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Michael Sampson

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Michael Sampson
Department of Economics
Concordia University
1455 de Maisonneuve Blvd W.
Montreal, Quebec
Canada H3G 1M8

E-mail: GCZFI17@vax2.concordia.ca

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Abstract

This paper considers the intertemporal consumption/savings decision when income follows a random walk with drift and the drift coefficient is unknown. Instead agents are Bayesian learners, combining prior and sample information to form a posterior for the drift coefficient and future income. This parameter uncertainty increases by an order of magnitude the uncertainty of future income over that generated by unknown future shocks to income and can lead agents to have much more precautionary savings and hence to accumulate more wealth than otherwise. In a calibration exercise it is shown that for a plausible specification of the level of prior information and real interest rate, that the level of aggregate wealth due to this parameter uncertainty could be larger than that generated by unknown future shocks to income, the latter of which has been estimated elsewhere to potentially account for 60 percent of US aggregate wealth.

Keywords: Permanent Income Hypothesis, Precautionary Savings, Parameter Uncertainty, Bayesian learning, Wealth Accumulation.

1. Introduction:

In most permanent income models of consumption agents are assumed to know the parameters of the stochastic process generating income (see for example Caballero, 1991, Deaton, 1991, Skinner, 1988 and Zeldes, 1989). This means that in these models the only source of uncertainty for agents is that of unknown future shocks to income.

If for example, as in Caballero (1991) income is assumed to follow a random walk with drift coefficient of say μ, then future income is uncertain because agents do not know if over the course of their lifetime they will be getting a series of above or below average increases in income (i.e. future shocks) around a known average income increase of μ.

The parameter μ can naturally be thought of as a proxy for skill levels that are acquired either through nature or nurture. Lucky agents who draw a high μ will have on average higher income growth over their lifetimes than agents who draw low values of μ. There is obviously considerable variation in μ across the population and the educational system is at least partly designed to provide workers and employers with information about the skill levels and hence μ that particular workers have been allocated. However, it seems at least plausible that when an agent enters the workforce, that there will still exist considerable uncertainty about the μ that he has been allocated and that this parameter uncertainty could be the dominant source of uncertainty vis-a-vis future income. Thus the question of whether someone just finishing law school is a brilliant, average, or below average lawyer could easily be responsible for more uncertainty about future income than whether the pay raises relative to his or her particular skill level will be above or below average.

This notion that uncertainty about μ is more important that unknown future shocks to income is derived formally in the paper. In particular it is shown that whereas when μ is known the conditional forecast variance of future income increases linearly with the forecast horizon, when μ is unknown it increases with the square of the horizon. Thus given a long enough horizon an unknown μ will be responsible for more uncertainty about future income than unknown future shocks.

The paper then examines the implications of this higher order uncertainty for precautionary savings and wealth accumulation. Given a convex marginal utility of consumption function, the additional uncertainty stemming from an unknown μ will increase the incentive agents have to hold precautionary balances and hence to accumulate wealth. Estimates in Caballero (1991) and Skinner (1988) are that precautionary savings due to an unknown future shocks to income could account for up to 60 percent of US wealth. Since an unknown μ can potentially generate more uncertainty about future income, one might expect it to be an important, perhaps more important, factor in explaining precautionary savings and wealth.

In the paper I separate aggregate wealth accumulation due to unknown future shocks and unknown μ. For the model I consider the relative importance of these two components of aggregate wealth depends only on the rate of interest and the amount of prior information agents have on μ. I show that for plausible parameter values that the amount of aggregate wealth accumulated because of an unknown μ can easily be of comparable magnitude to that due to unknown future shocks. Thus the paper argues that learning and parameter uncertainty may play an important factor in explaining savings and wealth.

The plan of the paper is as follows: In Section 2 of the paper I examine the uncertainty of future income when μ is unknown and show that there are potentially very large welfare gains from precautionary savings. Section 3 then deals with the intertemporal consumption/savings decision when μ is unknown. In Section 4 the effects of future shocks and unknown μ on aggregate wealth are examined and their relative importance assessed using a calibration exercise. The paper ends with some brief concluding remarks.

2. The Uncertainty of Future Income and its Welfare Costs:

Consider an agent living for T+1 periods t=0,1,2,...T. At age t the household's utility is:

$$1)\ V_t = E_t \sum_{\tau=0}^{T-t} U(C_{t+\tau})e^{-\delta\tau} \qquad \delta \geq 0$$

where the felicity function is assumed to exhibit constant absolute risk aversion:

$$2)\ U(C) = -\exp(-\theta C)/\theta \qquad \theta > 0.$$

The income of the household Y_t follows a random walk with drift:

$$3)\ Y_t = \mu + Y_{t-1} + a_t \qquad ,\ a_t \sim N[\ 0,\ \sigma^2]\ .$$

To begin the discussion assume for the moment that μ is known (I will assume that σ^2 is known for the entire paper) so that the only source of uncertainty are future shocks to income a_t. In this case since

$$4)\ Y_{t+k} = Y_t + k\ \mu + \sum_{j=1}^{k} a_{t+j}$$

Y_{t+k} differs from its forecast $Y_t+k\mu$ because of the sum of the k future shocks

a_{t+j} j=1,2,...k. Consequently the conditional forecast variance is:

$$\mathrm{Var}_t[Y_{t+k}] = \sigma^2 k,$$

which grows linearly with the horizon k and the distribution of Y_{t+k} conditional on the information set at time t is:

$$5)\ Y_{t+k} \mid Y_t\hat{}\mu\hat{}\sigma^2 \sim N\left[Y_t + k\,\mu\ ,\ \sigma^2 k\right].$$

Now consider the case where μ is unknown but is instead randomly drawn by nature or nurture from some probability distribution. At t=0, the beginning of their productive lives, individuals have prior information about their μ, which I will denote by P_o, given by:

$$6)\ \mu \sim N[\hat{\mu}_o,\ \sigma_o^2].$$

An alternative way of expressing the prior variance, which is more convenient for our purposes, is to define $t_o \equiv \sigma^2/\sigma_o^2$ and express 6) as

$$6')\ \mu \sim N[\hat{\mu}_o,\ \sigma^2/t_o].$$

Since this is a conjugate prior it is equivalent to observing an additional sample of size t_o with sample mean $\hat{\mu}_o$.

An example of the source of this prior information could be the educational system which, by testing the skill levels of future workers and to some extent simulating the work environment, provides both workers and employers with information about the μ that a particular worker has been allocated. In this context t_o could be interpreted as the equivalent number of work years of experience that the educational system provides while $\hat{\mu}_o$ would be the educational system's assessment (or estimate) of the individual's ability level or μ.

Once the agent begins working he will observe his past and present income history and use this information to learn about μ. Denote this sample information at age t as S(t) where:

7) $S(t) = \{Y_o, Y_1, Y_2 \ldots Y_t\}$.

Since ΔY_t is independent and normally distributed with mean μ and variance σ^2, the posterior for μ comes from standard Bayesian results (see Box and Tiao, 1973 for example) and is:

8) $\mu \mid S(t) \hat{} P_o \hat{} \sigma^2 \sim N\left[\hat{\mu}_t , \sigma^2/(t+t_o)\right]$

where the posterior mean $\hat{\mu}_t$ is:

8') $\hat{\mu}_t = \hat{\mu}_{t-1} + \left[\Delta Y_t - \hat{\mu}_{t-1}\right]/(t+t_o)$

or

8'') $\hat{\mu}_t = \bar{\mu}_t + t_o\left[\hat{\mu}_o - \bar{\mu}_t\right]/(t+t_o)$

and where $\bar{\mu}_t = (Y_t - Y_o)/t$ is the sample mean of ΔY_t over $S(t)$. From a classical viewpoint as long as $t_o < \infty$ then

$$\lim_{t\to\infty} \hat{\mu}_t = \mu, \text{ almost surely}$$

so there is no concern here with the issue of the convergence of the learning process, as for example, found in Woodford (1990) and the references therein.

It turns out that an uncertain μ fundamentally alters the qualitative nature of uncertainty. To see this note that from 4), 8) and the fact that μ is independent of future shocks that:

9) $$\mathrm{Var}_t[Y_{t+k}] = \mathrm{Var}\left[\sum_{j=1}^{k} a_{t+j}\right] + k^2\, \mathrm{Var}_t[\mu]$$
$$= \sigma^2 k + \sigma^2 k^2/(t+t_o)$$

so that the posterior for Y_{t+k} conditional on the information at t is:

10) $Y_{t+k} \mid S(t) \hat{} P_o \hat{} \sigma^2 \sim N\left[Y_t + k\,\hat{\mu}_t , \sigma^2 k + \sigma^2 k^2/(t+t_o)\right]$.

Comparing 10) with 5) we see that the conditional variance goes from being $O(k)$ when μ is known to $O(k^2)$ when μ is unknown. In particular for $k > t+t_o$ parameter uncertainty will dominate unknown future shocks as a source of uncertainty about future income.

The fact that $Var_t[Y_{t+k}]$ is $O(k^2)$ when μ is unknown holds generally. This can be seen by noting that if a series is growing at a rate $\mu(t+k)$, that the forecast error caused by μ being unknown for a k period horizon is $(\mu-\hat{\mu})k$ which has a variance $Var[\mu]k^2$ in the Bayesian case and $Var[\hat{\mu}]k^2$ in the classical case. Different specifications of the stochastic process will only alter the form of $Var[\mu]$ or $Var[\hat{\mu}]$ so that for large enough k this dominates the effects of unknown future shocks. (See Sampson 1991, 1993a and 1993b for a fuller discussion of these issues).

That precautionary savings in the face of this greater uncertainty has potentially large welfare implications for the agent can be seen by considering an agent without access to precautionary savings so that $C_t=Y_t$ (a so-called Keynesian consumption function). In this case by combining 1) and 10) and using the fact that if $X\sim N[\mu,\sigma^2]$ then $E[\exp(X)]=\exp(\mu+\sigma^2/2)$, it follows that welfare at time t is:

$$11) \quad V_t^1 = U(C_t)\sum_{\tau=0}^{T-t}\exp\left[-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right]$$

where:

$$11') \quad \lambda_{1t} = \delta + \theta\,(\hat{\mu}_t-\theta\sigma^2/2) \ ,$$

$$\lambda_{2t} = \frac{\theta^2\sigma^2}{2(t+t_o)} > 0 \ .$$

Since the term $\lambda_{2t}\tau^2$ in 11) is positive, it must eventually dominate the term $-\lambda_{1t}\tau$ for large enough T. Thus

$$12) \quad \operatorname*{Limit}_{T\to\infty} V_t^1(t_o,T) = -\infty$$

so that for long enough horizons the negative impact of parameter uncertainty dominates all other factors influencing welfare.

Alternatively, the cost of uncertainty can be measured by the amount of consumption $\tilde{\gamma}_t(t_o,T)$ an agent would be willing to sacrifice in return for knowing that future income will be $Y_t + \hat{\mu}_t\tau$ with certainty. This is:

$$U(C_t)\sum_{\tau=o}^{T-t}\exp\left[-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right] = U\left[C_t - \tilde{\gamma}_t(t_o,T)\right]\sum_{\tau=o}^{T-t}\exp\left[-\lambda_{1t}\tau\right]$$

or:

$$13)\quad \tilde{\gamma}_t(t_o,T) = \ln\left[\sum_{\tau=o}^{T-t}\exp\left[-\lambda_{1t}\tau + \lambda_{2t}\tau^2\right] \Big/ \sum_{\tau=o}^{T-t}\exp\left[-\lambda_{1t}\tau\right]\right]/\theta.$$

The same reasoning as above leads to the conclusion that:

$$14)\quad \underset{T\to\infty}{\text{Limit}}\ \tilde{\gamma}_t(t_o,T) = \infty$$

so that the welfare costs of an unknown μ increase without bound if the agent cannot or will not engage in precautionary savings. Thus there are potentially very large incentives for agents to engage in precautionary savings.

3. Precautionary Savings with Parameter Uncertainty:

I will now investigate the extent to which the use of precautionary savings could avoid the potentially large welfare costs associated with an unknown μ. Suppose that the agent can borrow and save at a constant real rate of interest so that wealth A_t evolves according to:

$$15)\quad A_t = e^r(A_{t-1} + Y_{t-1} - C_{t-1}) \quad , \ r > 0 \ , \quad A_o = 0$$

and where r is the real rate of interest.[1] There is no bequest motive so that $A_{T+1}=0$ or alternatively:

$$16)\quad C_T = Y_T + A_T.$$

[1] I use e^r rather than 1+r for notational convenience. One can, however, convert by simply replacing all occurrences of e^r with 1+r. Nothing depends on this notational convention.

Maximizing welfare at time t and assuming that $r=\delta$ leads to the standard Euler equation:

$$17)\quad U'(C_t) = E_t\left[U'(C_{t+1})\right] \text{ or}$$

$$17')\quad \exp(-\theta C_t) = E_t\left[\exp(-\theta C_{t+1})\right].$$

In the appendix it is shown that the general solution to 17) subject to the income generating process 3), the posterior for future income 10) and the wealth equations 15) and 16) is:

$$18)\quad C_t = Y_t + \alpha_t(T)\,A_t + \beta_t(T)\,\hat{\mu}_t - \gamma_t(t_o,T)$$

where:

$$19)\quad \alpha_t(T) = (1 - e^{-r})/(1 - e^{-r(T+1-t)}),$$

$$20)\quad \beta_t(T) = \frac{e^{-r}}{(1 - e^{-r})}\left[\frac{1 - e^{-r(T-t)} - (T-t)\,e^{-r(T-t)}(1 - e^{-r})}{1 - e^{-r(T-t+1)}}\right] \text{ and}$$

$$21)\quad \gamma_t(t_o,T) = \left[\alpha_t(T)\,\theta\sigma^2/2\right] \times$$

$$\sum_{k=1}^{T-t} e^{-rk}\left[1 + \frac{\beta_{t+k+1}(T)}{t+t_o+k}\right]^2\left[1 + \frac{1}{t+t_o+k-1}\right]/\alpha_{t+k+1}(T).$$

Note that $\alpha_t(T)$, $\beta_t(T)$ and $\gamma_t(t_o,T)$ are all positive.

From 18) it follows that savings $S_t \equiv Y_t - C_t$ is:

$$22)\quad S_t = -\alpha_t(T)\,A_t - \beta_t(T)\,\hat{\mu}_t + \gamma_t(t_o,T)$$

so that $\gamma_t(t_o,T)$ determines the level of precautionary savings due to uncertain future income for a given level of wealth and posterior mean $\hat{\mu}_t$.

The value function can by determined by using the law of iterated expectations and the fact that the marginal utility of consumption is proportional to the utility of consumption to obtain:

$$23)\quad V_t^2(t_o,T) = U(C_t)/\alpha_t(T)$$

From 23) it follows that an alternative interpretation of $\gamma_t(t_o,T)$ is the welfare cost of uncertain future consumption. In particular at time t the individual would be indifferent between his present random income process and a deterministic income process of:

$$Y_{t+\tau} = Y_t + \tau \hat{\mu}_t - \gamma_t(t_o,T) \qquad \tau=0,1,2...T-t.$$

Thus $\gamma_t(t_o,T)$ can be compared with $\tilde{\gamma}_t(t_o,T)$ in 13) to assess the relative welfare costs of future income uncertainty with and without precautionary savings.

Unlike the case where agents do not have access to precautionary savings, the effect of uncertain future income on welfare is bounded as the horizon T approaches infinity. This can be seen by inspecting 19), 20), and 21). As long as $r > 0$, the exponential discounting insures that the limits of as $T \to \infty$ of $\alpha_t(T)$, $\beta_t(T)$ and $\gamma_t(t_o,T)$ are all finite. These are given respectively by:

24) $\alpha = (1 - e^{-r})$,

25) $\beta = e^{-r}/(1 - e^{-r})$ and

26) $\gamma_t(t_o) = \theta\sigma^2/2 \sum_{k=1}^{\infty} e^{-rk}\left[1 + \frac{\beta}{t+t_o+k} \right]^2\left[1 + \frac{1}{t+t_o+k-1}\right]$.

In particular since $\gamma_t(t_o) < \infty$ the welfare costs of uncertain income are finite with an infinite horizon even though they were infinite (i.e. $\tilde{\gamma}_t(t_o,\infty) = \infty$) when the agent did not have access to precautionary savings.

The reason for the sharp divergence in the effects of uncertainty on welfare with and without precautionary savings is that precautionary savings allows the agent to detach the consumption process from the highly uncertain income process. With or without precautionary savings $Var_t[Y_{t+k}] = O(k^2)$. Without precautionary savings $C_{t+k} = Y_{t+k}$ so that $Var_t[C_{t+k}] = O(k^2)$ and this then causes the infinite welfare loss when the horizon is infinite.

However, with precautionary savings $\mathrm{Var}_t[C_{t+k}]$ increases at a lower rate. In the appendix I show that in this case $\mathrm{Var}_t[C_{t+k}]$ is given by:[2]

$$\mathrm{Var}_t[C_{t+k}] = (2/\theta)\left[(e^r-1)\sum_{j=0}^{k-1}\gamma_{t+j}(t_o) + \gamma_t(t_o) - \gamma_{t+k}(t_o)\right].$$

Since $\gamma_t(t_o)$ is decreasing in t:

$$\mathrm{Var}_t[C_{t+k}] \le (2/\theta)(e^r-1)\gamma_t(t_o)(k+1).$$

and hence $\mathrm{Var}_t[C_{t+k}]$ is O(k). Thus precautionary savings reduces the uncertainty of future consumption to the same order of magnitude as when μ is known.[3]

Since $Y_{t+k} = C_{t+k} + S_{t+k}$ and $\mathrm{Var}_t[Y_{t+k}]$ is $O(k^2)$, it follows that $\mathrm{Var}_t[S_{t+k}] = O(k^2)$ so that precautionary savings shifts the higher order $O(k^2)$ uncertainty from consumption to savings. From 22) and the fact that $\hat{\mu}_{t+k}$ is $O_p(1)$, it follows that $\mathrm{Var}_t[A_{t+k}]$ is $O(k^2)$ as well so that wealth shares the same order of uncertainty as savings.

4. Wealth Accumulation with an Infinite Horizon:

In this section of the paper I will investigate the implications of parameter uncertainty for wealth accumulation. To simplify the calculations I

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[2]Note that this expression is independent of $\hat{\mu}_t$ and hence is deterministic and that since γ_t is proportional to θ, $\mathrm{Var}_t[C_{t+k}]$ is independent of θ as well.

[3]When μ is known, it is shown below that $\gamma_t(t_o)=\beta\theta\sigma^2/2$ and the O(k) result then follows directly.

will assume an infinite horizon. I will first derive the wealth that an individual of the generation having age t will have accumulated and then aggregate over all generations.

From 18) with $T=\infty$ consumption for someone of generation t will be:

27) $C_t = Y_t + \alpha A_t + \beta \hat{\mu}_t - \gamma_t(t_o)$,

or, using 24) and $S_t \equiv Y_t - C_t$:

27') $A_t + S_t = e^{-r} A_t - \beta \hat{\mu}_t + \gamma_t(t_o)$.

Since $A_{t+1} = e^r(A_t + S_t)$:

28) $A_{t+1} - A_t = -\hat{\mu}_t/(1-e^{-r}) + e^r \gamma_t(t_o)$

and since $A_o = 0$:

28') $A_t = - \mu t/(1-e^{-r}) + \sum_{k=1}^{t} \left[(\mu-\hat{\mu}_{t-k})/(1-e^{-r}) + e^r \gamma_{t-k}(t_o) \right]$.

The term $- \mu t/(1-e^{-r})$ reflects the amount of wealth that would be accumulated without any uncertainty. The two terms in the brackets of the summation in 28') capture the effect of uncertainty at time t-k on wealth at time t. The first of these $(\mu-\hat{\mu}_{t-k})/(1-e^{-r})$ reflects errors made in the estimation of μ. The second of these terms $e^r \gamma_{t-k}(t_o)$ reflects the precautionary savings motive at time t-k.

Note from either 28) or 28') that the effect of uncertainty on wealth accumulation is permanent; that is, the effects of $(\mu-\hat{\mu}_{t-k})/(1-e^{-r})$ and $e^r \gamma_{t-k}(t_o)$ on future wealth do not diminish over time. Thus even though the agent is learning and hence his uncertainty is diminishing, there is no tendency for him to attempt to undo the effects of past errors or savings decisions on his present level of wealth.

$\gamma_t(t_o)$ reflects the two sources of uncertainty facing the agent: 1) unknown future shocks to income and 2) the fact that μ is unknown. It is possible to decompose $\gamma_t(t_o)$ accordingly as:

$$29)\ \gamma_t(t_o) = \gamma_1 + \gamma_{2t}(t_o)$$

where

$$30)\ \gamma_1 = \beta\theta\sigma^2/2 > 0$$

reflects the uncertainty of unknown future shocks to income and

$$31)\ \gamma_{2t}(t_o) = \theta\sigma^2/2 \sum_{k=1}^{\infty} e^{-rk}\left[\left[1 + \frac{\beta}{t+t_o+k}\right]^2\left[1 + \frac{1}{t+t_o+k-1}\right] - 1\right]$$

reflects the uncertainty stemming from μ being unknown.

It also is possible to decompose wealth along similar lines. From 28') and 29) and 30):

$$32)\ A_t = -t\left[\mu - \theta\sigma^2/2\right]/(1-e^{-r}) - \sum_{k=1}^{t}\left[\hat{\mu}_{t-k}-\mu\right]/(1-e^{-r})$$
$$+ e^{r}\sum_{k=1}^{t}\gamma_{2t-k}(t_o).$$

so that the effect of unknown future shocks on wealth is to adjust the effective μ down by $\theta\sigma^2/2$.

I will now aggregate over individuals and generations in order to assess the potential importance of an unknown μ on aggregate wealth accumulation. I will take it as given that wealth accumulation stemming from unknown future shocks to income is significant, for example appealing to Caballero(1991) and Skinner(1988) and their estimate that this can potentially account for 60 percent of US wealth. What I will attempt to do is measure the relative importance of wealth accumulation due precautionary savings caused by μ being unknown to that caused by unknown future shocks to income. To do this without getting bogged down in calibration exercises over too many dimensions, in particular with respect to μ, σ, and θ, I will make a number of simplifying aggregation assumptions. These are:

Aggregation Assumptions

A1. The population mean of μ across generations and individuals is zero.

A2. The prior information that individuals possess is given by:

$$\mu \sim N[\mu_o, \sigma^2/t_o]$$

where σ^2 and t_o is the same for all individuals. μ_o may vary across individuals but it is assumed to be an unbiased estimate of μ in the sense that across generations and individuals the population mean of $\mu_o - \mu$ is zero.

A3. The proportion of the population with age t is $e^{-rt}(1-e^{-r})$.

A4. The number of individuals having age t is large enough to allow use of the law of large numbers so that sample means and population means are identical.

A1 rules out the negative (positive) effect on aggregate wealth of a positive (negative) population mean of μ. Caballero (1991) in his calibration exercises also assumed that (the known value of) μ is zero. Alternatively, one can interpret the results below as being for wealth net of the effects of any nonzero μ.

A2 can be thought of as assuming that the educational system provides unbiased estimates of individual's skill levels and that the equivalent number of work years that the educational system provides is the same for all individuals. In addition all individuals have equally variable income.

A3 can be interpreted two ways. The first is that individuals live forever but that the number of individuals born each period grows at a rate of r each year so that r, in addition to being the discount rate and interest rate, is also the rate of population growth. Alternatively, each year the same number of individuals are born but at each subsequent year they face a constant probability of death of $1-e^{-r} \approx r$. In this case each individual's life expectancy is $e^{-r}/(1-e^{-r}) \approx 1/r$.

Denote aggregate wealth by A and let $A = A^1 + A^2$ where A^1 is the level of aggregate wealth that would be accumulated due to unknown future income shocks if μ were known, while A^2 is the level of aggregate wealth that can be attributed to uncertainty about μ.

From A1 and A4 we can for aggregate calculations set $\mu=0$ in 32). The first term in 32) then yields wealth accumulated because of unknown future shocks by individuals having age t. Using A3 to weight the contributions of all generations then yields:

$$33)\ A^1 = \theta\sigma^2/2 \sum_{t=0}^{\infty} te^{-rt} = \left[\theta\sigma^2/2\right] e^{-r}/(1-e^{-r})^2 .$$

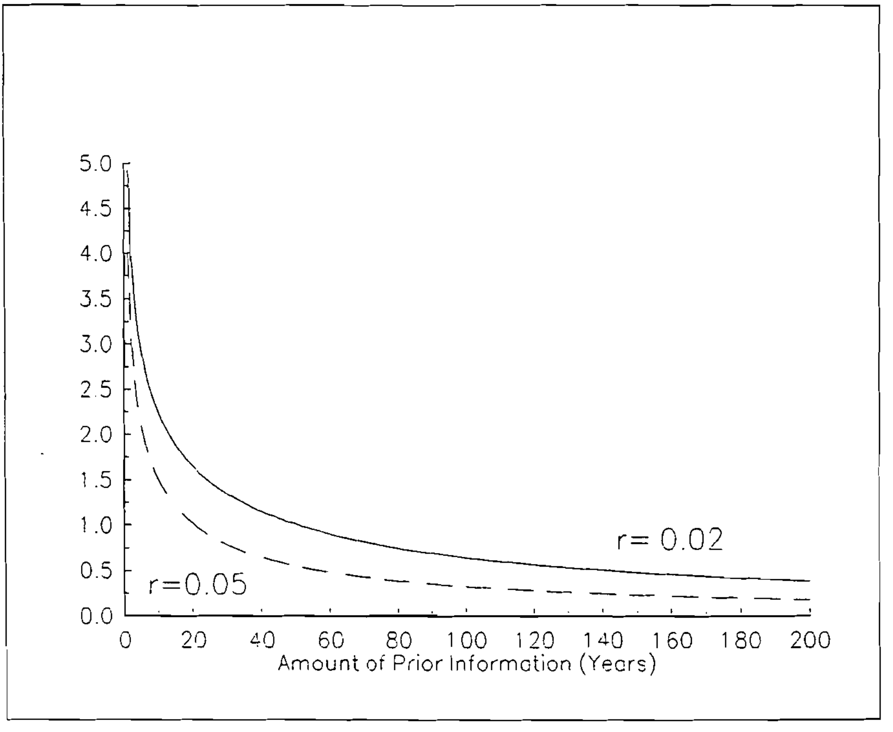
From A2 there is no systematic bias in prior beliefs. Since from 8'') the posterior mean is a weighted combination of the unbiased sample mean and the prior mean, it follows that the there will be no bias across individuals in the posterior mean or , from A4), the sample mean of $\hat{\mu}_t-\mu$ across individuals will be zero. Hence we can ignore the second term in 32) as far as aggregate calculations are concerned.

From A2 the third term in 32) is identical for all individuals in generation t since t_o and σ^2 do not vary across individuals. Using A3 to aggregate this term over all generations then yields:

$$34)\ A^2 = e^{r}(1-e^{-r}) \sum_{t=0}^{\infty} e^{-rt} \sum_{k=1}^{t} \gamma_{2t-k}(t_o).$$

Reversing the order of the double summation and replacing the t index by t-k and simplifying then yields:

Diagram 1 A Plot of $\rho(t_o,r)$ for r = 0.02 and r=0.05



35) $A^2 = \sum_{t=0}^{\infty} e^{-rt}\gamma_{2t}(t_o).$

Now, by substituting the definition of $\gamma_{2t}(t_o)$ in 31), and replacing this double sum over t and k by a single sum over t+k yields:

$$36)\ A^2 = \left[\theta\sigma^2/2\right]\sum_{k=1}^{\infty} k\ e^{-rk}\left[\left[1 + \frac{\beta}{k+t_o}\right]^2\left[1 + \frac{1}{k+t_o-1}\right] - 1\right] .$$

Note that both A_1 and A_2 are proportional to $\theta\sigma^2/2$ so that the relative importance of the two is independent of θ and σ^2.

Define $\rho(t_o,r) \equiv A^2/A^1$ as the ratio of the two components of aggregate wealth which from 35) and 36) is:

$$37)\ \rho(t_o,r) = e^r(1-e^{-r})^2\sum_{k=1}^{\infty} k\ e^{-rk}\left[\left[1 + \frac{\beta}{k+t_o}\right]^2\left[1 + \frac{1}{k+t_o-1}\right] - 1\right].$$

The function $\rho(t_o,r)$ is plotted in Diagram 1 with a solid line for r=0.02 and a dashed line for r=0.05 for t_o ranging from 1 to 200 years. The value of r=0.02 would imply a income generating life expectancy of 50 years and a population growth rate of 2 percent, which is about right for modern developed economies. A value of r=0.05 may be more realistic for the real rate of interest and the discount factor but would imply an income generating life expectancy of only 20 years and a much too high 5 percent rate of population growth.

As an example consider the possibility that agents enter the work force at age 20 years having received an equivalent of 20 years working experience from the educational system. Using t_o=20 in Diagram 1 indicates that with an interest rate of two percent, wealth accumulated through precautionary savings because of μ being unknown will be 65 percent larger than wealth accumulated because of unknown future shocks (i.e. $\rho(20,0.02)=1.65$.)

Increasing the rate of interest to 5 percent lowers this so that the two sources of wealth accumulation are of about the same magnitude, in particular $\rho(20,0.05)=1.02$. Hence in this case an unknown μ is at least as important and likely a more important determinant of aggregate wealth.

Let us now increase t_o by a factor of 10 so that the educational system provides an equivalent of $t_o=200$ years of working experience. In this case from the calculations for Diagram 1 $\rho(200,0.02) = 0.38$ and $\rho(200,0.05) = 0.18$. Thus while unknown future shocks would be a more important source of aggregate wealth accumulation, the effect of an unknown μ would still be economically significant.

In general $\rho(t_o,r)$ decays slowly as t_o increases; in particular as $t_o \to \infty$:

38) $\rho(t_o,r) = (2\beta+1)/t_o + O(t_o^{-2})$.

Given this slow decay it would require fairly large values of t_o before wealth accumulation due to an unknown μ would be economically insignificant. One could, for example, define insignificance as being less than 1 percent of aggregate wealth due to unknown shocks or $\rho(t_o,r) \le 0.01$. Using the approximation in 38) for $r=0.02$ and $r=0.05$:

$$\rho(t_o,0.02) \approx 100/t_o$$

$$\rho(t_o,0.05) \approx 40/t_o$$

which would require respectively $t_o \ge 10{,}000$ years and $t_o \ge 4{,}000$ years.

4. Conclusions:

In this paper it has been shown that parameter uncertainty can be a potentially important factor in explaining precautionary savings and wealth accumulation. This is because parameter uncertainty increases by an order of magnitude the uncertainty of the income stream. Precautionary savings then allows the household to transfer this higher order of uncertainty away from consumption and into savings and hence insulates welfare from this higher order of uncertainty. This would appear to make even more puzzling the Carroll-Summers result that consumption often appears to closely follow income over the life-cycle.

The quantitative nature of this effect depends crucially on the amount of prior information agents have on μ and it is difficult to say exactly how this should be calibrated. Some idea of this could perhaps come from studies of the predictive power of education and performance in the educational system on future earnings growth.

Appendix:

1. Proof of 19) 20) and 21).

From the Euler equation 17') for t-1 substitute 18) in for C_t to yield:

$$\text{A.1)}\quad \exp(-\theta C_{t-1}) = E_{t-1}\left[\exp\left[-\theta\ (\ Y_t + \alpha_t A_t + \beta_t \hat{\mu}_t - \gamma_t)\right]\right]$$

$$= E_{t-1}\left[\exp\left[-\theta\ (\ Y_{t-1} + \Delta Y_t + \alpha_t e^{r}(A_{t-1} + Y_{t-1} - C_{t-1}) + \beta_t(\hat{\mu}_{t-1} + (\Delta Y_t - \hat{\mu}_{t-1})/(t+t_o) - \gamma_t)\right]\ \right]$$

where the second equality follows $Y_t = Y_{t-1} + \Delta Y_t$, 8') and 15) and where I suppress the dependence of the coefficients on t_o and T. Collect the terms which are in the information set and those which are not. The term not in the information set will be

$$\text{A.2)}\quad E_{t-1}\left[\exp(-\theta(1+\beta_t/(t+t_o))\Delta Y_t)\right] =$$

$$\exp\left[-\theta(1+\beta_t/(t+t_o))\hat{\mu}_{t-1} + (\theta^2\sigma^2/2)\left[\left[1+\beta_t/(t+t_o)\right]^2\left[1+1/(t+t_o-1)\right]\right]\right]$$

since the distribution of ΔY_t conditional on the information set at t-1 is:

$$\Delta Y_t \,|\, S(t-1)\wedge P_o\wedge\sigma^2 \;\sim\; N\left[\hat{\mu}_{t-1},\ \sigma^2\left[1+1/(t+t_o-1)\right]\right]$$

which follows from 10) with t replaced by t-1 and k=1.

Using A.2) in A.1) and solving for C_{t-1} then yields:

A.3) $C_{t-1} = Y_{t-1} + \alpha_{t-1}A_{t-1} + \beta_{t-1}\hat{\mu}_{t-1} - \gamma_{t-1}$

where:

A.4) $\alpha_{t-1} = \alpha_t e^r/(1+\alpha_t e^r)$,

A.5) $\beta_{t-1} = (1+\beta_t)/(1+\alpha_t e^r)$ and

A.6) $\gamma_{t-1} = \left[\gamma_t + (\theta\sigma^2/2)(1+\beta_t/(t+t_o))^2(1+1/(t+t_o-1))\right]/(1+\alpha_t e^r)$.

From A.4) it follows that

A.7) $\alpha_{t-1}^{-1} = e^{-r}\alpha_t^{-1} + 1$

so that using $\alpha_T = 1$ and solving A.7) forwards results in 19). To obtain 20) and 21) note that from A.4)

A.8) $1/(1+\alpha_t e^r) = e^{-r}\alpha_{t-1}/\alpha_t$

so that if $\tilde{\beta}_t = \beta_t/\alpha_t$ and $\tilde{\gamma}_t = \gamma_t/\alpha_t$ then A.5) and A.6) can be rewritten as:

A.9) $\tilde{\beta}_{t-1} = e^{-r}\tilde{\beta}_t + e^{-r}\alpha_t^{-1}$

A.10) $\tilde{\gamma}_{t-1} = e^{-r}\tilde{\gamma}_t + e^{-r}(\theta\sigma^2/2)(1+\beta_t/(t+t_o))^2(1+1/(t+t_o-1))/\alpha_t$.

Again solving forwards and using $\tilde{\beta}_T = \tilde{\gamma}_T = 0$ then yields, after some straightforward manipulation, the required results.

2. Derivation of $Var_t[C_{t+k}]$.

From the Euler equation in 17') and the conditional normality of C_{t+k} it follows that:

$$E_t[\exp(-\theta C_{t+k})] = \exp(-\theta C_t)$$

so that from the conditional normality of C_{t+k} it follows that:

A.11) $\mathrm{Var}_t[C_{t+k}] = (2/\theta)\ E_t[C_{t+k} - C_t].$

From 18) with $T=\infty$:

$$E_t[C_{t+k} - C_t] = k\ \hat{\mu}_t + (1-e^{-r})\ E_t[A_{t+k} - A_t] + \gamma_t(t_o) - \gamma_{t+k}(t_o).$$

$$= (e^{r}-1)\sum_{j=o}^{k-1}\gamma_{t+j}(t_o) + \gamma_t(t_o) - \gamma_{t+k}(t_o)$$

where the last equality follows from using 29) to evaluate $E_t[A_{t+k} - A_t]$.

Hence $\mathrm{Var}_t[C_{t+k}]$ is:

A.12) $\mathrm{Var}_t[C_{t+k}] = (2/\theta)\left[(e^{r}-1)\sum_{j=o}^{k-1}\gamma_{t+j}(t_o) + \gamma_t(t_o) - \gamma_{t+k}(t_o)\right]$

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