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by

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**Abstract**

In this paper the implicit early unwind option of a risk neutral arbitrageur is valued. The problem is analyzed in a market microstructure framework where four different groups of market participants interact. Within this model the equilibrium price relationship between stock and futures markets is determined. Since the underlying of the option is influenced by arbitrage trading, the underlying of the option depends, contrary to standard option pricing theory, on the unwind option itself. The non-Markovian stochastic process of the basis is characterized, and the results of an extensive comparative static analysis of the option value are presented.

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# 1. Introduction

The model of cash and carry arbitrage is the most popular approach for pricing futures on storable goods. Of all of the underlying assumptions for this approach, the two fundamental assumptions are that arbitrageurs alone determine the difference between spot and futures prices, and that arbitrage positions are held until maturity.

There are good reasons to question these assumptions. First, arbitrageurs can determine the spread between futures and spot prices, the basis, only if arbitrage is risk free, arbitrage capital is unlimited, and capital markets are frictionless. If any of these assumptions does not hold, other traders will also influence the basis. Second, as futures are standardized contracts which are traded at organized exchanges, open futures positions can be closed at any time before maturity, and that means that there is no obligation for arbitrageurs to hold arbitrage positions until maturity. The arbitrageur has the option to unwind arbitrage positions before maturity whenever it is favourable to him or her, and this is a strategy frequently used. Sofianos (1993) provides evidence that about 70 % of all arbitrage positions are unwound before maturity. Merrick (1989) estimates that early unwinding profits represent up to 44 % of total arbitrage profits.

With these results in mind, our paper will relax the standard assumptions in the following way: different groups of investors, including arbitrageurs, determine the basis, and arbitrage positions may be closed before maturity.

Using these modified assumptions, we can value the early unwind option of an arbitrageur in the framework of a market microstructure model. Contrary to standard option pricing theory, the *underlying of this option*, the basis, is an *endogenous* variable which depends on the trading behavior of all groups of investors. Therefore, *the basis depends on the early unwind option of the arbitrageur itself.*

There are a few approaches in the literature to value the early unwind option. Brennan and Schwartz (1988, 1990) analyze the value of the option with an *exogenous* basis assuming that the basis follows a Brownian bridge process. Within this model, Duffie (1990) derives analytically the optimal stopping time of the early unwind option. Tuckman and Vila (1992) using a similar setup study the optimal arbitrage strategy without analyzing the value of the early unwind option. The most important criticism against these approaches, already pointed out by Brennan and Schwartz (1990), is that the basis should be determined endogenously as the typical role of arbitrageurs is to influence the basis. This is done in two papers by Holden (1990) and Cooper and Mello (1990). Holden focuses, like Tuckman and Vila (1992), on the optimal arbitrage strategy. He derives the optimal trading rule of an arbitrageur and the impact of arbitrage trading on the endogenous basis. As Holden (1990) considers neither asymmetric transaction costs nor capital restrictions, he cannot study the value of an early unwind option. Using a model similar to Holden (1990), Cooper and Mello (1990) allow for asymmetric transaction costs and determine the value of the early unwind option. But, there are two assumptions in their model of the market which are difficult to justify. First, there is a chronological segmentation in the execution of orders so that arbitrage trading is risk free. Second, the basis follows a mean reverting process to its cash and carry value even if arbitrageurs do not trade. Thus, arbitrageurs have no specific role in the market. Both assumptions are avoided in our model.

Our model extends the literature on the valuation of the unwind option by bringing together a market microstructure equilibrium model of the price relationship between futures and spot markets with an option pricing approach. The option valuation is done by comparing the expected profits of an optimal arbitrage strategy with and without early unwinding in a dynamic equilibrium setting. The spot price is endogenously determined by the trading of four different groups of investors, the futures price process is taken as exogenous. So, the endogenous part of the basis is the spot price which is determined by the interaction of four groups of investors with different motives for trading. We model the demand functions of a monopolistic risk neutral arbitrageur, two groups of risk averse speculators differing in the price forecasting model they use, and noise traders. Contrary to the arbitrageur, speculators are not assumed to form rational expectations: one group uses historical spot price series, the other group uses the current price in the futures market as a predictor of future spot price changes. Consequently, the exogenous futures price has an impact on the spot price not only through arbitrage trading, but also through its informational role for the speculative demand. This modelling of speculative demand is similar to De Long et al. (1990), Cutler, Poterba and Summers (1990) and De Grauwe and Dewachter (1993). It accounts for the empirical finding of several authors that expectations in financial markets are not formed in a uniform way.[1]

The basic structure of the model is similar to Fremault (1991) where the impact of stock index arbitrage on prices, volatilities and welfare is analyzed in a static rational expectation equilibrium. In her model three groups of rational investors with different motives for trading and unequal access to markets interact. However, as Fremault (1991) models a one period problem, she cannot address the questions raised in this paper as she uses a static market setting. In order to analyze the process of opening and closing arbitrage positions over time, a dynamic model for the optimal arbitrage strategy has to be developed.

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1 Frankel/Froot (1986, 1988) and Allen/Taylor (1989, 1992) provide empirical evidence about the weight of chartists and fundamentalists in foreign exchange markets.

The dynamic structure of our model is similar to the model of Kyle (1985). As in Kyle (1985), one monopolistic investor, the arbitrageur in our case, is fully rational and determines his optimal solution to a dynamic optimization problem given the behavior of all other traders. A specific feature of our model is an order execution lag which all traders have to consider when placing their orders. As a result, investors do not know the prices at which their orders will be executed, and arbitrage is, in fact, not risk free. In addition, the arbitrageur has to pay transaction costs for each opening and closing transaction.[2] As unwinding a position is assumed to result in lower transaction costs, the current arbitrage position has an impact on these costs and, therefore, on the trading strategy of the arbitrageur. Like Kyle (1985), we model the trading process as a sequence of auctions in a pure exchange economy. But, contrary to him, trading is basically not caused by asymmetric information of the investors, but by different motives for trading and different price expectations.

Finally, our model is related to microstructure models analyzing several markets, for example Bhushan (1991), Chowdrhy/Nanda (1991), Subrahmanyam (1991). There is an essential difference concerning the interaction of the markets between our model and these approaches. In their models the markets are connected by traders who choose the best market to place their orders which results in a shifting demand between competing markets. Contrary to that, in our model speculators watch both markets, but are restricted to trade only in the spot market, whereas arbitrageurs trade in both markets simultaneously or do not trade at all.

Within the equilibrium model outlined above, the expected profits of an optimal arbitrage strategy with and without early unwinding are determined. The value of the unwind option is defined as the present value of the difference between the expected profits of these strategies. It depends mainly on the futures price, the spot price and the net long or short position the arbitrageur holds. In addition, it is shown by numerical comparative statics how the option value depends on time to maturity, volatility of the exogenous variable, market liquidity, and transaction costs.

The main results concerning the value of the option are summarized below. Other things being equal, the option value increases with time to maturity, transaction costs, and market liquidity. Contrary to standard option pricing models, the impact of changes in the volatility of the exogenous variable on the option value differs for in- and out-of-the-money-options. Furthermore, the option value increases with the number of open arbitrage positions held by the arbitrageur, but at a decreasing rate. The justification for this result is that closing out one unit of

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2 Holding costs which are analyzed by Tuckman/Vila (1992) are not considered in the model. However, the model could easily be extended with respect to different types of costs the arbitrageur faces.

the position affects the endogenous basis adversely and therefore reduces the value of the option.

The effect of the current basis on the option value is complex. In general, the numerical results show that the value of the unwind option is mainly determined by the size of the cash and carry basis relative to the number of open arbitrage positions, but the impact of the cash and carry basis on the option value is typically not monotonic. Also, even if the basis and the net number of arbitrage positions are fixed, the value of the option depends on the current spot price.

The model presented here improves the understanding of the arbitrageur´s behavior between spot und futures markets by considering the impact of the unwind option and may explain the path dependency of the basis found by MacKinlay and Ramaswamy (1988). Since the current behavior of the arbitrageur is influenced by his trades in the past, which were mainly determined by the basis at that time, future changes in the basis depend on the path of the basis in the past. Also, the results may be seen in another context. Unwinding is a strategy usable in futures markets where standardized contracts are traded. In contrast, forwards are individual agreements between partners so that unwinding is much harder to realize. Thus, the unwind option may be a reason for price differences between futures and forwards even when interest rates are not stochastic.[3]

The remainder of the paper is organized as follows. In section 2 the microstructure model is developed, and the equilibrium price process is determined. In section 3 the unwind option is defined, and the determinants of its value are discussed. Section 4 contains the main body of results. In the final section the most important findings are summarized, and avenues of further research are outlined.

## 2. Microstructure Model

### 2.1. Basic Setup

Three markets are considered in the model: the futures market, the spot market, and the money market. The futures market is exogenous, and only one futures contract is traded at a time. The price of that contract is assumed to follow a random walk process. There are no transaction costs, and the market is free of

[3] It is wellknown that in an economy with non-stochastic interest rates and zero transaction costs, the prices of futures and forwards are equal in absence of arbitrage opportunities. See for example Cox/Ingersoll/Ross (1981).

restrictions. It is assumed that futures are settled at the equilibrium spot price on the maturity date, and that there is neither a marking to market nor an initial margin payment.[4]

In the money market, the investors can borrow or lend a riskless asset without restrictions. The return of the asset is assumed to be constant and exogenous.

In the spot market, the arbitrageur, noise traders, and speculators trade a single asset. The class of speculators is divided into two groups which differ in the information they use to forecast future spot price changes. One group of speculators consists of positive feedback traders, and the other of information traders.

*Positive feedback traders* are risk averse investors who buy assets when prices rise and sell them when prices fall. This kind of behavior may result from *extrapolative expectations* about future spot price changes in the following sense: an asset which has strongly performed in the past is expected to outperform in the future. Using this forecast model, positive feedback traders choose a portfolio consisting of the spot market instrument and the risk free asset which maximizes wealth in a mean-variance framework. They do not take positions in the futures market. Transaction costs are assumed to have neither an impact on the demand of this group of traders nor on the demand of information traders.

*Information traders* are also assumed to be risk averse, to choose a mean variance efficient portfolio, and not to take positions in the futures market. They differ from positive feedback traders by the forecasting model for future spot price changes which they use.[5] With respect to the relationship of futures and spot prices, their expectations are *almost rational* as they assume that the spot price exhibits a mean reversion behavior to the spot price implied by the futures price through cash and carry arbitrage. That suggests that information traders expect an almost parallel movement of spot and futures prices and use the current futures price as an information source for their trades in the spot market. This behavior can be rationalized by the empirical finding of several papers that the futures market leads the spot market.[6] Therefore, if the current basis is above the cash and carry basis, the information traders expect that the basis will decrease and the spot price will increase in the future. In this case information traders exhibit a smaller demand in the spot market as in the case of a current basis which is below

4 This assumption is taken for convenience and could easily be omitted since the riskless rate of return is assumed to be constant.

5 Information traders should not be confused with informed traders which usually denote investors who have better information than other investors. For a model with informed traders see for example Kyle (1985).

6 For example, see Stoll/Whaley (1990), Chan (1992), and Grünbichler/Longstaff/Schwartz (1992).

the cash and carry basis. This information effect which is "the big benefit from futures markets", as Black (1976) has pointed out, provides a first link between futures and spot market.

A single *monopolistic arbitrageur* trades in the spot market and in the futures market. He observes the current basis and builds *rational expectations* about the price changes in the spot and futures market. His trades in both markets are based on these expectations. The arbitrageur is assumed to be risk neutral. His objective is to maximize total expected arbitrage profit over a fixed finite time interval [0,T]. The arbitrageur has to pay proportional transaction costs in the spot market, but there are no transaction costs in the futures market. If he builds up a new arbitrage position he is obliged to pay round trip transaction costs. If he unwinds an arbitrage position he has to pay unwind transaction costs which are assumed to be lower than round trip costs.[7] This asymmetry between transaction costs for opening and closing positions has two major consequences. First, transaction cost asymmetry makes the early unwind option valuable because closing a long (short) position and opening a new short (long) position of the same size would yield different profits. Second, the decision of the arbitrageur depends on his current position, or, equivalently, on his trades in the past. The arbitrageur considers the impact of his demand on the basis. Therefore, despite his risk neutrality, he does not exhibit an unlimited arbitrage demand as soon as the expected profit becomes positive.

Finally, there are *noise traders* in the spot market. Noise traders are introduced n order to model the basis risk. Without noise traders, the futures and spot markets would be locally perfectly correlated. Their demand is described by an exogenous stochastic variable.

The model is a short period model in that it only analyzes the period of time [0,T] over which one futures contract with expiration date $T+\Delta t$ is traded. The period [0,T] is divided into N intervals of the length $\Delta t$. The individual trading dates are denoted by $t_n$, $1 \leq n \leq N$. The first trade takes place at time $t_1 = \Delta t$, and the last occurs at $t_N = T$. At time $T+\Delta t$ there is no trading in the futures market, the futures contract is settled at the spot price, and no arbitrage profits or losses can be realized. Therefore, there is no optimization problem to be solved by the arbitrageur at that time. Consequently, that date will have no impact on the value of the early unwind option and will not be considered in the model explicitly.

The trading sequence is as follows. An investor places his order at time $t_{n-1}$. After a positive execution lag, $\Delta t$, a Walrasian auction takes place at $t_n$. After this auction at time $t_n$, investors may give further orders to be executed at time $t_{n+1}$. So, there is a sequence of N auction equilibria.

7 For a justification of this assumption see Brennan/Schwartz (1990).

Two types of orders are considered in the model. An investor may specify his demand depending on the realization of the unknown future spot price (limit orders) or the investor may demand a certain quantity (market order). The groups of speculators are assumed to place limit orders, so that the quantity they will buy or sell is unknown, but optimal, given the realization of the price in the next period. In contrast, the arbitrageur submits market orders in order to get a known quantity. Otherwise he would run the risk of open positions in the spot or futures market. Like arbitrageurs, noise traders are assumed to place market orders.

## 2.2 Demand Specification

In the following section the demand functions of the different groups of investors for strictly positive spot prices are derived. [8]

### (I) Demand function of positive feedback traders

Positive feedback traders use historical spot prices to forecast future spot prices $\tilde{S}_E^F(t_n)$.[9] A simple forecasting model of that type is given by [10]

$$\text{(1)} \qquad \tilde{S}_E^F(t_n) = S(t_{n-1}) + \left\{ \mu_F S(t_{n-1}) + \beta_F \left[ S(t_{n-1}) - S(t_0) e^{\mu_F (t_{n-1} - t_0)} \right] \right\} \Delta t + \sigma_F \tilde{Z}_F(t_n)$$

where $n = 1,\ldots,N$. Here, $\mu_F$ denotes a fixed growth rate of the spot price per year which is assumed to be constant and positive. $\beta_F$ is a constant and positive factor of adaption. Hence, if the spot price has increased at a rate larger than $\mu_F$ in the period from the fixed point of time $t_0$ in the past to the current date $t_n$, the second term is positive, and positive feedback traders adjust their expectations. An asset with a performance better than $\mu_F$ is expected to outperform in the future, too. The term, $\sigma_F \tilde{Z}_F(t_n)$, is a normally distributed random variable with an expected value of zero and variance $\sigma_F^2 \Delta t$. $\tilde{Z}_F(t_i)$ and $\tilde{Z}_F(t_k)$ are assumed to be uncorrelated whenever $t_i \neq t_k$ (i,k = 1,...,N).

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8 Additional provisions have to be considered when the spot price approaches to zero in order to assure that only positive equilibrium prices are obtained. This matter is not further pursued here.

9 $\tilde{S}_E^F(t_n)$ denotes the spot price forecasted by positive feedback traders using the forecast model (1).

10 Tildes indicate stochastic variables. Note that the forcasting model is defined conditional on the spot price $S(t_{n-1})$.

The demand $D_F[t_n, S(t_n)]$ of an individual positive feedback trader for the spot instrument at time $t_n$ is defined as the change of the optimal risky investment between time $t_{n-1}$ and time $t_n$. To determine this demand function the optimal portfolio of a feedback trader at time $t_n$ has to be characterized.

As positive feedback traders place limit orders, their orders submited at time $t_{n-1}$ will depend on the unknown spot price $\tilde{S}(t_n)$. Conditional on the realization $S(t_n)$ of $\tilde{S}(t_n)$ they will invest the proportion $w_F[t_n, S(t_n)]$ of their wealth $W_F[t_n, S(t_n)]$ in the risky asset and $1 - w_F[t_n, S(t_n)]$ in the risk free asset with the constant rate of return r.

By assumption, positive feedback traders maximize their end of period wealth, $\tilde{W}_E^F(t_{n+1})$, in a state independent mean-variance framework.[11] In addition, they behave myopically. Using the constant factor, $\gamma_F$, of the absolute risk aversion, their optimization problem conditional on the spot price $S(t_n)$ can be specified as

(2)
$$\max_{w_F[t_n, S(t_n)]} \left\{ E\left[\tilde{W}_E^F(t_{n+1})\right] - \frac{1}{2}\gamma_F \, Var\left[\tilde{W}_E^F(t_n)\right]\right\},$$

*where*

$$\tilde{W}_E^F(t_{n+1}) = w_F[t_n, S(t_n)] W_F[t_n, S(t_n)] \frac{\tilde{S}_E^F(t_{n+1})}{S(t_n)} + \{1 - w_F[t_n, S(t_n)]\} W_F[t_n, S(t_n)](1 + r\Delta t)$$

and $S(t_n) > 0, n = 1, \ldots, N$. From the optimal solution, $w_F^*[t_n, S(t_n)]$, the optimal number, $x_F^*[t_n, S(t_n)]$, of the risky asset in his portfolio follows as

(3)
$$\begin{aligned} x_F^*[t_n, S(t_n)] &= \frac{w_F^*[t_n, S(t_n)] W_F[t_n, S(t_n)]}{S(t_n)} \\ &= \frac{1}{\gamma\sigma_F^2}\{[\mu_F - r]S(t_n) + \beta_F[S(t_n) - S_F(t_n)]\}, \end{aligned}$$

where $S_F(t_n) \equiv S(t_0)e^{\mu_F(t_n - t_0)}$ denotes the reference spot price in the forecasting model of the feedback trader. From the definition of the demand function, $D_F[t_n, S(t_n)]$, it follows:[12]

11 $\tilde{W}_E^F(t_n)$ denotes the wealth forecasted by positive feedback traders using the forecast model (1).

12 As the demand function only depends on the price change, $\Delta S(t_n) \equiv S(t_n) - S(t_{n-1})$, for convenience, the price change is used as argument of the demand function.

$$
\text{(4)}\qquad
\begin{aligned}
D_F[t_n, \Delta S(t_n)] &= x_F^*[t_n, S(t_n)] - x_F^*[t_{n-1}, S(t_{n-1})] \\
&= \frac{1}{\gamma_F \sigma_F^2}\{[\mu_F - r + \beta_F]\Delta S(t_n) - \beta_F \Delta S_F(t_n)\}.
\end{aligned}
$$

This demand function is linear in the spot price change. Since positive feedback traders are defined as investors who buy when the price rises, the slope of their demand function, $\alpha_F$,

$$
\text{(5)}\qquad \alpha_F \equiv \frac{[\mu_F - r + \beta_F]}{\gamma_F \sigma_F^2}
$$

is assumed to be positive. The second part of the demand function in equation (4) shows that positive feedback traders will sell the risky assets when there is no change in the spot price as the reference price $S_F(t_{n-1})$ increases over time. Defining the parameter, $\lambda_F \equiv \beta_F / \gamma_F \sigma_F^2$, the demand function can be rewritten as

$$
\text{(6)}\qquad D_F[t_n, \Delta S(t_n)] = \alpha_F \Delta S(t_n) - \lambda_F \Delta S_F(t_n), \qquad\qquad (\mathrm{n} = 1,\ldots,\mathrm{N}).
$$

## (II) Demand function of information traders

In contrast to positive feedback traders, information traders use the futures price to forecast spot price changes. The spot price implied by the current futures price through cash and carry arbitrage is subsequently denoted as implied spot price, $S_I(t_{n-1})$. It is the discounted current futures price and is defined as[13]

$$
\text{(7)}\qquad S_I(t_{n-1}) = F(t_{n-1}, T + \Delta t)e^{-r(T + \Delta t - t_{n-1})}.
$$

The cash and carry, or, implied basis is defined as the difference $F(t_{n-1}, T + \Delta t) - S_I(t_{n-1})$. This implied basis differs from the (endogenous) basis which is defined as $F(t_{n-1}, T + \Delta t) - S(t_{n-1})$. Finally, the difference between the endogenous basis and the cash and carry basis can be considered as the mispricing

$$
\text{(8)}\qquad MIS(t_n) \equiv S_I(t_n) - S(t_n)
$$

of the spot instrument. Information traders expect the spot price to rise when the futures price or, equivalently, the implied spot price rises. In addition, if the current endogenous spot price differs from the implied price, information traders

13 It is assumed that the spot instrument does not pay any income. This assumption is fulfilled for the German Stock Index DAX as it is a performance index reinvesting all dividends. Note that the implied spot price will in general not be equal to the equilibrium price of the model.

expect the spot price to adjust to this value. The following simple forecasting model for the future spot price, $\tilde{S}_E^I(t_n)$, has the desired properties:

$$(9) \qquad \tilde{S}_E^I(t_n) = S(t_{n-1}) + \Delta\tilde{S}_I(t_n) + \beta_I\{S_I(t_{n-1}) - S(t_{n-1})\}\Delta t + \sigma_I\tilde{Z}_I(t_n).$$

$\beta_I$ denotes a constant and positive factor of adaption, and $\sigma_I\tilde{Z}_I(t_n)$ is a random disturbance with identical properties as $\sigma_F\tilde{Z}_F(t_n)$. From (9) follows that the mispricing expected by an information trader exhibits a mean reversion behavior around zero.

The implied spot price, which is perfectly linked with the exogenous futures price, is assumed to follow a stochastic process denoted by[14]

$$(10) \qquad \Delta\tilde{S}_I(t_n) = rS_I(t_{n-1})\Delta t + \sigma\tilde{Z}(t_n),$$

where $\sigma\tilde{Z}(t_n)$ is a normally distributed random variable with an expected value of zero and variance $\sigma^2\Delta t$. The random variables $\tilde{Z}(t_i)$ and $\tilde{Z}(t_k)$ are assumed to be uncorrelated whenever $t_i \neq t_k$ (i,k = 1,...,N).

Information traders behave like positive feedback traders, but they use the forecast model in equation (9) and the exogenous implied spot price as defined in equation (10) for determining their optimal demand. Using the same arguments as in part (1), their optimization problem is specified as

$$\max_{w_I[t_n,S(t_n)]}\left\{E\left[\tilde{W}_E^I(t_{n+1})\right] - \frac{1}{2}\gamma_I\, Var\left[\tilde{W}_E^I(t_n)\right]\right\},$$

(11) *where*

$$\tilde{W}_E^I(t_{n+1}) = w_I[t_n,S(t_n)]W_I[t_n,S(t_n)]\frac{\tilde{S}_E^I(t_{n+1})}{S(t_n)} + \{1 - w_I[t_n,S(t_n)]\}W_I[t_n,S(t_n)](1+r\Delta t)$$

and $S(t_n) > 0, n = 1,\ldots,N$. Standard calculations yield the optimal demand of an individual information trader. It depends on the spot price and the futures price at time $t_n$. Therefore, the limit order of each information trader is a function of $S(t_n)$ and $\tilde{F}(t_n,T+\Delta t)$ or, equivalently, of $S(t_n)$ and $\tilde{S}_I(t_n)$. As the demand function is linear in the spot price differences, $S(t_n) - S(t_{n-1})$, and in the differences of the implied spot price, $\tilde{S}_I(t_n) - S_I(t_{n-1})$, again, for convenience, in the demand function

[14] This formulation with a constant volatility of the implied price change $\Delta\tilde{S}_I(t_n)$ is chosen for convenience. Of course, the model could also be formulated with a constant volatility of $\Delta\tilde{F}(t_n,T+\Delta t)$ and a volatility of $\Delta\tilde{S}_I(t_n)$ decreasing with time to maturity.

the arguments $\Delta S(t_n) = S(t_n) - S(t_{n-1})$ and $\Delta \tilde{S}_I(t_n) = \tilde{S}_I(t_n) - S_I(t_{n-1})$ are used.[15] Then, the demand function, $D_I\left[t_n, \Delta \tilde{S}_I(t_n), \Delta S(t_n)\right]$, of an individual information trader has the following structure:

$$(12) \quad D_I\left[t_n, \Delta \tilde{S}_I(t_n), \Delta S(t_n)\right] = \alpha_I \left\{\Delta \tilde{S}_I(t_n) - \Delta S(t_n)\right\}, \qquad n = 1, \ldots, N.$$

The parameter

$$(13) \quad \alpha_I \equiv \frac{1}{\gamma_I} \frac{(r + \beta_I)}{\sigma_I^2 + \sigma^2} > 0$$

turns out to be positive, and, therefore, the slope, $-\alpha_I$, of the demand function is negative. The demand of information traders depends on price changes of the exogenous variable. This is the first reason why changes of the futures price have an impact on the equilibrium spot price.

### (III) Demand of the monopolistic arbitrageur

The monpolistic arbitrageur is assumed to maximize the present value of his expected total arbitrage profit in the period of time [0,T]. His trading strategy is defined as a sequence of N market orders, $D_A(t_n)$, which are placed at point of time $t_{n-1}$ (n = 1,...,N). The result of a long or short arbitrage position is obtained at point of time $T + \Delta t$. It depends on the futures price, $\tilde{F}(t_n, T + \Delta t)$, the endogenous spot price, $\tilde{S}(t_n)$, and transaction costs, $T\tilde{C}(t_n)$. These quantities are uncertain since the arbitrageur, like speculators, suffers an execution lag. The result of $D_A(t_n)$ arbitrage positions discounted to the execution date $t_n$ is denoted as the arbitrage profit, $\tilde{G}(t_n)$, of this position. Its precise definition is[16]

$$(14) \quad \tilde{G}(t_n) = D_A(t_n)\left\{\tilde{F}(t_n, T + \Delta t)e^{-r(T + \Delta t - t_n)} - \tilde{S}(t_n)\right\} - T\tilde{C}(t_n), \qquad n = 1, \ldots, N.$$

Using the definition of the implied spot price, equation (14) can be rewritten as

$$(15) \quad \tilde{G}(t_n) = D_A(t_n)\left\{\tilde{S}_I(t_n) - \tilde{S}(t_n)\right\} - T\tilde{C}(t_n).$$

15 The notation should be understood as follows: The spot price is considered as a variable as long as no equilibrium conditions are imposed. The stochastic process of the equilibrium spot price will be discussed in section 2.3. This distinction in not necessary for the exogenous implied spot price.

16 Note that $\tilde{G}(T + \Delta t)$ is zero, independent of the trading strategy.

Thus, the profit of the arbitrageur is determined by the mispricing of the spot instrument, $MIS(t_n) \equiv \tilde{S}_I(t_n) - \tilde{S}(t_n)$, and the transaction costs. The implied spot price, $\tilde{S}_I(t_n)$, is determined by the exogenous futures price and does not depend on the trades of positive feedback traders, information traders, noise traders, or arbitrageurs. Contrary, the second term, $\tilde{S}(t_n)$, is influenced by the demands of the feedback traders, information traders, and noise traders. In addition, the demand of the monopolistic arbitrageur has an impact on $\tilde{S}(t_n)$ which the arbitrageur takes into account when he decides on the number of his arbitrage transactions. This price impact is characterized by a yet unknown positive constant factor $\alpha_A$ which is related to the liquidity of the spot market. This factor, $\alpha_A$, will be characterized in equation (29) below. Therefore, the spot price, $\tilde{S}(t_n)$, can be decomposed into a spot price without arbitrage trading, $\tilde{S}_{NA}(t_n)$, and the impact of arbitrage trading, that is

(16) $\tilde{S}(t_n) = \tilde{S}_{NA}(t_n) + \alpha_A D_A(t_n)$.

The total transaction costs, $T\tilde{C}(t_n)$, in equation (14) are assumed to be proportional to the trading volume of the arbitrageur, $D_A(t_n)$, and the endogenous spot price, $\tilde{S}(t_n)$. Thus, total transaction costs are

(17) $T\tilde{C}(t_n) = \left| D_A(t_n) * \tilde{S}(t_n) * TCU \right|$,

where $TCU$ are the transaction costs per unit. Due to asymmetric transaction costs at opening and unwinding, the value $TCU$ is determined through the current demand, $D_A(t_n)$, of the arbitrageur and his former trades. If he is long in the spot instrument and short in the future (long arbitrage position) and wants to open another long arbitrage position, $D_A(t_n) > 0$, he has to pay opening round-trip unit costs, $c_O$. If he wants to go short, $D_A(t_n) < 0$, it is assumed that he first unwinds the long positions held at the lower unwinding unit costs, $c_{UW}$. Analogously, if he wants to switch from a net long position to a net short position, he first unwinds all existing positions at unit costs $c_{UW}$ and then opens new short positions at costs $c_O$. This assumption implies that $TCU$ depends on the current arbitrage demand, $D_A(t_n)$, and the net arbitrage position, $B(t_{n-1})$, which equals the sum of former demands:[17]

[17] Note that this assumption restricts the trading strategy of the arbitrageur. In some situations he may be better off by holding simultaneously long and short arbitrage positions. This property of the arbitrageur´s optimal strategy is in variance with the Brennan/Schwartz (1990) and Cooper/Mello (1990) models where it is always optimal to unwind a long (short) position before a short (long) position is opened. The main reason for this difference is that in the model presented here transaction costs depend on the endogenous spot price.

(18) $B(t_{n-1}) = \sum_{i=1}^{n-1} D_A(t_i)$.

The unwinding costs per unit, $c_{UW}$, apply as long as $|D_A(t_n)| \le |B(t_{n-1})|$, and the signs of both terms are different, i.e. $B(t_{n-1}) * D_A(t_n) < 0$. Therefore, $TCU$ can be written as

(19)

$$TCU[B(t_{n-1}), D_A(t_n)] = \begin{cases} c_O & \text{if} \quad B(t_{n-1}) * D_A(t_n) \ge 0 \\ c_{UW} & \text{if} \quad |D_A(t_n)| \le |B(t_{n-1})|, B(t_{n-1}) * D_A(t_n) < 0 \\ \dfrac{|B(t_{n-1})| c_{UW} + [|D_A(t_n)| - |B(t_{n-1})|] c_O}{|D_A(t_n)|} & \text{if} \quad |D_A(t_n)| > |B(t_{n-1})|, B(t_{n-1}) * D_A(t_n) < 0. \end{cases}$$

Using this definition of the transaction costs per unit, the total transaction costs will be

(20) $\tilde{TC}(t_n) = \left| D_A(t_n) * \left\{ \tilde{S}_{NA}(t_n) + \alpha_A D_A(t_n) \right\} * TCU[B(t_{n-1}), D_A(t_n)] \right|$.

The monopolistic arbitrageur is assumed to maximize the present value of his expected total arbitrage profit in the period of time [0,T].[18] That profit, $P_{UW}$, is defined as

(21) $P_{UW} = \max_{\{D_A(t_n)\}} E_0 \left\{ \sum_{n=1}^{N} e^{-rt_n} \tilde{G}(t_n) \right\}$,

where $E_0$ denotes the expectation with respect to the information available at time t=0 about the stochastic process $\left[F(0, T+\Delta t), \ldots, \tilde{F}(t_N, T+\Delta t)\right]$ of the exogenous variable $\tilde{F}(t_n, T+\Delta t)$.[19] The optimization problem can be solved using a dynamic programming approach where the state variable at $t_{n-1}$ is characterized by the price history, $F_H(t_{n-1}) = [F(0, T+\Delta t), \ldots, F(t_{n-1}, T+\Delta t)]$, of the futures price. Conditional on this price history, the expected *one period profit* as a function of the arbitrageur's market order, $D_A(t_n)$, will be

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Theoretically, it is easy to relax this assumption, but it makes the computation much harder as the number of state variables in order to determine the optimal strategy of the arbitrageur would increase from three to four. This will become clear later.

18 Note that at time $T+\Delta t$ the futures contract is settled and no arbitrage profits or losses occur. Therefore, that point of time can be neglected.

19 $\{F(t_n, T+\Delta t)\}$ is defined on the $\sigma$-algebra of N+1-dimensional Borel sets equipped with the Lebesgue measure.

$$(22)\quad \begin{aligned}E_{n-1}\left[\tilde{G}(t_n)\middle|F_H(t_{n-1})\right] &= D_A(t_n)*\left\{E_{n-1}\left[\tilde{S}_I(t_n)\middle|F_H(t_{n-1})\right]-E_{n-1}\left[\tilde{S}_{NA}(t_n)\middle|F_H(t_{n-1})\right]-\alpha_A D_A(t_n)\right\}\\ &\quad-\left|D_A(t_n)*\left\{E_{n-1}\left[\tilde{S}_{NA}(t_n)\middle|F_H(t_{n-1})\right]+\alpha_A D_A(t_n)\right\}*TCU\left[F_H(t_{n-1}),D_A(t_n)\right]\right|.\end{aligned}$$

The first term in equation (22) is the expected profit without transaction costs, and the second one characterizes the conditional expected transaction costs. The expected profit of the arbitrageur at any time $t_n$ depends on the whole history of the futures price, $F_H(t_{n-1})$. Therefore, the price path of the future determines the expected implied spot price, the expected spot price, the stock of open arbitrage positions, and the transaction costs per unit. So, the expected profit and, therefore, the behavior of the arbitrageur are path dependent.

The price history, $F_H(t_{n-1})$, matters only as far as it influences the endogenous spot price, $S(t_{n-1})$, the futures price, $F(t_{n-1},T+\Delta t)$, and the net arbitrage position, $B(t_{n-1})$. Therefore, a reduction (if n-1>3) of the dimension of the state variable $F_H(t_{n-1})$ from n-1 to these three state variables is possible. With this parsimonious modelling of the state variable, the conditional expected one period profit may be reformulated as

$$(23)\quad \begin{aligned}&E_{n-1}\left[\tilde{G}(t_n)\middle|F(t_{n-1}),S(t_{n-1}),B(t_{n-1})\right]=\\ &D_A(t_n)*\left\{E_{n-1}\left[\tilde{S}_I(t_n)\middle|F(t_{n-1})\right]-E_{n-1}\left[\tilde{S}_{NA}(t_n)\middle|F(t_{n-1}),S(t_{n-1})\right]-\alpha_A D_A(t_n)\right\}\\ &-\left|D_A(t_n)*\left\{E_{n-1}\left[\tilde{S}_{NA}(t_n)\middle|F(t_{n-1}),S(t_{n-1})\right]+\alpha_A D_A(t_n)\right\}*TCU\left[B(t_{n-1}),D_A(t_n)\right]\right|,\end{aligned}$$

where $t_{N-1}$ is the last point of time at which the arbitrageur determines his optimal demand $D_A^*(t_N)$. This demand is executed at time $t_N$. Positions not unwound until time $t_N$, $B(t_N)$, are closed at the maturity date $T+\Delta t$ of the futures contract.

It can be shown that a unique solution strategy $D_A^*(t_n)$, n=1,...,N, of the optimization problem (21) exists. The proof uses the fact that, without transaction costs, the objective function is quadratic and strictly convex in $D_A(t_n)$, n = 1,...,N.

The behavior of the arbitrageur in this model is quite different from a cash and carry arbitrageur. The arbitrageur considered here has to take into account the impact of his trading on future arbitrage opportunities when deciding about his current optimal market order, $D_A^*(t_n)$. This link is established through the impact of his decision on the equilibrium spot price, $\tilde{S}(t_n)$, and the resulting net arbitrage position, $B(t_n)$.

Finally, it should be noted that the arbitrageur differs in three important respects from the other market participants. One difference is that he is the only one who trades in the futures and in the spot market. Another difference is that he places market orders in order to avoid unbalanced positions in the futures and the spot instrument. The third difference is that the arbitrageur is fully rational in the sense that he uses the equilibrium model of the spot market to determine the expected spot price at time $t_n$.

### (IV) Demand of noise traders

The last group of investors considered in the model are noise traders. Their motivation to buy or sell the spot instrument is not modelled explicitely. It is assumed that their demand does neither depend on the spot price nor on the futures price. The demand of an individual noise trader, $\tilde{D}_L(t_n)$, at time $t_n$ is uncertain. It is described by the following stochastic process:

(24) $\tilde{D}_L(t_n) = \sigma_L \tilde{Z}_L(t_n).$

$\tilde{Z}_L(t_n)$ is a normally distributed random variable with zero mean and variance $\sigma_L^2 \Delta t$. $\tilde{Z}_L(t_i)$ and $\tilde{Z}_L(t_k)$ are assumed to be uncorrelated whenever $t_i \neq t_k$ (i,k = 1,...,N). In addition, $\tilde{Z}_L(t_n)$ (n = 1,...,N) are uncorrelated with the exogenous random variable $\tilde{Z}(t_i)$ (i = 1,...,N).

## 2.3. Equilibrium

In this section the equilibrium process of the spot price conditioned on the futures price is determined. The price of the spot instrument, S(0), at the beginning of the trading period is given exogenously. Trading takes place N times during the interval [Δt,T] at the points of time $t_n$ (n=1,...,N). At the same points of time, market clearing is achieved through N Walrasian auctions. Therefore, the equilibrium concept is an auction equilibrium where the traders differ by their motives for trading and by the price forecasting models which they use. The equilbrium is defined as an allocation of optimal demands, $D_F\left[t_n, \Delta\tilde{S}(t_n)\right], D_I\left[t_n, \Delta\tilde{S}_I(t_n), \Delta\tilde{S}(t_n)\right], \tilde{D}_N(t_n), D_A^*(t_n)$, and a sequence of random prices, $\tilde{S}(t_n)$, for n = 1,...,N such that the spot market clears.[20] Under the assumption that there are $y_F$ identical positive feedback traders, $y_I$ identical information traders, $y_N$ identical noise traders, and a monopolistic arbitrageur, the following market clearing condition has to be satisfied:

20 The random character of the equilibirum spot price is now considered in the notation.

(25) $y_F D_F\left[t_n, \Delta\tilde{S}(t_n)\right] + y_I D_I\left[t_n, \Delta\tilde{S}_I(t_n), \Delta\tilde{S}(t_n)\right] + y_L \tilde{D}_L(t_n) + D_A^*(t_n) = 0$.

This equation has to hold at each $t_n$ ( n = 1,...,N ) conditioned on the spot price $S(t_{n-1})$, the implied spot price $S_I(t_{n-1})$, and the number of open arbitrage positions $B(t_{n-1})$.

In section 2.2., the optimal demand of positive feedback traders, $D_F\left[t_n, \Delta\tilde{S}(t_n)\right]$, and the optimal demand of information traders, $D_I\left[t_n, \Delta\tilde{S}_I(t_n), \Delta\tilde{S}(t_n)\right]$, were derived as functions of the market clearing price. Furthermore, it was argued that an optimal strategy for the arbitrageur exists which takes into account the impact of the arbitrageur's trading on expected profits in later auctions. The arbitrageur submits optimal market orders given the information available at time $t_{n-1}$. Inserting these demand function into the left hand side of the market clearing condition (25) yields the aggregate conditional demand function:

(26) $D\left[t_n, \Delta\tilde{S}_I(t_n), \Delta\tilde{S}(t_n)\right] = y_F\left\{\alpha_F \Delta\tilde{S}(t_n) - \lambda_F \Delta S_F(t_n)\right\} + y_I \alpha_I \left\{\Delta\tilde{S}_I(t_n) - \Delta\tilde{S}(t_n)\right\} + y_L \sigma_L \tilde{Z}_L(t_n) + D_A^*(t_n)$.

The aggregate demand function is linear in the equilibrium price, $\tilde{S}(t_n)$, and has a constant slope of $-1/\alpha$ where $\alpha$ is given by

(27) $\alpha = \dfrac{1}{y_I \alpha_I - y_F \alpha_F}$.

The slope is assumed to be negative, so that the paramter, $\alpha$, which can be interpreted as the market liquidity parameter is positive. Using this parameter, the market clearing condition yields the equilibrium spot price conditional on $S(t_{n-1}), S_I(t_{n-1})$, and $B(t_{n-1})$:

(28) $\tilde{S}(t_n) = S(t_{n-1}) + \alpha\left\{y_I \alpha_I \Delta\tilde{S}_I(t_n) - y_F \lambda_F \Delta S_F(t_n) + y_L \sigma_L \tilde{Z}_L(t_n) + D_A^*(t_n)\right\}, \qquad n = 1, \ldots, N$.

There are four main properties of this equilibrium process. One is that the spot price rises with an increase in the implied spot price, $\tilde{S}_I(t_n)$, as the demand of information traders increases.[21] It also rises[22] with an increase in arbitrage demand, $D_A^*(t_n)$. The spot price decreases with an increase of the reference price, $S_F(t_n)$, of the positive feedback traders. Second, the conditional volatility of the spot price depends on the volatility of the implied spot price and on the basis risk introduced by the demand of noise traders. Third, without noise traders, the

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21 This monotonicity property should be understood in the following sense. Whenever the realization $\hat{S}_I(t_n)$ is larger than $\hat{\hat{S}}_I(t_n)$, then the same relation holds between the corresponding spot prices.

22 In the sense of stochastic dominance of first order.

conditional volatility ratio between the futures and the spot market is mainly determined by the slope of the demand functions of positive feedback traders and information traders as well as by the size of these groups. An increasing number of feedback traders raises the volatility of the spot price while an increasing number of information traders reduces the volatility of the spot price. The demand of noise traders has a straightforward influence on the volatility of $\tilde{S}(t_n)$. Finally, the endogenous spot price, $\tilde{S}(t_n)$, is not Markovian as it depends via the demand of the arbitrageur, $D_A^*(t_n)$, on the whole price history.

The undefined parameter, $\alpha_A$, in the optimization problem of the arbitrageur can now be specified. From equation (28), $\alpha_A$ is shown to be

$$(29)\quad \alpha_A = \alpha,$$

and the conditional expected spot price at time $t_n$ is

$$(30)\quad \begin{aligned} &E_{n-1}\left[\tilde{S}(t_n)\middle|S_I(t_{n-1}),S(t_{n-1}),B(t_{n-1})\right] = \\ &\qquad S(t_{n-1}) + \alpha\{y_I\alpha_I r S_I(t_{n-1}) - y_F\lambda_F\mu_F S_F(t_{n-1})\}\Delta t + \alpha D_A^*(t_n) \end{aligned}.$$

To discuss the role of the arbitrageur in the model, the endogenous spot price process is transformed into a process for mispricing, $MI\tilde{S}(t_n) \equiv \tilde{S}_I(t_n) - \tilde{S}(t_n)$. It follows from equation (28) that the change of the mispricing $\Delta MI\tilde{S}(t_n)$ is

$$(31)\quad \Delta MI\tilde{S}(t_n) = \alpha\left\{y_F\lambda_F\Delta S_F(t_n) - D_A^*(t_n) - y_N\sigma_N\tilde{Z}_N(t_n)\right\} - \alpha\, y_F\alpha_F\Delta\tilde{S}_I(t_n).$$

Without arbitrage trading, i.e. $D_A(t_n) \equiv 0$, and without noise trading, the first term is positive and non-stochastic. It raises a positive mispricing and reduces a negative one. The expected change of the implied spot price is positive and has, therefore, an opposite effect. So, contrary to the model of Cooper and Mello (1990), the basis does not even on average necessarily converge to its cash and carry value if the arbitrageur does not trade. Since noise traders have no systematic influence on the basis, it is the role of the arbitrageurs to assure a comovement of spot and futures market.

## 3. Value of the Early Unwind Option

To define the value of the early unwind option, two arbitrageurs are considered. The first one has the opportunity to unwind arbitrage positions and faces the optimization problem described in equation (21) with the optimal expected present value of the total arbitrage profit, $P_{UW}$. The second arbitrageur differs from the first one only by his restricted set of strategies. The second arbitrageur has to hold every arbitrage position until maturity date, $T+\Delta t$, of the future where it is settled against his spot position. Like the first arbitrageur, he is a monopolist with rational expectations, and his optimal expected present value of the total arbitrage profit is denoted by $P_O$. The value of the early unwind option, $V_{UW}$, is defined as the difference

(32) $V_{UW} = P_{UW} - P_O$

between these optimal present values. As all arbitrage positions are settled at maturity $T+\Delta t$, the non-negative value of the early unwind option is the result of different transaction costs for opening and closing transactions.[23] If the closing transaction cost rate, $c_{UW}$, equals the opening cost rate, $c_O$, the unwind option is worthless.

According to equations (21), (23), and (30), the expected present value of all arbitrage opportunities depends mainly on the following factors. First, the expected total arbitrage profit depends on the current spot price S(0) and on the current futures price $F(0,T+\Delta t)$ or, equivalently, on the current implied spot price $S_I(0)$. Next, the distribution of futures prices $\tilde{F}(t_n,T+\Delta t)$ or implied spot prices $\tilde{S}_I(t_n)$ (n = 1,...,N), especially the volatility, $\sigma$, in the stochastic process of the futures price, has an impact on the expected profit. The same is true for the volatility $\sigma_L$ of the noise trader's demand. The market liquidity parameter, $\alpha$, captures the influence of arbitrage trading on the spot price and therefore the expected total profit. The interest rate r determines the discount factor and the slopes of the demand functions. Given the length of the execution lag, $\Delta t$, the time to maturity, $T+\Delta t$, of the future determines the number of trading dates and therefore the expected total profit. Finally, transaction costs of opening an arbitrage position, $c_O$, have an impact on the expected total profit. In addition, if early unwinding is possible, the number, B(0), of open arbitrage positions at the beginning and transaction costs, $c_{UW}$, for closing an arbitrage position are relevant.

23 Another argument for the unwind option to have a positive value is that arbitrageurs may have a position limit. This is pointed out by Brennan/Schwartz (1988, 1990).

The influence of these factors on $P_{UW}$ and $P_O$ is easy to analyze in most cases. For example, both expected present values increase if the current absolute mispricing and time to maturity increase. They decrease with the market liquidity parameter and with transaction costs. In addition, $P_{UW}$ increases with the absolute number of open arbitrage positions as the arbitrageur has more chances to unwind positions at lower costs. However, the influence of these factors on the difference $P_{UW} - P_O$ and, therefore, on the option value is not obvious in most cases.

There are two basic effects which determine the value of the early unwind option. One effect occurs when an arbitrageur has an open arbitrage position and there is a favourable mispricing, e.g. an open long position and a negative mispricing. If this is the case the arbitrageur can obtain an immediate additional profit by unwinding the position at the lower transaction cost rate, $c_{UW}$. This effect relates to the intrinsic value of an option. The second effect is that, even if the arbitrageur has currently no open arbitrage position, with every opening transaction he aquires the chance to save transaction costs in the future. This second effect is an indirect one as it does not lead immediately to transaction costs savings compared with a hold to maturity strategy. It is related to the time value of an option.

Standard options are said to be in-the-money when the immediate exercise of the option yields a profit, the well-defined intrinsic value of the option. Applying this criterion to the early unwind option is difficult for three reasons. First, whenever the arbitrageur decides to unwind a position he does not know the profit resulting from this decision as there is an execution lag. Second, when the arbitrageur decides to unwind, the profit depends on the mispricing of the spot instrument and on transaction costs which he influences through his decision. Third, by trading, the arbitrageur also changes his net number of arbitrage positions and, by that, the value of the unwind option he holds. Therefore, the term "in-the-money" will not be used in the usual sense. Rather, the unwind option is said to be in-the-money whenever there is positive (negative) mispricing and a negative (positive) number of open arbitrage positions. Thus, the term just indicates whether, given a certain net arbitrage position, the sign of the mispricing is favourable to the arbitrageur. According to this definition, the option is at-the-money when there is no mispricing and out-of-the-money when the mispricing and the number of arbitrage positions show the same sign.

## 4. Numerical Results

### 4.1. Parameter Values

The parameter values of the stochastic price processes introduced in section 2.2 are estimated using price quotations for the German stock index DAX and the corresponding futures contracts. The formulation of the stochastic processes demands a specification of the volatilities in absolute terms (index points). The percentage numbers are based on an index value of 2,000. Using daily transaction prices the following estimated values are obtained:

| | |
|---|---|
| Drift of the DAX | : $\mu_F = 0.12$ |
| Factor of adaption for the spot price | : $\beta_F = 0.36$ |
| Volatility of the disturbance term | : $\sigma_F = 0.16$ |
| Factor of adaption for the mispricing | : $\beta_I = 9$ |
| Volatility of the disturbance term | : $\sigma_I = 0.02$ |
| Volatility of the implied spot price | : $\sigma = 0.2$ |

For the remaining parameters the following basic constellation is assumed:

| | |
|---|---|
| Volatility of noise trader's demand | : $\sigma_L = 6*10^{-6}$ |
| Interest rate | : $r = 0.08$ |
| Number of trading periods | : $N = 10$ |
| Trading interval | : $\Delta t = 1/360$ ( = 1 day ) |
| Opening costs per unit | : $c_O = 0.5\%$ |
| Unwinding costs per unit | : $c_{UW} = 0.125\%$ |
| Risk parameters | : $\gamma_F = \gamma_I = 1$ |
| Number of traders | : $y_F = y_I = y_L = 15000$ |

From these parameter the slopes of the demand functions and the market liquidity parameter are computed as:

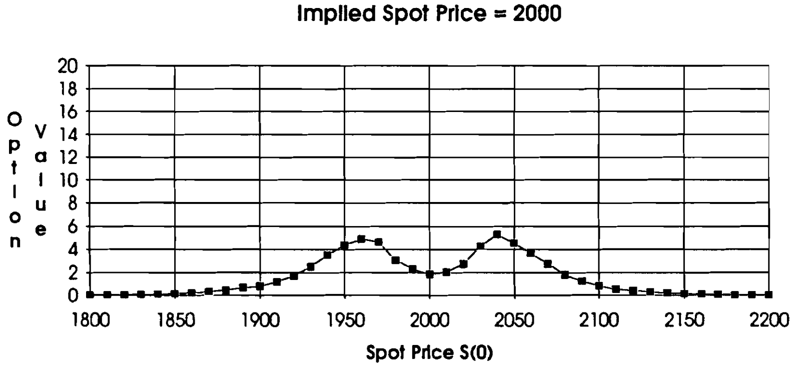
| | |
|---|---|
| Positive feedback trader demand | : $y_F \alpha_F = 0.06$ |
| Information trader demand | : $-y_I \alpha_I = -0.93$ |
| Market liquidity parameter | : $\alpha = 1.15$ |

The market liquidity parameter implies that one unit arbitrage demand raises the spot price by 1.15 index points.

## 4.2. Comparative Static Analysis

The relationship between the option value and the state variables spot price, implied spot price, and the number of arbitrage positions is complex as there are several effects influencing this value. Figures 1 and 2 present the option value as a function of the current spot price, $S(0)$, for a given implied spot price $S_I(0)$ and three different net arbitrage positions, $B(0)$. The spot price varies between 1800 and 2200, and this corresponds to the mispricing interval, $S_I(0)-S(0)$, ranging from +200 to -200 index points.

Figure 1: Value of the Early Unwind Option as a Function of the Current Spot Price: Number of Open Arbitrage Positions B(0) = 0




For the discussion of figure 1 the reader is reminded that the value of the *early unwind option is defined as the difference* of the expected present value of total arbitrage profits with and without early unwinding possibility.

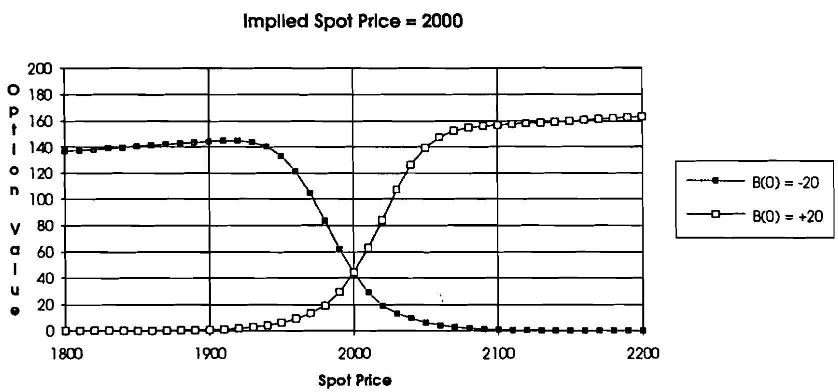
If the arbitrageur holds no open position, and if the current spot price equals its implied value, the early unwind option has a positive value. This is true since at future points of time it may be advantageous to open an arbitrage position and to unwind it at a later date.[24] If the absolute value of the mispricing increases, the

[24] Therefore, in a situation where at first early unwinding is not allowed and where then this regulation is lifted, the arbitrageur receives a valuable option free even if he holds no open positions and the mispricing is zero. Since the value of this option comes from the different transaction costs at opening and closing positions, this free option is given to the arbitrageur by the broker and the exchange on which the spot instrument is traded. They loose transaction costs when positions are closed compared with the case that a new position of the same size is opened.

value of the option first increases and then decreases. This behavior can be explained by the fact that, given the number of open positions, the value of the option depends on the probability that this option will be in-the-money at a future point of time. For the option to be in-the-money in the future when the current number of open positions is zero, two conditions have to be met. The first is that there has to be a mispricing such that the arbitrageur opens an arbitrage position. At that time the option becomes alive and is out-of-the-money. The second condition is that the sign of the mispricing has to change so that the option will be in-the-money. Thus, the early unwind option is path dependent and somewhat similar to a drop-in-option. If the absolute mispricing is large, the arbitrageur will open a large number of arbitrage positions, but the probability that the sign of the mispricing changes and this position can be unwound early will be small. However, if the absolute mispricing is small the arbitrageur will only open a few arbitrage positions, but it is very likely that the sign of the mispricing will change and permit the arbitrageur to close the position early. It can be seen in figure 1 that, for the parameters chosen, the two opposite effects lead to a maximum at an absolute mispricing of about 40 index points.

A completely different figure is obtained when analyzing an arbitrageur who already holds a non-zero number of open arbitrage positions.

Figure 2: Value of the Early Unwind Option as a Function of the Current Spot Price Number of Open Arbitrage Positions B(0) = +20 and B(0) = -20



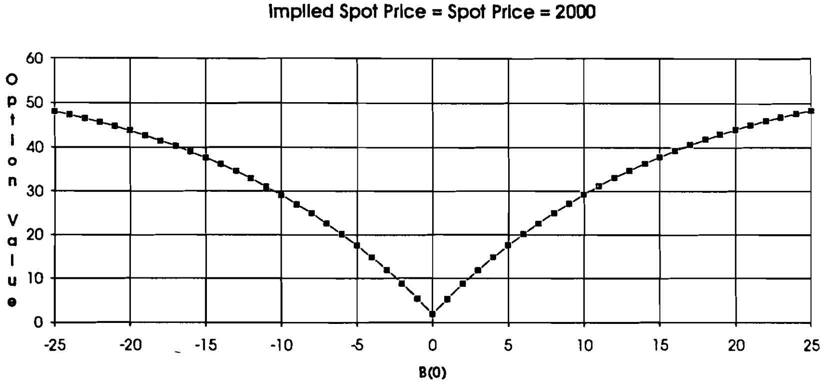

*Case 1*, $B(0) = +20$: If the arbitrageur is long in the spot instrument and short in futures, the value of the option monotonically increases with the current spot price or, equivalently, decreases with the mispricing. The maximum value of the early unwind option at a spot price of 2,200 can be explained by the expected savings of transaction costs at the next trading date. These savings can be approximated by assuming that the arbitrageur has no execution risk. Immediate

unwinding results in savings of the amount $(c_O - c_{UW}) * S(0) * B(0) = 165$ which approximates the exact value of 163.38 very closely. For negative values of the mispricing, the option is in-the-money as the long position can be closed out with an expected positive profit. For positive values of the mispricing, the value of the option is still higher than without open positions, but it converges to the value in figure 1 if the option is deep out-of-the-money. This is due to the fact that the probability of unwinding these 20 open positions in the future converges to zero.

*Case 2,* $B(0) = -20$: Surprisingly, given open short positions and a certain mispricing $MIS(0)$, the option value does not equal the option value of the same number of long positions at the mispricing $-MIS(0)$. For example, figure 2 shows that the maximum option value for the short position is lower than the corresponding value of the long position, and that the value of the option starts to decline for large positive values of the mispricing. If, again, the execution risk of the arbitrageur is neglected, the expected savings $(c_O - c_{UW}) * E_0[\tilde{S}(t_1)] * |D_A^*(0)|$ from unwinding $|D_A^*(0)|$ positions can be approximated by $(c_O - c_{UW}) * S(0) * |D_A^*(0)|$. If the msipricing increases from 0 to 200 index points, $|D_A^*(0)|$ increases until all open positions are unwound. This will be the case for a mispricing of 80 index points or, equivalently, $S(0) = 1920$. Unwinding 20 position will result in transaction cost savings of approximately $0.00375 * 1920 * 20 = 144$ index points. If mispricing increases further, the optimal demand of the arbitrageur will increase, too. As only 20 units can be unwound early, the approximate savings in transaction costs $0.00375 * S(0) * 20$ decreases with the spot price. By the same argument, in case 1 the value of the early unwind option increases in $S(0)$ even if $D_A^*(0) > B(0)$.

It was argued above that the value of the option depends on the number of net arbitrage positions *and* the mispricing. In order to separate these two effects, figure 3 demonstrates that, given a mispricing of zero, the value of the early unwind option depends on the number of arbitrage positions.

Figure 3: Value of the Early Unwind Option as a Function of the Current Number of Open Positions: No Current Mispricing



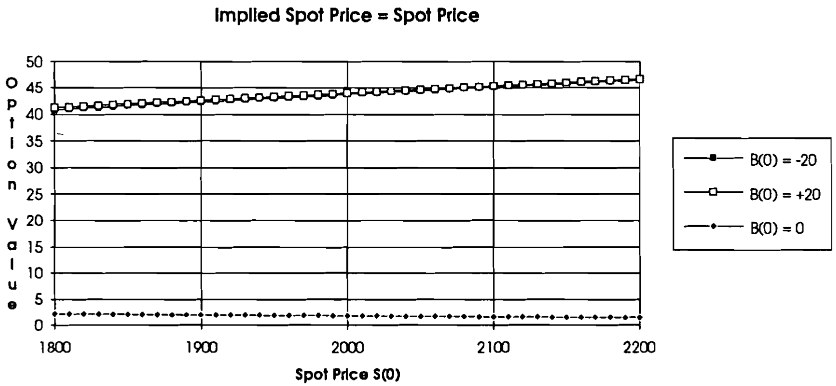

It can be seen that the option value is minimal when the arbitrageur has no open positions. Without any open arbitrage position, the arbitrageur has no opportunity to earn an immediate unwind profit. He first has to open a new position before he gets a chance to unwind it. The value of the option in the case $B(0)=0$ is 2 index points or 0.1 % of the index value.

The option value increases monotonically with the absolute number of arbitrage positions. The more positions the arbitrageur holds, the more he can unwind if in the future the option will be in-the-money which subsequently increases the expected savings in transaction costs and the option value as well. The marginal option value of one additional arbitrage position decreases resulting from the fact that the spot price is endogenous. Unwinding a single arbitrage position at time $t_n$ reduces the expected absolute value of the mispricing and therefore the expected profit of unwinding an additional position. Furthermore, the probability of unwinding another open position at the next point of time, $t_{n+1}$, declines. Contrary to standard option pricing models, exercising the option influences the stochastic process of the underlying asset.

Figure 3 is not symmetric. The option value for a positive number of open positions is slightly larger than for the same number of negative open positions. The reason for this asymmetry is that a long arbitrage position will be unwound if the spot price increases relative to the implied spot price, a short arbitrage position will be unwound if the opposite is true. Therefore, on average, savings of transaction costs will be larger for long positions than for short ones.

When discussing figure 2, it was argued that the spot price has an impact on the option value even if the mispricing and the net number of arbitrage positions are given. The impact of the spot price is now discussed in more detail. Figure 4 shows the value of the early unwind option as a function of the spot price, $S(0)$. This figure differs from figures 1 and 2 by the assumption that there is no current mispricing, i.e. $S(0) = S_I(0)$.

Figure 4: Value of the Early Unwind Option as a Function of the Spot Price: No Current Mispricing



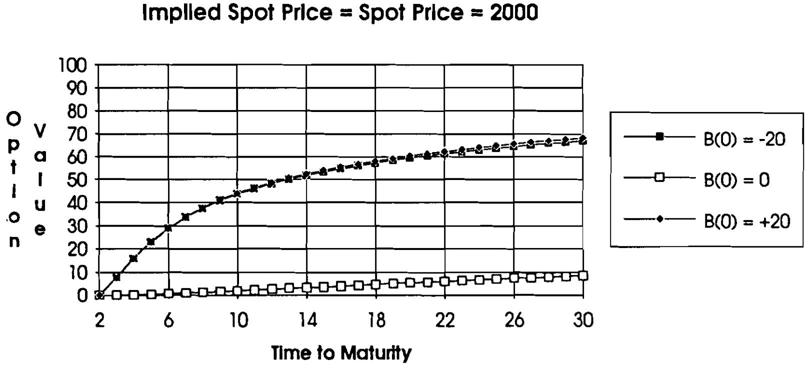

In correspondence with figure 3, the option value is lowest when the arbitrageur has no open arbitrage position. Specifically, figure 4 shows that the option value increases with the spot price if the arbitrageur holds a non zero position. If he holds no position the value of the option decreases slightly with the spot price. Other things being equal, unwinding current arbitrage positions will occur at higher future spot prices if the current spot price is high.[25] Since the transaction cost savings are proportional to transaction costs the savings increase with the future spot price and, therefore, with the current spot price, $S(0)$, as well. For that reason, a higher current spot price leads to a higher option value. This argument does not hold when the arbitrageur has no open position. In that case, a second effect dominates. As transaction costs increase with the spot price even if transaction costs per unit are fixed, other things being equal, higher transaction costs reduce the trading volume. Without any open positions the arbitrageur first has to open some positions before the option becomes alive. Higher future spot prices, therefore, result in less trading, in a lower number of future open arbitrage positions, and in less transaction cost savings. This effect dominates when the arbitrageur currently holds no open position. Thus, the option value will not only be determined by the mispricing and the number of open positions, but also by the

25 Again, future spot prices are compared by stochastic dominance of first order.

spot price itself. Whether the option value increases or decreases with the current spot price, depends on the number of open positions and the size of mispricing.

From standard option pricing theory, it is wellknown that the price of an american type option increases with time to maturity. The early unwind option is an american option as the arbitrageur may exercise it at every trading date. Thus, it can be expected that its value increases with time to maturity. This hypothesis is supported by figure 5 which shows the value of an at-the-money option for different net arbitrage positions as a function of time to maturity of the futures contract.

Figure 5: Value of the Early Unwind Option as a Function of Time to Maturity: No Current Mispricing

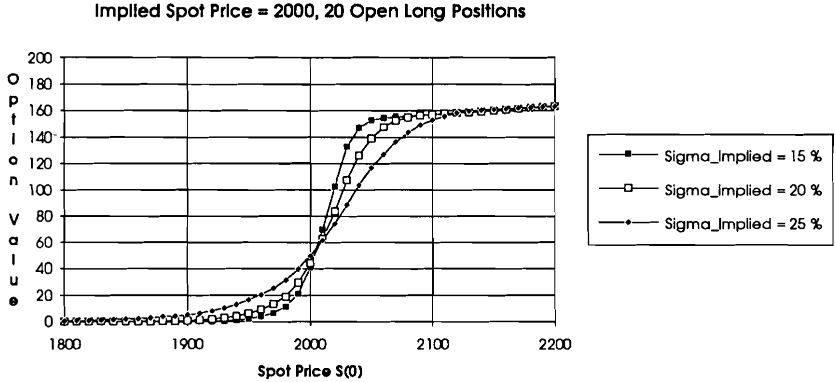



The option is worthless at time $t_{N-1}$ when there are only 2 days left until the futures contract is settled. This results from the fact that it does not pay for the arbitrageur to place a closing order at $t_{N-1}$ as the actual mispricing is assumed to be zero and the closing order executed at time $t_N = T$ has an adverse price effect. Instead, he will hold the position until $T + \Delta t$ where the future is settled against the spot position.

Without any open arbitrage position, the arbitrageur will not even trade when there are a few days left. The reason is that he first has to open an arbitrage position resulting in transaction costs and an adverse price effect. If the mispricing is zero at time $t_{N-2}$, this leads to an average arbitrage loss at $t_{N-1}$ which the arbitrageur only can be compensated for if the mispricing at $t_N$ changes so much that the expected future unwinding profits are very high. However, given the parameters of the model, especially the volatility, this does not happen, and that implies that the expected profit of this strategy is negative, that the arbitrageur will not trade, and that the option is worthless. With a higher volatility of the exogenous random variables the option might have had a positive value three days before maturity.

To discuss the effect of the volatility of the exogenous variable on the option value, figure 6 shows the option value for three different volatilities of the implied spot price.

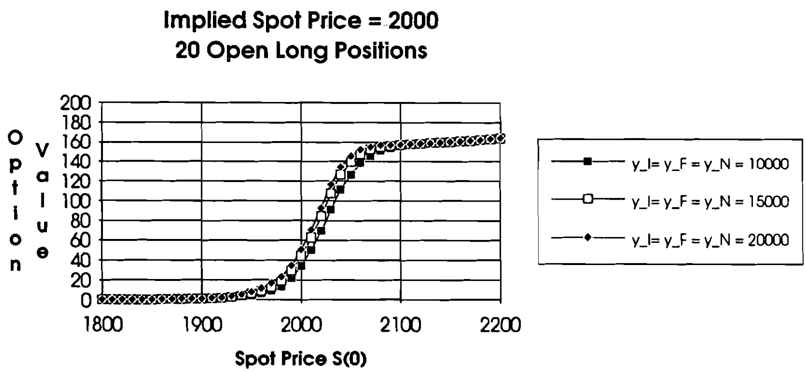
Figure 6: Value of the Early Unwind Option as a Function of the Spot Price: Different Volatilities of the Implied Spot Price




The figure suggests that the option value does not necessarily increase as the volatility of the implied spot price increases. If the early unwind option is in-the-money, an increase of $\sigma$ results in a decrease of the option value, if the option is out-of-the-money, the opposite effect results. These ambiguous consequences can be explained by the fact that a variation of $\sigma$ affects the drift and the conditional volatility of the mispricing process (31) simultaneously as the parameter $\alpha$ depends on $\sigma$. If $\sigma$ increases, by equation (13), the absolute value of the slope, $\alpha_I$, of the demand function of information traders decreases. Since $\alpha$ decreases with $\alpha_I$, as long as the denominator of $\alpha$ in equation (27) is positive, $\alpha$ increases, and the market liquidity $1/\alpha$ decreases with $\sigma$. Therefore, the conditional volatiliy of the mispricing and the impact of arbitrage trading on the mispricing increases with $\sigma$. Whereas the first effect results in higher option value, the second one reduces the value of the early unwind option.

Before figure 6 will be discussed in detail, it will be shown that the volatility effect can be seperated from the market liquidity effect. From equation (31) follows that a reduction of the group size $y_F, y_I, y_L$ under the restriction $y_F = y_I = y_L$ increases $\alpha$, but does not change the coefficients of $\Delta S_F$, $\tilde{Z}_L$, and $\Delta\tilde{S}_I$. Therefore, the conditional volatility of the mispricing process remains unchanged whereas the market impact of arbitrage trading increases. As a consequence, the option value will decrease since unwinding the same numer of open positions will cause larger adverse price movements. This effect is shown in figure 7.

Figure 7: Value of the Early Unwind Option as a Function of the Spot Price: Different Market Liquidity




The option values in figure 7 differ most if the option is in-the-money, but not deep in-the-money. This can be justified by the argument that the arbitrageur is able to unwind immediately less positions with a profit if his order has a larger impact on the spot price. If the option is deep in-the-money, he will unwind all positions, even if the market liquidity is low. Therefore, the disadvantage of lower market liquidity is smaller. If the option is out-of-the-money, the arbitrageur makes no immediate unwinding profit.

The influence of $\sigma$ on the market liquidity explains why in figure 6 the option value for in-the-money options increases with decreasing volatility. However, figure 6 also shows that for options out-of-the-money and deep in-the-money the ranking of the curves changes. This results from the second effect of an increasing volatiliy. If the option is out-of-the-money, i.e. $S(0) < S_I(0)$, its value depends essentially on the probability that the mispricing changes its sign. If the volatility of the implied spot price increases, it follows from equation (31) that the conditional volatility of the mispricing, the chance of the out-of-the-money option to move in-the-money, and, therefore, the value of the option increases.

If the option is deep in-the-money all positions will be unwound and a net short position will be build up. Related with this new short position is a deep out-of-the-money option since the spot price is considerably larger than the implied spot price. As argued above, the value of this option increases with $\sigma$. This effect dominates the market liquidity effect if the absolute mispricing is high.

Thus, the impact of the volatility on the option value is different for in- and out-of-the-money-options and can be summarized as follows. If the option is out-of-the-money the option value increases with volatility as there is a better chance for the option to move in-the-money. If the option is in-the-money there are two opposite effects. First, the lower the volatility, the lower will be the impact of the unwinding on the mispricing, and the more positions the arbitrageur can unwind immediately. The consequence of this higher market liquidity is smaller when the mispricing is so large that all positions will be unwound even if the market is less liquid. Second, the smaller the volatility, the smaller is the value of the deep out-of-the-money option the arbitrageur receives when he opens new long or short positions. This second effect dominates in the case of deep out-of-the-money options.

Different transaction costs for opening and closing arbitrage positions are the basic reason for the positive value of the option. Therefore, the size of these transaction costs should have a strong impact on the option value. To analyze this impact, three questions will be addressed. How does the value of the early unwind option changes when (1) the round trip transaction costs per unit $c_O$ are fixed and the difference to the unwinding transaction costs per unit $c_{UW}$ increase, (2) $c_O$ increases and the difference $c_O - c_{UW} = 0.00375$ remains unchanged, and (3) $c_O$ increases and the ratio $c_O / c_{UW} = 4$ remains fixed ?

The first and second questions can be answered qualitatively. Neglecting the influence of the unwind option on the demand strategy, the immediate unwinding profit is determined by $|D_A^*(0)| * (c_O - c_{UW}) * \tilde{S}(t_1)$. If the unwinding costs, $c_{UW}$, decrease, the unwinding profit and the option value will increase. The answer to the second question follows from the observation that higher transaction costs lead to less arbitrage trading, so $|D_A^*(0)|$ decreases. If the difference $(c_O - c_{UW})$ is constant, less trading leads to less immediate unwind profit and, therefore, to a lower option value. Both qualitative arguments have been supported by numerical results.

The answer to the third question is not obvious as both effects described above occur simultaneously. An increase in the round trip costs per unit reduces the trading volume, but increases the transaction costs difference, $(c_O - c_{UW})$, as the unwind costs are proportional to the round trip costs. Figure 8 shows the value of the early unwind option as a function of the current spot price for three different round trip transaction costs per unit. The unwinding costs per unit are one quarter of the round trip costs.

Figure 8: Value of the Early Unwind Option as a Function of the Spot Price: Different Round Trip Transaction Costs Per Unit

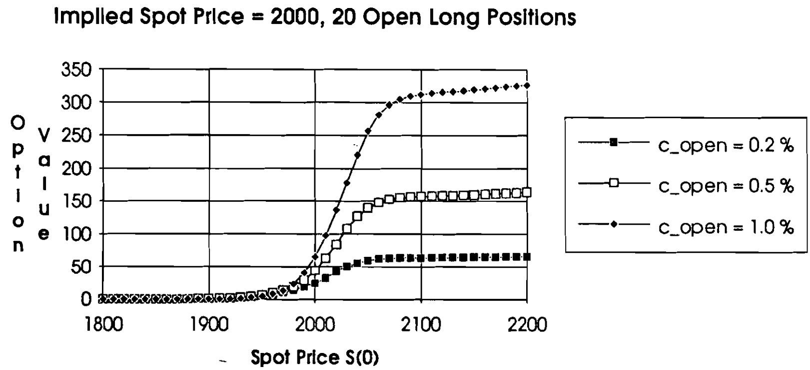



Figure 8 demonstrates that the effect of increasing transaction costs differences dominates the trade volume effect. So, assuming that the unwinding transaction costs per unit are proportional to the round trip costs per unit, higher transaction costs lead to a higher option value. Numerically, changing transaction costs per unit have the most important impact on the option value. This result could be expected because the basic reason for a positive option value is the difference in transaction costs per unit.

## 5. Conclusions

In this paper the value of the unwind option implied in open arbitrage positions has been determined where the underlying of the option has been assumed to be an endogenous variable. The stochastic process of this variable emerges from a dynamic market microstructure model. This model is mainly characterized by four specific features. First, there are two different groups of speculative investors, noise traders, and one monopolistic arbitrageur. The investors differ by their motives for trading, by the price forecast model they use, by the type of orders they place, and by the instruments they trade. Second, the futures market has an impact on the spot market by arbitrage trading as well as by trading of speculative investors who use the futures price as an information in their trading decision. Third, arbitrageurs are subject to an order execution lag which prevents risk free arbitrage. Finally, as the stochastic processes of the spot price and the basis are determined endogenously by the demands of the all investors, the basis depends on the early unwind option of the arbitrageur itself.

With respect to the endogenous basis process, the most important result of the paper is that this process is not Markovian. The reason for the path dependency lies in the structure of the arbitrageur´s demand which depends on his whole trading history. Concerning the value of the early unwind option, it is shown numerically that the early unwind option has a positive value even if the arbitrageur holds no open arbitrage position currently. A positive or negative number of open arbitrage positions makes the option more valuable. The option value is not symmetric with respect to the mispricing as the transaction costs depend on the level of the spot price, and the option value does not change monotonically with the mispricing. These results are in contrast to the Brennan and Schwartz (1990) model where the underlying of the option is an exogenous variable and transaction costs are constant. Contrary to standard option pricing theory, an increase of the volatility of the exogenous variable does not necessarily increase the option value. The reason for the ambiguous effect of changing volatility is that both the market liquidity and the volatility of the mispricing change simultaneously. Finally, although increasing transaction costs per unit reduce the profit and the trading volume of the arbitrageur, the value of the early unwind option increases.

The results thus far suggest additional lines of research to be pursued in the future. Four different spot price processes will be analyzed: the implied spot price, the endogenous spot price without arbitrageur, the endogenous spot price with an arbitrageur holding positions until maturity, and the endogenous spot price with an arbitrageur who has the early unwind option. This investigation will be supplemented by a comparative study of the arbitrageur´s optimal demand strategy. Special interest will be paid to his optimal unwind decision, especially to the question how the critical values of the mispricing where the arbitrageur closes out positions relate to the number of open arbitrage positions he holds. Finally, a comparison of the endogenous basis with and without arbitrage will offer some further insight into the role of the arbitrageur in the model.

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