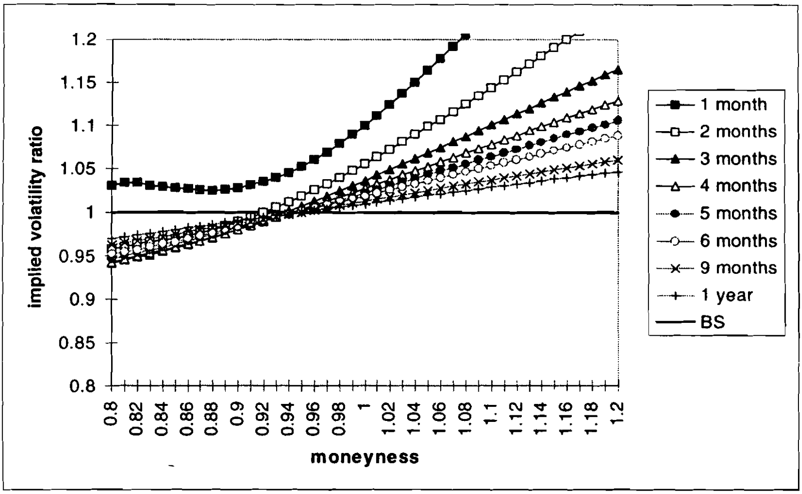

Figure 8: high initial volatility

It is quite remarkable that the volatility patterns of 1-month options and 6-months option in Figure 8 show a striking similarity to the fitted implied volatility pattern of traded DAX options as plotted in Figure 1.

6.2 Behaviour of Price Deviations

In this section I study EGARCH option effects in the metric of prices and take DAX options as an example. The DAX spot index value is fixed at $S = 2000$ and the exercise price is varied from $X = 2500$ to $X = 1667$ to give values of moneyness between $S / X = 0.8$ and $S / X = 1.2$. The EGARCH parameters are specified as $a_0 = -0.70$, $a_{1a} = -0.10$, $a_{1b} = 0.20$ and $b_1 = 0.92$ and represent approximate estimates for the DAX time series. Note that the same parameter values were used in the implied volatility structure of Figure 6.

The standard errors of the EGARCH price simulations are plotted in Figure 9. Due to the large number of simulation runs, the standard errors are in almost all cases smaller than the tick size of 0.1 index points, the only exceptions being deep-in-the-money options with a time-to-maturity of one year. But these options have large theoretical values of more than 260 points under the EGARCH model and the Black-Scholes model. As expected, standard errors grow with time to maturity since the length of sample path is extended. The conclusion to be drawn from Figure 9 is that the simulations with 200,000 repetitions, using both the antithetic and control-variate methods, provide highly accurate EGARCH option prices.

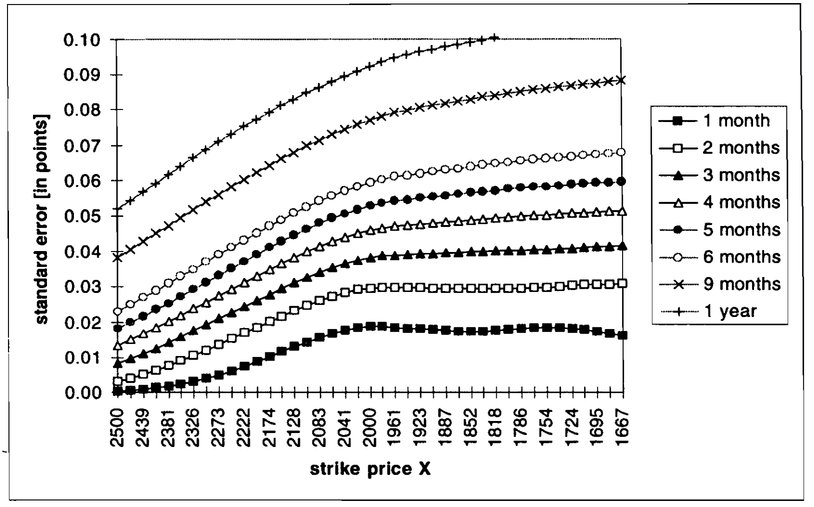

Figure 9: Standard Error of EGARCH option prices in points

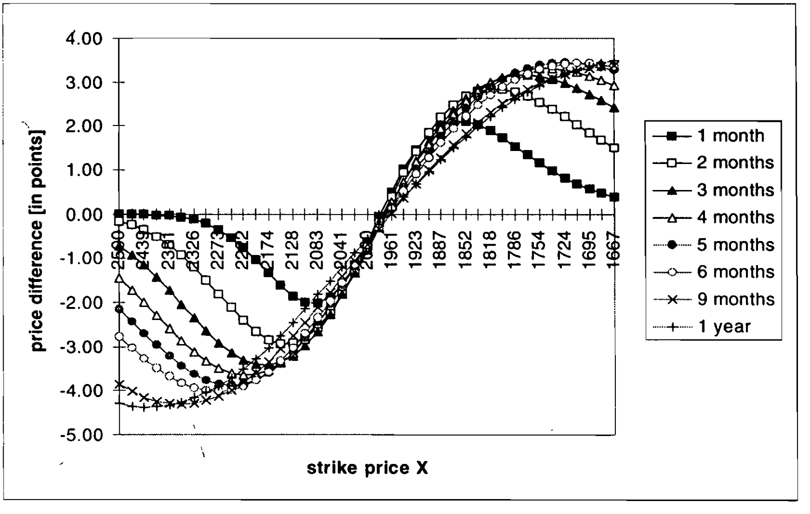

Figure 10: Deviation of EGARCH option prices [in points]

Figure 10 shows the price deviations between EGARCH prices and Black-Scholes prices as measured in index points. The price difference varies between –4.5 points (–45 German marks per contract) for out-of-the-money options and +4.0 points (40 German marks) for in-the-money options. For greater values of a_{1b} and more negative values of a_{1a} these deviations can rise up to 8 points (not shown here).

There are quite substantial differences for both in-the-money and out-of-the-money options and the pattern of price differences in Figure 10 is in line with the implied volatility patterns of Figure 6. Since option prices are a positive function of volatility, the price bias is negative (positive) when the implied volatility ratio is negative (positive).

To examine the *percentage* price difference between EGARCH prices and Black-Scholes prices, I use a slightly different approach and compare the time values of the corresponding option prices as opposed to the total option prices. Note that the (total) price of an option can be decomposed into the intrinsic value and the time value. The intrinsic value of an call option reflects the moneyness of the option and is defined[30] as $\max[S - X, 0]$. The time value is then the difference between the option price and its intrinsic value. This time value is that part of an option price that remains to be explained by an option pricing model. For out-of-the-money options, the intrinsic value is zero and, therefore, the time value is equal to the option premium.

Previous studies of GARCH option price effects have compared total prices and, as a result, the authors typically find large percentage deviations between GARCH and Black-Scholes prices for out-of-the-money options, whereas for in-the-money options the difference is often less than two percent.[31] Therefore, these studies conclude that ARCH-type option pricing models are mostly relevant for out-of-the-money options but less so for in-the-money options.

Figure 11 shows that this can be a misleading conclusion. Figure 11 plots the percentage differences between the time values of the EGARCH prices and Black-Scholes prices. For out-of-the-money call options $(X > 2000)$, the price deviations of up to 4 points (see Figure 10) imply EGARCH option prices (or time premia) that are up to 65 percent smaller than the corrresponding Black-Scholes prices. As time to maturity increases, these differences decrease. For in-the-money options, EGARCH prices are less than 1.7 percent higher than the Black-Scholes prices and this would confirm results from previous studies.[32] But comparing the time value of option prices leads to completely different results. Especially for short-term in-the-money options, the time values of EGARCH option prices are much greater than the time values for Black-Scholes prices. This clearly shows that EGARCH models would also be of relevance for in-the-money options. The more an option is in-the-money the greater is the rela-

[30] From an economic point of view, the intrinsic value for European call options should be defined as $\max[S - X \cdot e^{-r\cdot(T-t)}, 0]$ since this is the lower boundary for the value of an European call option, otherwise arbitrage opportunities would arise. Since I set the risk free interest rate to zero, this definition of the intrinsic value reduces to the conventional definition given in the text.

[31] See, for example, Duan (1995), Chaudhury and Wei (1995), and Geyer and Schwaiger (1995).

[32] The figure for (total) percentage price deviations between EGARCH and Black-Scholes is not displayed here but is available from the author upon request.

tive difference of the time values of the two option pricing models. Again, these time-value effects decrease for options with a longer time to maturity.

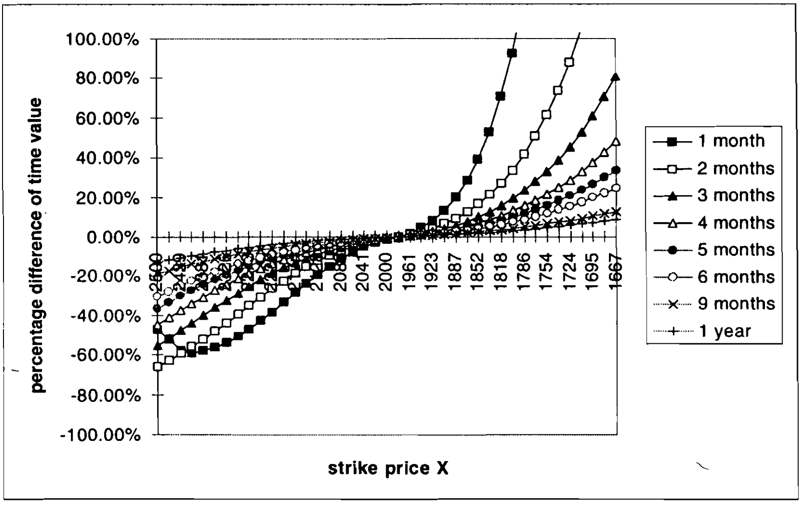

Figure 11: Price Deviation as percentage of the Black-Scholes time values

7 Conclusion

In this paper, I studied the behaviour of European call option prices when the stock return process follows a EGARCH(1,1) model. The motivation for this model is the fact that the EGARCH model is compatible with the stylised facts of non-normality, heteroskedasticity and volatility skew in implied volatilities, which is often found in stock (index) option prices after the '87 crash. Since the GARCH option pricing of Duan (1995) is not able to capture this volatility pattern, the GARCH model is extended to the EGARCH option pricing model. The resulting EGARCH option prices are evaluated by Monte Carlo simulations because the distribution of the terminal stock (index) prices cannot be derived analytically.

Because of its ability to take the leverage effect into account, the EGARCH option pricing is not only able to explain the volatility smile but also the skewness in implied volatilities. By comparing the time values of the option pricing models, instead of the absolute prices, I showed that the EGARCH option pricing model is not only suitable and relevant for out-of-the-money options but also for in-the-money options. However, this result would also hold for other option pricing models such as the GARCH option model.

Further but preliminary results, which are not presented here, show that for DAX options the skewness and smile in implied volatilities is larger than predicted by the simulated options prices when EGARCH parameters are estimated from time series of the stock returns. Since market prices for traded options are readily available, one could apply the EGARCH option pricing model to infer EGARCH parameters from option prices.[33] This would be an alternative way to measure the leverage effect. Exploring this avenue is left for future research.

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[33] The method of estimation was introduced by Engle and Mustafa (1992) for the GARCH model.

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