Discussion Paper

Discussion Paper No. 93-25

On the Modelling of Speculative Prices by Stable Paretian Distributions and Regularly Varying Tails

Juergen Kaehler

ZEW

Zentrum für Europäische Wirtschaftsforschung GmbH

International Finance Series

On the Modelling of Speculative Prices by Stable Paretian Distributions and Regularly Varying Tails

by

Juergen Kaehler

Zentrum für Europäische Wirtschaftsforschung (ZEW)
and
University of Mannheim

December 1993

Abstract

Earlier studies which applied the family of stable Paretian distributions to financial data are inconclusive and contradictory. In this article I estimate the parameters of the model by the Feuerverger-McDunnough method which enables the application of maximum likelihood methods. Based on inferential statistics, stable Paretian distributions can be rejected with monthly data. In order to confirm this result, the model is extended to the family of distributions with regularly varying tails. The result that stable Paretian distributions are not applicable is indeed confirmed by estimating the coefficient of regular variation.

I. Introduction

The concept of choice under uncertainty is the cornerstone of financial theory. Therefore, the stochastic specification of financial models is of fundamental importance in almost any branch of modern finance. A convenient and natural choice for the underlying probability model is the normal distribution in the static context and the corresponding Wiener process in the continuous-time context. However, triggered by the seminal papers of Mandelbrot (1963) and Fama (1965), financial economists began to question the assumption of normality. In empirical work, it has repeatedly been found that short-run price and return dynamics do not have normal distributions. It turned out that excessive mass at the centre and in the tails of empirical distributions, i.e. leptokurtosis, is a strong and robust stylized fact of daily and weekly price dynamics in speculative markets.

It was also Mandelbrot (1963) who introduced the model of stable Paretian distributions into finance in order to capture these empirical regularities. Over the years, a number of other probability models have been suggested which also imply leptokurtosis, but the stable Paretian distributions have one distinctive advantage over those rival models: the stable Paretian distributions are related to a generalization of the central limit theorem. If one drops the assumption of finite variance in the central limit theorem, one arrives at the family of stable Paretian distributions as the only limit distributions for sums of independent and identically distributed random variables (with appropriate standardization). The normal distribution is just a special stable distribution. As with the normal distribution, the sum of stable-distributed random variables also has a stable distribution. This sum-stability is, of course, a very desirable property of the model. It means that a model for daily data would also be applicable for weekly or monthly data whereas this does not hold, for instance, for Student's t distribution which is one of the popular rival models.

The applicability of the stable Paretian distributions to stock prices[1], exchange rates (see below) and future prices[2] has been examined in numerous studies. The evidence is mixed and those studies which claim to find evidence against stable Paretian distributions are in most cases not conclusive because they employ crude estimation methods and are not based on statistical inference. Only recently, the studies by Akgiray and Booth (1988), Kofman and de Vries (1990), and Jansen and de Vries (1991) cast serious doubt on the applicability of stable Paretian distributions in empirical finance. However, they only examine the tail behaviour of empirical distributions in the context

[1]See, among others, Officer (1972), Barnea and Downes (1973), Leitch and Paulson (1975), Fielitz and Rozelle (1983), Kon (1984), Akgiray and Booth (1988), Akgiray, Booth and Loistl (1989), Lau, Lau and Wingender (1990), and Jansen and de Vries (1991).

[2]See, among others, Cornew, Town and Crowson (1984), So (1987), Hall, Brorsen and Irwin (1989), and Kofman and de Vries (1990).

of the generalized Pareto distribution and of extreme-value distributions. They cannot, therefore, resolve the puzzle of why in direct estimation of the model it could never be convincingly rejected.

In this paper, I present evidence which helps to resove this puzzle. I apply the stable Paretian distributions to exchange-rate data, but it should be emphasized that the approach taken here is readily extended to stock prices and future prices. Since there is very strong similarity in the empirical properties of different speculative prices, similar results can be expected for other financial data.

Since the late 1970's, the model of stable distributions has often been applied to exchange-rate data. However, the results from these studies are inconclusive and contradictory. Westerfield (1977), Calderon-Rossell and Ben-Horim (1982), McFarland et a. (1982) and So (1987) conclude that the model is applicable to exchange rates, but Friedman and Vandersteel (1982), Schlittgen et al. (1982), Gaab (1983) and Kaehler (1988) reject the model. Koedijk et al. (1990) find weak evidence in favour of the model whereas Boothe and Glassman (1987) and Akgiray and Booth (1988) find mixed evidence that casts some doubt on the applicability of the model. Rogalski and Vinso (1978) and Tucker and Pond (1988) do not reject the model but they favour other models (Student's t distribution and compound Poisson process) which seem to fit the data better. This paper aims to resolve this uncertainty about the applicability of stable Paretian distributions to exchange-rate data.

II. Some statistical properties of exchange-rate dynamics

As mentioned in the Introduction, the great popularity of the stable distributions is due to the fact that they are related to a generalization of the central limit theorem. From an economic point of view, a stochastic model which is based on the summation of random variables fits well into the broad framework of asset-market theories of the exchange rate (see e.g. Mussa (1984)).

There is another property of stable distributions which makes them attractive for modelling speculative prices. In continuous time, a stochastic process which is driven by Gaussian increments, the so-called Wiener process, is itself continuous and cannot, therefore, explain jumps in the observed series. The sample path of a stochastic process driven by increments from a stable distribution, on the other hand, is everywhere discontinuous. A common property of short-run speculative price series is that they include jumps which are incompatible with a normal distribution in the sense that they would be extremely unlikely under this distribution. For instance, the U.S. dollar depreciated against the German mark on Monday, 23rd September 1985 by 5.75 percent following Sunday's Plaza-agreement to bring the dollar down. Under the normal distribution, with the population variance replaced by the sample variance, a depreciation of this magnitude would be expected to occur once in about 70,000 years. Thus, there is an obvious need to choose a model which attaches more probability to extreme observations, i.e. to adopt a fat-tailed distribution. Stable distributions are in fact fat tailed.

The property of fat tails is related to the concept of leptokurtosis. A distribution is said to be leptokurtic if β_2, the ratio of the fourth central moment to the square of the variance, is greater than 3. It is called mesokurtic if $\beta_2 = 3$ and platykurtic if $\beta_2 < 3$. It can be shown that leptokurtosis can be caused by excessive mass (compared with the normal distribution) both at the center of the distribution and in the tails (see Ruppert (1987) and Balanda and MacGillivray (1988)). As mentioned before, leptokurtosis is a very strong and robust statistical property of short-run price dynamics in speculative markets. For exchange-rate fluctuations, earlier studies found significant leptokurtosis in daily and weekly data but not in monthly or quarterly data (see Boothe and Glassman (1987)). In order to gain more insight into the statistical properties of exchange-rate dynamics which are relevant for the stable Paretian model, this section provides some analysis of the distribution and moments of exchange-rate fluctuations.

The data to be analyzed are the exchange rates of the U.S. dollar against the German mark, the British pound, the Swiss franc and the Japanese yen. The data are on a daily basis but weekly, monthly and quarterly data are also used. In these cases, end-of-period data were derived from the daily exchange rates. The data range from July 1st, 1974 to December 31st, 1987. Because of bank holidays, the number of daily observations is different for each country: 3386 for the mark, 3417 for the pound, 3392 for the franc and 3365 for the yen. For all currencies, the number of observations in the weekly series is 704, in the monthly series it is 161 and in the quarterly series it is 53. Data source is the IMF's International Financial Statistics and the monthly reports of the Swiss National Bank. The exchange-rate dynamics are analyzed in the form of $x_t = 100(e_t - e_{t-1})$, where e_t is the logarithm of the exchange rate at time t.

Table 1 shows the first four moment statistics together with two non-parametric tests for the exchange-rate series at daily, weekly, monthly, and quarterly intervals. The null hypothesis (H_0) of a zero mean is tested by a standard *t*-test. At the 5 percent significance level, H_0 can only be rejected for the yen and at the 10 percent level it can be rejected for the weekly and monthly franc series. Thus there is no strong evidence against a zero mean.

For daily data, the variance is between 0.381 (for the yen) and 0.674 (for the franc). Stable Paretian distributions have infinite variances but, of course, the sample variances from these distributions would always be finite. Granger and Orr (1972) suggested a graphical "convergence variance test" in order to explore the applicability of stable Paretian distributions. Following Granger and Orr, I calculated sequential variances

Table 1. Moments and distributional properties of exchange-rate movements

		mark	pound	franc	yen
day	mean	-0.014	0.007	-0.025	-0.025 **
	variance	0.464	0.437	0.674	0.381
	skewness	-0.370 ***	-0.143 ***	-0.106 **	-0.611 ***
	kurtosis	8.324 ***	8.363 ***	8.893 ***	7.997 ***
	median	-0.004	0.0	-0.009	0.0
	symmetry	-0.010	-0.031 *	-0.015	-0.028 **
week	mean	-0.068	0.035	-0.120 *	-0.120 **
	variance	2.162	2.060	2.953	1.641
	skewness	-0.299 ***	0.012	-0.248 ***	-0.988 ***
	kurtosis	5.845 ***	7.381 ***	4.955 ***	6.999 ***
	median	-0.045	-0.006	-0.053	0.0
	symmetry	-0.007	0.005	-0.010	-0.028
month	mean	-0.306	0.148	-0.525 *	-0.547 **
	variance	11.49	10.50	14.99	11.36
	skewness	6	6 ***	8	2
	kurtosis	-0.025 **	-0.555 **	0.190 **	-0.274
	median	3.869	4.147	4.194	3.624
	symmetry	-0.237	0.046	-0.390	-0.099
		-0.010	-0.017	-0.015	-0.039
quarter	mean	-0.976	0.415	-1.576	1.665
	variance	38.93	31.52	54.13	38.22
	skewness	1	6	6	4
	kurtosis	0.145	-0.004	-0.435	-0.460
	median	2.671	2.725	2.766	2.622
	symmetry	-1.317	-0.311	-0.763	-0.770
		0.015	0.011	-0.039	-0.051

Significance levels: *10 percent, **5 percent, ***1 percent.

(1)
$$s_{T_n}^2 = \frac{1}{T_n - 1} \sum_{t=1}^{T_n} (x_t - \overline{x}_n)^2$$

stepwise by setting $T_1 = 10$ and added 10 more observations at each following step. This gives a sequence of variances for the first 10, 20, 30, ... observations. Granger and Orr (1972) proposed to plot this sequence of variances against T_n. If all x_t come from the same distribution with variance σ_X^2, then $s_{T_n}^2$ should converge to σ_X^2. A failure of convergence could be a sign of the fact that x_t does not have finite variance. Granger and Orr noted, however, that this graphical test does not give a sufficient condition for the presence of infinite variance since non-convergence can also be caused by non-stationarity or non-independence. Although no strong conclusion can be drawn from the sequential variances, it is still instructive to look at the plots within an exploratory data analysis. The plots are displayed in Figure 1.

The sfr-dollar series and the yen-dollar series, somewhat later than the sfr-dollar rate, appear to have reached almost stationary values of their variances. The mark-dollar series and the pound-dollar series, on the other hand, show an upward tendency in the sequential variances and this could indicate the presence of stable Paretian distributions.

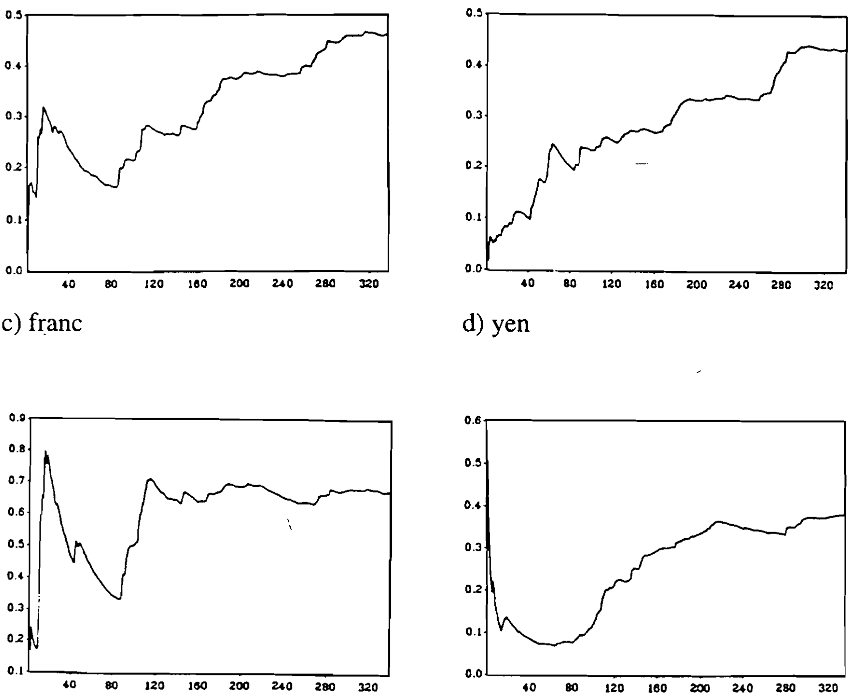
Figure 1. Sequential variances: daily exchange rates

a) mark

b) pound

c) franc

d) yen



Turning next to a measure of asymmetry in distribution, table 1 shows that the skewness, defined as the third standardized moment, is significantly different from zero for most daily and weekly series. All significant skewness statistics are negative implying that there are more strong dollar depreciations than strong dollar appreciation in the data.

As noted before, stable distributions have often been applied to financial time series because these distributions are leptokurtic. Table 1 shows that there is strong leptokurtosis in daily and weekly exchange-rate data but there is only moderate leptokurtosis in monthly data and the quarterly data have even a platykurtic distribution (although the platykurtosis is not significant). This convergence to normality casts doubts of the applicability of stable Paretetian distributionsto the exchange-rate series. Furthermore, the kurtosis statistics are well behaved in the sense that there is relatively little variation in these statistics between the four exchange rates. Lau et al. (1990) observed that for typical parameter estimates of stable distributions, the sample kurtosis would show great variation and the mean sample kurtosis was always larger than 100 in their simulations. Obviously, the kurtosis statistics of the exchange-rate data do not follow this pattern.

The highly significant leptokurtosis statistics for short-run exchange-rate dynamics cautions against the application of parametric tests. It is, therefore, advisable to supplement the parametric tests for zero mean and for symmetry by some non-parametric tests.

The median, which is more robust in the context of fat-tailed distribution than the mean, is reported in table 1. Note that for all series the median is closer to zero than the mean. For the daily pound series and the daily and weekly yen series the median is exactly zero. A non-parametric test for the H_0 of a centre of location at zero is given by the simple median sign test (see Kendall and Stuart (1979), pp. 542-546)[3]. This test does not reject the H_0 for any of the exchange rates at any time interval. Thus we may maintain the hypothesis of a center of location at zero[4].

The non-normality of the data also calls for a robust test for symmetry. Randles et al. (1980) suggested a non-parametric U-test based on order statistics. The U-test statistics are shown in the last row of each panel in table 1. The results of this U-test are drastically different from those of the skewness test. The evidence of asymmetry disappears virtually. Only in the daily yen series is the U-test statistic different from

---

[3]The test statistic is $z = \left(| B - \frac{1}{2} T_{(0)} | - \frac{1}{2} \right) / \frac{1}{2} T_{(0)}^{\frac{1}{2}}$ where B is the number of observations which are smaller than zero and $T_{(0)}$ is T minus the number of observations which are exactly zero. The test statistic z follows asymptotically a standard normal distribution.

[4]The same conclusion would be drawn from applying the more general biweight mean, which is a M-estimator of location (see Iglewicz (1983)). Kaehler (1989)) shows that a test based on the biweight mean cannot reject the H_0 of a zero mean for the four exchange rates.

zero at the 5 percent level and the statistic for the daily pound series is significant at the 10 per cent level. All other statistics are not significantly different from zero and, therefore, the H_0 of symmetry cannot be rejected. The discrepancy between the results from the skewness test and the U-test can be explained with the correlation between kurtosis and skewness. Fat tail distributions tend to produce stronger (positive or negative) skewness than thin tail distributions (see Bowman and Shenton (1986)). Therefore, the results from the non-parametric U-test are more trustworthy than those from the skewness test.

To sum up, there is mixed evidence in the stylized facts of exchange-rate dynamics on the applicability of stable Paretian distributions to exchange-rate data. First, the empirical leptokurtosis is compatible with stable Paretian distributions. However, many other popular stochastic models in empirical finance such as the mixture of normal distributions, the compound Poisson process or Student's *t* distribution also imply leptokurtosis. Second, the analysis of sequential variances provides some evidence in favour of stable Paretian distributions for only two of the four exchange-rate series. Third, the hypothesis of a centre of location at zero and of symmetry cannot be rejected by non-parametric tests for the exchange-rate data[5] and this simplifies the later computations.

III. Properties of stable Paretian distributions

There are two reasons which appear to make stable distributions unattractive for applications. First, not everybody is ready to accept the implications of infinite variance since variance is a widespread concept in statistics and economics. One would have to rework many areas of statistics and economics to incorporate stable distributions. In fact, there were some attempts to formulate portfolio analysis for underlying stable distributions of returns by Fama and Samuelson following the apparent success of these distributions to fit stock-price data. However, stable distributions are rather awkward to work with analytically as will be shown shortly.[6]

Second, stable distributions are also awkward to work with empirically because they cannot, in general, be described in closed forms of the density or the distribution function. Instead, they are usually described by their log-characteristic function

[5]In contrast, stock price dynamics often have non-zero means and asymmetric distributions

[6]Also, infinite variance is sometimes seen to be implausible. It is true that every empirical variance must be finite because every empirical support is finite but this cannot be advanced as an argument against stable distributions because they share with many other distributions the property of infinite support, including the normal distribution. The difference to finite-variance distributions is simply the increase of probability in the tails of the distributions.

(2) $$\log \Phi_X(u) = i\delta u - \gamma \mid u \mid^{\alpha} [1 + i\beta(u/\mid u \mid)\omega(u,\alpha)]$$

where $i = \sqrt{-1}$ and

(3) $$\omega(u,\alpha) = \begin{cases} \tan(\pi\alpha/2) & \text{if} \quad \alpha \neq 1 \\ 2\log(\mid u \mid)/\pi & \text{if} \quad \alpha = 1. \end{cases}$$

and u is an auxiliary variable.

The characteristic function is determined by four parameters which can be related to the first four moments. First, δ is a location parameter $(-\infty < \delta < \infty)$ which is equal to the expected value of X if $1 < \alpha \leq 2$. It is equal to the median and mode if $\beta = 0$. Second, γ is a scale parameter ($\gamma > 0$) which measures the spread of the distribution. If $\alpha = 2$ (the case of the normal distribution), $\gamma = \sigma^2/2$, where σ^2 is the variance. For $\alpha < 2$, γ is some other measure of spread, for instance if $\alpha = 1$ and $\beta = 0$ (the case of the Cauchy distribution) γ is the semi-interquartile range. Third, β is a skewness parameter ($-1 \leq \beta \leq 1$). If $\beta = 0$, then the distribution of X is symmetric, for $\beta > 0$ it is skewed to the left, and for $\beta < 0$ it is skewed to the right. Together with the characteristic exponent α, β determines the type of distribution. The characteristic exponent ($0 < \alpha \leq 2$) determines the highest order of finite moments within this family. If $\alpha < 2$, then the variance is infinite, i.e. the normal distribution with log-characteristic function

(4) $$\log \Phi_X(u) = i\delta u - \frac{1}{2}\sigma^2 u^2$$

is the only member in this family with finite variance and finite moments of any (positive integer) order. The expected value is not finite (and a fortiori all higher moments are not finite) if $\alpha \leq 1$.

The characteristic exponent is related to kurtosis in the following way. Recall that kurtosis measures both peakedness and tail weight. For a symmetric stable random variable which is standardized by $x' = (x - \delta)/\gamma$, the density at the origin is given by (see Holt and Crow (1973))

(5) $$f(0 \mid \alpha) = \frac{1}{\pi\alpha}\Gamma\left(\frac{1}{\alpha}\right)$$

where $\Gamma(.)$ denotes the gamma function. Obviously, the density in (5) is a decreasing function of α. Hence, all symmetric non-Gaussian stable distributions are peaked when compared to the normal distribution.

The tail behaviour of the symmetric non-Gaussian stable distributions can be described by

$$F(x) = C \mid x \mid^{-\alpha} \qquad \text{for} \quad x \to -\infty \tag{6}$$

$$1 - F(x) = Cx^{-\alpha} \qquad \text{for} \quad x \to +\infty, \tag{7}$$

where F denotes the distribution function and $C > 0$, whereas $C = 0$ for the normal distribution (see Mandelbrot (1963)). This means that non-Gaussian stable distributions have fatter tails than the normal distribution and the smaller the characteristic exponent α is, the fatter the tails are. The properties of (6) and (7) are quite important in economics and probability theory. In economics, the Pareto distribution, which is often applied to model income distributions, has a distribution function which satisfies (7). This prompted Mandelbrot to introduce the name "stable-Paretian distributions" for the non-Gaussian stable distributions.

In probability theory, distribution functions which satisfy (6) and (7) are called distributions with regularly varying tails. They play an important role in the concept of the domain of attraction. The common distribution F of independent random variables X_j is defined to belong to the domain of attraction of a distribution G if the sum of the appropriately standardized X_j tends in distribution to G. The classical central limit theorem is based on the fact that F belongs to the domain of attraction of the normal distribution if F has finite variance. On the other hand, a distribution belongs to the domain of attraction of stable Paretian distributions if it satisfies (6) and (7) with $0 < \alpha < 2$. In more informal terms, this implies that stable Paretian distributions can only attract distributions which are "similar" to themselves whereas the normal distribution can attract distributions with widely varying shapes (see Galambos (1988), chapter 6). I will come back to the concept of regularly varying tails in Section V.

Stable Paretian distributions are "self-attracting" in the sense that the sum of independent and identically distributed stable variables has also a stable distribution with the same α and β. This stability of the shape parameters α and β under addition gave rise to the name of this family of distributions.

In the following, I will restrict the model of stable Paretian distributions to the symmetric case, i.e. $\beta = 0$, with $\delta = 0$. The non-parametric test results of the previous section justify these restrictions. Furthermore, α is restricted to the interval (1,2]. An estimated value of α in the interval (0,1] would obviously put me in an unpleasant position of having to reconcile such a result, which implies that the expected value is not finite, with the finding that the null hypothesis of a constant mean at zero cannot be rejected. Anyway, in the actual estimations of α, there was never a convergence to the value of 1. Also, in previous applications of stable distributions to exchange rates, all estimates of α were above 1.

What makes stable Paretian distributions awkward to work with empirically is the fact that, in general, closed forms for the corresponding densities are not available. Apart from the Cauchy distribution, which was mentioned above, the only other non-normal stable distributions with known closed-form densities are the Holtsmark-Levy-Smirnov distribution with $\alpha = 1/2$ and $\beta = \pm 1$ and the Mitra distributions (see Csörgö (1984)). Of course, the probability law of a random variable can be described by the distribution function, by the density function or by the characteristic function and all three ways are perfectly equivalent. Furthermore, the three functions are related through the operations of differentiation, integration and Fourier transform. However, in order to apply maximum likelihood (ML) methods, one needs to compute the densities.

The lack of general closed forms of the densities has led to the suggestion of numerous estimators for the parameters, especially for α, which is the decisive parameter[7]. For exchange-rate data and other speculative prices, the most popular estimator of α has been the one suggested by Fama and Roll (1968, 1971). Their estimator is based on matching empirical and theoretical fractiles and exploits the fact that tail weight is a function of α. McCulloch (1986) generalized the Fama-Roll estimator to cover non-symmetric distributions and to remove the asymptotic bias in the Fama-Roll method [8]. McCulloch's method has been applied to exchange-rate data by So (1987) and Tucker and Pond (1988).

A method based on the empirical characteristic function has been introduced by Koutrouvelis (1980) and was applied to exchange-rate data by Akgiray and Booth (1988). This method depends crucially on the values of the auxiliary variable u chosen. Koutrouvelis suggested selecting different values of u_i and estimating α from a regression of $\log(-\log|\Phi_T(u_i)|^2)$ on $\log|u_i|$, where $\Phi_T(u_i)$ is the empirical characteristic function based on T observations [9].

Finally, Feuerverger and McDunnough (1981) proposed to overcome the problem of lacking densities by estimating the densities via the fast Fourier transform (FFT). This method implies some computational burden. However, it is the most elegant and convincing method of all the ones which have been proposed, and it permits one to apply ML methods on the estimated densities. This estimator has been applied by Boothe and Glassman (1987) to exchange-rate data.

---

[7] A good overview is provided by Csörgö (1984).

[8] There is a downward bias in the Fama-Roll estimator of α.

[9] Akgiray and Lamoureux (1989) compared the McCulloch estimator and the Koutrouvelis estimator in a Monte-Carlo study and found that the Koutrouvelis estimator performed better than the McCulloch estimator in terms of bias and precision for any sample size and values of α and β. Both methods were quite accurate in estimating α.

IV. Estimation of parameters

A central issue in the fitting of stable distributions to speculative prices is the stability of α under time aggregation. According to this model, only γ and δ (if non-zero) should change under time aggregation. In earlier applications to exchange-rate data, however, there was an apparent increase in the estimated α's under time aggregation. Table 2 gives an overview of the studies.[10] McFarland et al. (1982) and Boothe and Glassman (1987) observed an "instability" of α and interpreted it as evidence against this model. At this stage, however, this conclusion is not very convincing. First, an increase of α could also occur in a model which is a mixture of stable distributions. That is, one does not have to abandon the family of stable distributions in order to reconcile rising α's with the model. Second and more importantly, those earlier studies listed in table 2 do not present any test statistics on which proper statistical inference could be based to test the stability of α. The only study which reports standard errors of α is the one by So but he analyses only daily data. Therefore, the stability of α, and thus the applicability of this model, is still an open question.

Following Feuerverger and McDunnough (1981), I obtained densities via FFT[11]. I then used a numerical gradient method to get the ML estimates $\hat{\alpha}$ and $\hat{\gamma}$. The results from applying the Feuerverger-McDunnough approach to the estimation of stable Paretian distributions are reported in table 3. Starting values for $\hat{\alpha}$ and $\hat{\gamma}$ in the iterations were obtained from the Koutrouvelis estimators. The estimates of α from the Koutrouvelis method ranged between 1.70 and 1.78 for daily data and 1.73 and 1.80 for weekly data.

Compared with these earlier studies, my Koutrouvelis estimates of α for daily data are surprisingly high. I also applied the Fama-Roll method to estimate α and got values between 1.41 and 1.55 for the four daily series. These estimates from the Fama-Roll method are more in line with earlier studies. Since the comparative simulation study by Akgiray and Lamoureux (1989) showed that the Koutrouvelis estimators

[10]From the 13 studies mentioned in the Introduction, 5 studies are not included in the table. In the article of Rogalski and Vinso (1978), it is unclear which data they used to estimate the stable distributions. Schlittgen et al. (1982) and Calderon-Rossell and Ben-Horim (1982) did not estimate the characteristic exponent. Akgiray and Booth (1988) reported only likelihoods for the estimated stable distributions. However, these likelihoods are not plausible. For other candidate models, their log-likelihoods are between 7944.2 and 8192.5 but for the stable distribution they are between -24302.5 and -17307.1. According to my own experience, the likelihoods of all their candidate models should be similar in magnitude (see Kaehler (1990)). Finally, Koedijk et al. (1990) employ the same methods as I do in Section V, but they use data from the European Monetary System. Presumably, the data generating process of a flexible exchange-rate system is quite different from the one which governs a system with limited exchange-rate variablility but occasional realignments.

[11]The Feuerverger-McDunnough approach is described in the appendix.

Table 2. Studies applying stable distributions to exchange-rate data

Authors	Period	Estimates of α	Method	Currencies
Westerfield (1977)	day	1.33-1.51	Fama-Roll	DM, £, SFr
Friedman, Vandersteel (1982)	day week # month #	1.11-1.45 1.30-1.50 1.33-1.63	Fama-Roll	DM, £, SFr, ¥
McFarland, Pettit, Sung (1982)	day week #	1.12-1.40 1.50-1.92	Fama-Roll	DM, £, SFr, ¥
Gaab (1983)	day week # month #	1.50 1.45 1.78	Fama-Roll	DM
So (1987)	day	1.10-1.16	McCulloch	£, ¥
Boothe, Glassman (1987)	day week month	1.27-1.62 1.54-1.72 1.37-2.00	Feuerverger-McDunnough	DM, £, ¥
Kaehler (1988)	day week # month #	1.5 -1.7 1.4 -1.7 1.6 -1.9	Fama-Roll	DM
Tucker, Pond (1988)	day week	1.12-1.39 1.26-1.55	McCulloch	DM, £, SFr, ¥

Quasi-weekly or quasi-monthly data obtained from sums of 5 and 20 daily data, respectively.
Note that only those currencies are listed in the table which are also analysed in this study.

of α is superior to the one by McCulloch (and hence also to the one by Fama and Roll) in terms of bias and precision, one may conclude that earlier studies probably underestimated α.

As table 3 shows, the ML estimates of the Feuerverger-McDunnough approach do not differ very much from the Koutrouvelis estimates but the estimates of α tend to be somewhat smaller. Standard errors of the parameters are reported in brackets. I also estimated stable Paretian distributions for monthly data because stability of α under time aggregation is central to this model.

Table 3. Estimates of stable distributions by the Feuerverger-McDunnough method

		mark	pound	sfr	yen
day	α	1.74 (0.03)	1.56 (0.03)	1.68 (0.03)	1.60 (0.03)
	γ	0.40 (0.01)	0.35 (0.01)	0.46 (0.01)	0.33 (0.01)
	LR	371.8 ***	511.1 ***	497.1 ***	473.7 ***
	χ^2 (97)	226.7 ***	418.4 ***	220.0 ***	741.7 ***
week	α	1.68 (0.07)	1.74 (0.07)	1.68 (0.07)	1.65 (0.07)
	γ	0.84 (0.04)	0.84 (0.03)	1.00 (0.05)	0.71 (0.03)
	LR	60.6 ***	74.4 ***	38.9 ***	81.6 ***
	χ^2 (47)	57.8	77.5 ***	67.0 **	61.9 *
month	α	1.81 (0.14)	1.92 (0.07)	1.91 (0.08)	1.87 (0.17)
	γ	2.18 (0.19)	2.16 (0.13)	2.60 (0.16)	2.26 (0.21)
	LR	3.56 *	0.99	4.31 **	7.49 ***
	χ^2 (27)	21.4	27.8	33.7	47.9 ***

Significance levels: see table 1

There are several remarkable findings. First, for short-run exchange-rate dynamics, the estimates of α are significantly below 2 and there is no obvious increase of $\hat{\alpha}$ in weekly data as compared with daily data. According to the likelihood-ratio statistic, stable distributions achieve a much better fit in comparison with the normal distribution. However, the χ^2 test of goodness-of-fit rejects all daily models at the 1 percent significance level and one of the weekly models at the same level. This rejection by the χ^2 test is very similar to the rejection of the previous three models (mixture of normal distributions, compounded Poisson distribution and generalized t -distribution) by this test. The results for the monthly data, however, are drastically different. None of the $\hat{\alpha}$'s is significantly different from 2 as judged from their standard errors. Accordingly, the LR test rejects the H_0 of normality only for the yen. Thus, there is strong evidence for convergence towards normality. I also estimated the model with quarterly data and got point estimates of $\hat{\alpha}=2$ for all four series. For the quarterly mark and pound, the starting values from the Koutrouvelis method were also equal to 2. The results from table 6 are broadly in line with earlier studies.

V. Regularly varying tails

The fact that α is significantly smaller than 2 for short-run data but not for monthly or quarterly data indicates that the exchange rates do not follow stable distributions but other fat-tailed distributions. In order to examine this possibility, I come back to the concept of regularly varying tails. As mentioned above, only (symmetric) distributions whose tail probabilities follow the function $Cx^{-\alpha}$, with $0 < \alpha < 2$ belong to the domain of attraction of (symmetric) stable Paretian distributions. If $\alpha > 2$, then the distribution belongs to the domain of attraction of the normal distribution. I, therefore, extend the model of stable distributions to the class of distributions with regularly varying tails of which stable distributions are a sub-class which obtains when the tail probabilities follow $Cx^{-\alpha}$ with $\alpha < 2$ (recall that stable distributions are self-attracting). All other fat-tailed distributions with tail probability $Cx^{-\alpha}$ and $\alpha > 2$ do not belong to the family of stable Paretian distributions. In contrast to stable Paretian distributions they converge to the normal distribution under addition.

Within the class of distributions with regularly varying tails one can, therefore, discriminate between fat-tailed stable and non-stable distributions by estimating the coefficient of regular variation α. One can reject the model of stable Paretian distributions if α turns out to be larger than 2. Note that α can have two meanings in this context. First, it denotes the coefficient of regular variation which determines the tail behaviour of the distributions function and, second, it denotes in addition the characteristic exponent of stable distributions if $\alpha < 2$.

Analysing the family of distributions with regularly varying tails can enable us to reject the sub-class of stable Paretian distributions, but it does not help to identify a specific distribution function if α is estimated to be greater than 2. Some distributions like Student's distribution are known to have regularly varying tails with $\alpha > 2$ but one cannot associate a specific $\alpha > 2$ with a specific distribution function.

Hill (1975) proposed a conditional ML method to estimate α. The estimator is given by:

(8)
$$\hat{\alpha}(q) = \left[\frac{1}{q} \sum_{j=1}^{q} \log |x_{(T-j+1)}| - \log |x_{(T-q)}| \right]^{-1}$$

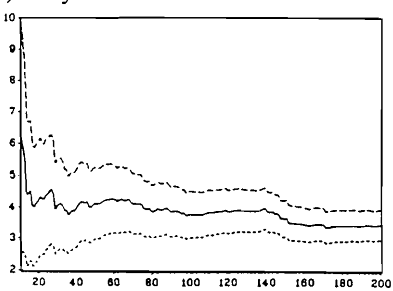
where $x_{(i)}$ denotes the i-th order statistic in descending order. It is called a conditional ML estimator because it is a function of the chosen integer q. Hall (1982) established the asymptotic normality of $\hat{\alpha}(q)$ and showed that the asymptotic standard errors of $\hat{\alpha}(q)$ under the H_0 of $\alpha = \alpha_0$ are equal to $\hat{\alpha}(q)/\sqrt{q}$. In the estimation of α, the choice of q is crucial. Following the work of Hall (1982), Phillips and Loretan (1990) suggested applying a range of values of q centred around $q^* = T^{2/3}/\log(\log T)$.

The estimator in (8) can be applied to both the lower and the upper tail of a distribution but also to both tails simultaneously. In the latter case, one has to take absolute values first before the observations are ordered according to magnitude. For symmetric distributions it is preferable to apply the 2-tailed version. In the sequel, I will only report results from this 2-tailed version because the results from analyzing the lower and upper tails separately do not differ substantially from the results of the 2-tail version[12].

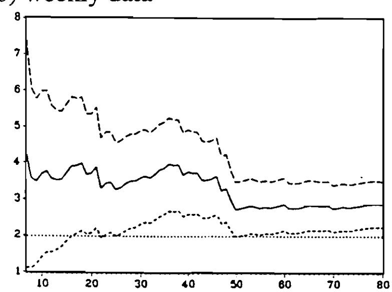
The rule that q should be centred around q^* leads to approximate values of q^* of 100, 40, 20, and 10 for the daily, weekly, monthly and quarterly data, respectively. Figure 3 plots values of $\hat{\alpha}(q)$ centred around the approximate values of q^* with the upper bound of q equal to $2q^*$. Some low values of q have been truncated because $\hat{\alpha}(q)$ is very erratic for these values of q . To save space, only the plots for the four yen series are shown.

Figure 2. Coefficient of regular variation as a function of q : yen-dollar series

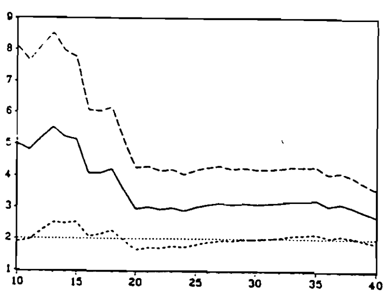
a) daily data



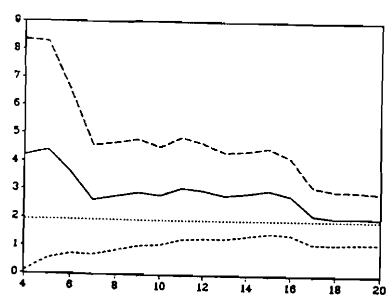
b) weekly data



c) monthly data



d) quarterly data



[12] There is only a tendency in short-run data for $\hat{a}(p)$ to be somewhat lower when it is estimated from the lower tail than when it is estimated from the upper tail or both tails.

The solid lines in Figure 3 show $\hat{\alpha}$ as a function of q and the dashed lines mark 95-percent confidence intervals. In all four plots, every single value of $\hat{\alpha}$ is above the critical line of $\alpha = 2$. The estimated $\hat{\alpha}$'s converge apparently to values significantly above 2 in short-run data. In monthly and quarterly data, however, the confidence intervals include the value of 2. This could be attributed to a decrease in power of the test under decreasing sample size or to convergence to normality under time-aggregation. The decisive result is, however, that the H_0 of $\alpha < 2$ can be firmly rejected, and this is very strong evidence against stable Paretian distributions.

Based on the 2-tail version, the estimates of $\hat{\alpha}(q^*)$ and their standard errors, are reported for all series in Table 3. For all series, $\hat{\alpha}(q^*)$ is above 2, and for all short-run data it is significantly above 2. In most cases, the exponent of regular variation lies in the interval from 3 to 5.

Table 4. Estimates of the exponent of regular variation

		mark	pound	franc	yen
day	$\hat{\alpha}$ (100)	3.81 (0.38)	3.86 (0.39)	3.50 (0.35)	3.77 (0.38)
	χ^2 (18)	26.0 *	19.6	21.6	1.24
week	$\hat{\alpha}$ (40)	3.51 (0.56)	3.35 (0.53)	4.80 (0.76)	3.76 (0.59)
	χ^2 (6)	3.6	4.0	5.2	6.0
month	$\hat{\alpha}$ (20)	3.02 (0.68)	4.34 (0.97)	5.37 (1.20)	2.96 (0.669)
	χ^2 (2)	2.8	1.2	0.0	1.6
quarter	$\hat{\alpha}$ (10)	3.46 (1.09)	4.97 (1.57)	2.16 (0.68)	2.80 (0.88)
	χ^2 (2)	0.8	2.0	0.4	1.2

Also reported in Table 3 is a test for goodness-of-fit. Hill (1975) showed that the variable $W_q = \alpha q \log(X_{(q)}/X_{(q+1)})$ has an exponential distribution with unit expectation under the assumption of regularly varying tails. This suggests to examine the appropriateness of the model with a χ^2 goodness-of-fit test for $W_1, \ldots, W_{q^*}$. The results from the χ^2 test in Table 3 show that there is no evidence against this model. The only rejection of an exponential distribution for W_q is for the daily mark series at the 10 percent significance level.

VI. Conclusions

Mandelbrot, who introduced the model of stable Paretian distributions into economics and finance, concluded about the applicability of this model: "I know of no other comparably successful prediction in economics" (Mandelbrot (1983), p.339). As regards the application to financial data, previous studies have been inconclusive because either they were not based on methods of statistical inference or they produced only indirect evidence against the model. However, this study provides conclusive evidence, in terms of statistical inference, that stable Paretian distributions can be rejected as a model. This conclusion is based on ML estimates of the characteristic exponent and on the estimates of the exponent of regular variation. This result is good news for those who have feared that traditional statistical methods and concepts in financial economics are not applicable to speculative prices because they were thought to follow distributions with infinite variance.

Appendix

In order to understand the Feuerverger-McDunnough approach, note that a density function $f(x)$ can be obtained from a characteristic function $\Phi(u)$ via the Fourier transform

$$(A1) \qquad f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\{-iux\}\Phi(u)du.$$

If the FFT algorithm is applied to evaluate the intergral in (A1) then x takes on values on the equi-spaced grid $0, \pm\Delta x, \pm 2\Delta x, \ldots, \pm N\Delta x/2$. For the auxiliary variable one gets $\Delta u = 2\pi/(N\Delta x)$ and the algorithm is applied to the sequence $1/2, \Phi(\Delta u), \ldots, \Phi((N-1)\Delta u)$. If the output sequence is multiplied by $2/(N\Delta x)$, one obtains $f(0), f(\Delta x), \ldots, f(N\Delta x/2)$ and the corresponding densities at the negative values of the Δx grid points (which do not contribute additional information under symmetric distributions). In order to apply this method, one has to choose values of N and Δx. Following the suggestions of Feuerverger and McDunnough, I set $N = 1024$ and $\Delta x = 0.05$.

The consequence of applying the discrete FFT approximation

$$(A2) \qquad f(k\Delta x) = \frac{1}{\sqrt{N}}\sum_{j=0}^{N-1} \Phi(j\Delta u)\exp\{-i2\pi jk/N\} \quad k = 0, 1, \ldots, N-1$$

of the Fourier integral is the so-called aliasing effect. Feuerverger and McDunnough found that the aliasing error stays essentially constant and can thus be determined by the difference between the estimated density and the exact density at $x = 0$ which is given in (A2). This requires to standardize the data by x_t/γ with the current estimate of γ. In order to get the densities at the actual values of x_t, I used cubic Hermite interpolation. I applied these methods and checked the calculated densities with the densities tabulated by Holt and Crow (1973) and found complete agreement. Having obtained densities, I then used a numerical gradient method with local search at suspected maxima to get the ML estimates $\hat{\alpha}$ and $\hat{\gamma}$.

References

Akgiray, V., and Booth, G.G. (1988), Mixed diffusion-jump process modeling of exchange rate movements. *Review of Economics and Statistics* 70: 631-637.

Akgiray, V., and Booth, G.G. (1988), The stable-law model of stock returns. *[FUSS-NOTE]Journal of Business and Economic Statistics* 6(1): 51-57.

Akgiray, V., Booth, G.G., and Loistl, O. (1989), Statistical models of German stock returns. *Journal of Economics* 50(1): 17-33.

Akgiray, V., and Lamoureux, C.G. (1989), Estimation of stable-law parameters: A comparative study. *Journal of Business and Economic Statistics* 7 (1): 85-93.

Balanda, K.P., and MacGillivray, H.L. (1988), Kurtosis: A critical review. *The American Statistician* 42 (2): 111-119.

Barnea, A., and Downes, D.H. (1973), A reexamination of the empirical distribution of stock price changes. *Journal of the American Statistical Association* 68(342): 348-350.

Boothe, P., and Glassman, D. (1987), The statistical distribution of exchange rates. *Journal of International Economics* 22: 297-319.

Bowman, K.O. and L.R. Shenton (1986), Moment $((\sqrt{b_1}, b_2)$ techniques. In: R.B. D'Agostino and M.A. Stephens, Goodness-of-Fit Techniques. New York: Marcel Dekker.

Calderon-Rossell, J.R., and Ben-Horim, M. (1982), The behavior of foreign exchange rates. *Journal of International Business Studies* 13: 99-111.

Cornew, R.W., Town, D.E., and Crowson, L.D. (1984), Stable distributions, futures prices, and the measurement of trading performance. *Journal of Future Markets* 4(4): 531-557.

Csörgö, S. (1984), Adaptive estimation of the parameters of stable laws. In *Limit Theorems in Probability and Statistics, Vol. 1* (R. Revesz, ed.). Amsterdam: North-Holland, pp. 305-367.

Fama, E.F. (1965), The behavior of stock-market prices. *Journal of Business* 38: 34-105.

Fama, E.F., and Roll, R. (1968), Some properties of symmetric stable distributions. *Journal of the American Statistical Association* 63: 817-836.

Fama, E.F., and Roll, R. (1971), Parameter estimates for symmetric stable distributions. *Journal of the American Statistical Association* 66: 331-338.

Feuerverger, A., and McDunnough, P. (1981), On efficient inference in symmetric stable laws and processes. In *Statistics and Related Topics* (M. Csörgö, D.A. Dawson, J.N.K. Rao, and A.K.M. Saleh, eds.). Amsterdam: North-Holland, pp. 109-122.

Fielitz, B.D., and Rozelle, J.P. (1983), Stable distributions and the mixtures of distributions hypotheses for common stock returns. *Journal of the American Statistical Association* 78: 28-36.

Friedman, D., and Vandersteel, S. (1982), Short-run fluctuations in foreign exchange rates. *Journal of International Economics* 13: 171-186.

Gaab, W. (1983), *Devisenmärkte und Wechselkurse*. Berlin: Springer.

Galambos, J. (1988), *Advanced Probability Theory*. New York: M. Dekker.

Granger, C.W.J., and Orr, D. (1972), "Infinite variance" and research strategy in time series analysis. *Journal of the American Statistical Association* 67: 275-285.

Hall, J.A., Brorsen, B.W., and Irwin, S.H. (1989), The distribution of futures prices: A test of the stable Paretian and mixture of normals hypotheses. *Journal of Financial and Quantitative Analysis* 24(1): 105-116.

Hall, P. (1982), On some simple estimates of an exponent of regular variation. *Journal of the Royal Statistical Society Series B* 44 (1): 37-42.

Hill, B.M. (1975), A simple general approach to inference about the tail of a distribution. *Annals of Statistics* 3 (5): 1163-1174.

Holt, D.R., and Crow, E.L. (1973), Tables and graphs of the stable probability density functions. *Journal of Research of the National Bureau of Standards* 77B (3&4): 143-198.

Iglewicz, B. (1983), Robust scale estimators and confidence intervals for location. In: D.C. Hoaglin, F. Mosteller, J.W. Tukey (eds.), Understanding Robust and Exploratory Data Analysis. New York: Wiley.

Jansen, D.W., and de Vries, C.G. (1991), On the frequency of large stock returns: Putting booms and busts into perspective. *Review of Economics and Statistics* 73(1): 18-24.

Kaehler, J. (1988), Statistische Modelle für den DM-Dollar Wechselkurs. In *Empirische Wirtschaftsforschung* (H. Kräger, ed.). Frankfurt: Campus.

Kaehler, J. (1989), *Statistical Properties of Exchange Rates*. Institut für Volkswirtschaftslehre und Statistik, Universität Mannheim. Discussion Paper No. 386-89.

Kaehler, J. (1990), *Price Dynamics in Foreign Exchange Markets: An Empirical Study*. Institut für Volkswirtschaftslehre und Statistik, Universität Mannheim. Discussion Paper No. 427-90.

Kendall, M., Stuart, A. (1979), *The Advanced Theory of Statistics*. Vol. 2, 4th ed., London: Griffin.

Koedijk, K.G., Schafgans, M.M.A., and de Vries, C.G. (1990), The tail index of exchange rate returns. *Journal of International Economics* 29: 93-108.

Kofman, P., and de Vries, C.G. (1990), Potato futures returns: A tail investigation. *Review of Futures Markets* 8(2): 244-258.

Kon, S.J. (1984), Models of stock returns - A comparison. *Journal of Finance* 39(1): 147-165.

Koutrouvelis, I.A. (1980), Regression-type estimation of the parameters of stable laws. *Journal of the American Statistical Association* 75: 918-928.

Lau, A.H.-L., Lau, H.-S., Wingender, J.R. (1990), The distribution of stock returns: New evidence against the stable model. *Journal of Business and Economic Statistics* 8(2): 217-223.

Leitch, R.A., and Paulson, A.S. (1975), Estimation of stable law parameters: Stock price behavior application. *Journal of the American Statistical Association* 70: 690-697.

Mandelbrot, B. (1963), The variation of certain speculative prices. *Journal of Business* 36: 394-419.

Mandelbrot, B. (1983), *The Fractal Geometry of Nature*. New York: Freeman.

McCulloch, J.H. (1986), Simple consistent estimators of stable distribution parameters. *Communications in Statistics - Simulation and Computation* 15 (4): 1109-1136.

McFarland, J.W., Pettit, R.R., and Sung, S.K. (1982), The distribution of foreign exchange price changes: Trading day effects and risk measurement. *Journal of Finance* 37: 693-715.

Mussa, M. (1984), The theory of exchange rate determination. In *Exchange Rate Theory and Policy* (J.F.O. Bilson, and R.R. Marston, eds.). Chicago: University of Chicago Press, pp. 13-58.

Officer, R.R. (1972), The distribution of stock returns. *Journal of the American Statistical Association* 67: 807-812.

Phillips, P.C.B., and Loretan, M. (1990), *Testing Covariance Stationarity under Moment Condition Failure with an Application to Common Stock Returns.* Cowles Foundation for Research in Economics, Yale University, mimeo.

Rogalski, R.J., and Vinso, J.D. (1978), Empirical properties of foreign exchange rates. *Journal of International Business Studies* 9: 69-79.

Ruppert, D. Feb. (1987), What is kurtosis? *The American Statistician* 41 (1): 1-5.

Schlittgen, R., Hammann, D., and Lepinat, K. (1982), Zur Verteilung von Wechselkursveränderungen. *Allgemeines Statistisches Archiv* 66 (2): 162-173.

So, L.C. (1987), The sub-Gaussian distribution of currency futures: Stable Paretian or nonstationary? *Review of Economics and Statistics* 69: 100-107.

So, J.C. (1987), The distribution of foreign exchange price changes: A comment. *Journal of Finance* 42 (1): 181-188.

Tucker, A.L., and Pond, L. (1988), The probability distribution of foreign exchange price changes: Tests of candidate processes. *Review of Economics and Statistics* 70: 638-647.

Westerfield, J.M. (1977), An examination of foreign exchange risk under fixed and floating rate regimes. *Journal of International Economics* 7: 181-200.