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Strategies under Strategic Uncertainty

Helene Mass

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Strategies under Strategic Uncertainty

Helene Mass

Abstract

I investigate the decision problem of a player in a game of incomplete information who faces uncertainty about the other players' strategies. I propose a new decision criterion which works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which are not rationalizable. Second, I apply the maximin expected utility criterion. Using this decision criterion, one can derive predictions about outcomes and recommendations for players facing strategic uncertainty. A bidder following this decision criterion in a first-price auction expects all other bidders to bid their highest rationalizable bid given their valuation. As a consequence, the bidder never expects to win against an equal or higher type and resorts to win against lower types with certainty.

JEL classification: C72, D81, D82, D83

Keywords: Auctions, Incomplete Information, Informational Robustness, Rationalizability

1 Introduction

I investigate the decision problem of a player in a game of incomplete information who faces strategic uncertainty. Formally, a player faces strategic uncertainty if the smallest set of strategies such that the player knows that the other players' true strategy is an element of this set, is not a singleton. I propose a new decision criterion which works in two steps: First, I assume common knowledge of rationality and eliminate all actions which are not best replies. That is, the set of the other players' possible strategies is restricted to the set of rationalizable strategies. Afterwards, I apply the maximin expected utility criterion. Using this decision criterion, I can derive recommendations for a player facing strategic uncertainty. Furthermore, I analyze outcomes under the assumption that every player in the game uses this decision criterion. In sections 2-4 I consider a game with incomplete information under strategic uncertainty with common knowledge of type distributions. In an extension in section 5 I discuss how the proposed decision criterion can be applied under the presence of both, distributional and strategic uncertainty.

Before I explain the decision criterion in more detail, I argue why strategic uncertainty can occur in games (of complete or incomplete information). Consider a game and a player who has to decide about her strategy. There may exist strategy profiles which formally fulfill the conditions of a (Bayes-) Nash equilibrium. However, a player may be uncertain whether her opponents employ such strategies and consequently face strategic uncertainty. As stated by Pearce (1984), "some Nash equilibria are intuitively unreasonable and not all reasonable strategy profiles are Nash equilibria". He argues that if players cannot communicate, then a player will best reply to Nash equilibrium strategies only if she is able to deduce these equilibrium strategies. However, a player may consider more than one strategy of the other players' as possible. For example, this can occur under the existence of multiple Nash equilibria without one being focal or salient (Bernheim (1984)). Thus, a Nash equilibrium may not be a suitable decision criterion if a player does not observe or does not deduce a unique conjecture about the other players' strategies. Similarly, Renou and Schlag (2010) argue that "common knowledge of conjectures, mutual knowledge of rationality and utilities, and existence of a common prior" are required in order to justify Nash equilibria as a decision criterion.

So far, I argued that a player may not know which strategies are played by the other players. But a player may not consider all strategies of the other players as possible. The fact that rational players interact strategically given some commonly known rules of a game (e.g. the rules of a first-price auction), already contains information about the set of possible strategies. Therefore, in the first step of the decision criterion I propose to consider strategies which a player can deduce only from common knowledge of rationality. Under strategic uncertainty a player is rational if her action is a best reply given her type, the commonly known type distribution and a conjecture about the other players' strategies. A strategy which a player assumes to be played by another rational player has to be rational as well, i.e. the action prescribed by a strategy for a given type has to be a best reply given the type, the commonly known type distribution and a conjecture about the other players' strategies. This reasoning continues ad infinitum. Pearce (1984) and Bernheim (1984) (and Battigalli and Siniscalchi (2003b) for games of incomplete information) show that common knowledge of rationality is equivalent to bidders playing *rationalizable strategies* [1]. These are strategies which survive the iterated elimination of

[1]For games of incomplete information where also the type distribution is not known, i.e. only the type spaces and action spaces are common knowledge, Battigalli and Siniscalchi (2003b) use the term *belief-free rationalizable strategies.* If additional information about possible strategies or distributions is common knowledge, i.e. more than the type spaces and the action spaces is common knowledge, they use the term Δ-*rationalizable strategies.* If the type distribution is common knowledge but nothing besides the actions spaces is known about strategies, they use the term *rationalizable strategies.* I will use the term rationalizable strategies throughout the paper.

actions which are not best replies to some strategy which consists of actions which have not been eliminated in previous elimination rounds.

In the second step I apply the maximin expected utility criterion due to Gilboa and Schmeidler (1989). A player applying this criterion chooses the action which maximizes her minimum expected utility given her type. The application of the maximin expected utility criterion can be modeled as a simultaneous zero-sum game against an adverse nature whose action space consists of the other players' rationalizable strategies. Given the strategy of the adverse nature, the player applying the maximin criterion chooses the action which maximizes her expected utility. The adverse nature's utility is the player's expected utility multiplied by -1.

In other words, under the proposed decision criterion a player facing strategic uncertainty forms a subjective belief about the other players' strategies and acts optimally given this subjective belief. The first step of the decision criterion determines the set from which a player chooses her subjective belief. The second step determines how the subjective belief is chosen. The subjective belief is given by the adverse nature's equilibrium strategy, in the following called *subjective maximin belief*. In order to distinguish the Nash equilibrium in the simultaneous game between a player and the adverse nature and the Bayes-Nash equilibrium which may exist in a given game of incomplete information, I will refer to the Nash equilibrium in the former case as a *maximin equilibrium*.

By assuming common knowledge of rationality and applying the maximin expected utility criterion, I am able to derive recommendations for players facing distributional and strategic uncertainty. Moreover, I characterize outcomes under the assumption that every player follows the proposed decision criterion.

The following two examples illustrate two different reasons for why strategic uncertainty can occur and how the proposed decision criterion applies under strategic uncertainty. In the first example there exist multiple Nash equilibria without one being salient. In the second example a salient Nash equilibrium exists but is not the unique rationalizable action. In particular, the salient Nash equilibrium is not compatible with actions derived from the maximin utility or minimax regret criterion. Afterwards, I will summarize the results for first-price auctions under strategic uncertainty and provide the results for the extension of the decision criterion to both, distributional and strategic uncertainty.

For the first example consider a sender who has to deposit a package either in places A, B or C. A receiver has to decide to which places she sends one or two drivers in order to pick up the package. If the package is picked up, sender and receiver earn each a utility of P and zero otherwise. In addition, the receiver faces a cost of c if a driver travels to place A or B and a cost of $\tilde{c}$ if a driver travels to place C. The game is summarized in the following utility table:

	A	B	C	AB	AC	BC
A	$P; P-c$	$0; -c$	$0; -\tilde{c}$	$P; P-2c$	$P; P-c-\tilde{c}$	$0; -c-\tilde{c}$
B	$0; -c$	$P; P-c$	$0; -\tilde{c}$	$P; P-2c$	$0; -c-\tilde{c}$	$P; P-c-\tilde{c}$
C	$0; -c$	$0; -c$	$P; P-\tilde{c}$	$0; -2c$	$0; P-c-\tilde{c}$	$P; P-c-\tilde{c}$

Assume it is common knowledge that it holds $P - \tilde{c} < -c$ and $P - 2c > -c$. The Nash equilibria in this game are $(A; A)$, $(B; B)$ and both players mixing between A and B with probability $\frac{1}{2}$. Although Nash equilibria exist, the players may be uncertain about each other's strategy since there does not exist a particularly salient one. The application of the maximin criterion leaves both players indifferent between actions A and B. The maximin criterion does not yield to action AB for the receiver since by choosing AB she would face the risk that the sender deposits the package in C, leaving the receiver with the costs of two drivers $-2c$. However, the result of the maximin criterion changes after assuming common knowledge of rationality. Excluding actions which are not best replies leads to the elimination of strategies C, AC and BC for the receiver, leading to the elimination of action C for the sender:

	A	B	AB
A	$P; P-c$	$0; -c$	$P; P-2c$
B	$0; -c$	$P; P-c$	$P; P-2c$

Now the maximin criterion leads to action AB for the receiver. In other words, if the receiver anticipates that the sender anticipates that she will never send a driver to C, the application of the maximin criterion leads to action AB. In this case, the receiver earns a utility of $P - 2c$ with certainty. If she would follow a Nash equilibrium strategy or apply the maximin criterion directly, she would face the risk of getting a utility of $-c$.

As a second example consider the following utility table. It illustrates the decision problem of a player who is uncertain about which of the possible rationalizable actions her opponent will choose:

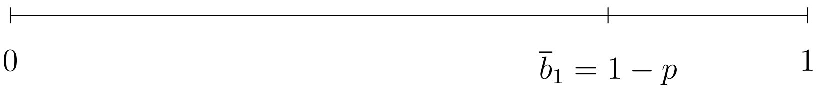
	X	Y	Z
A	10;10	0;9	0,0
B	5;1	5;9	0,0
C	4;1	4;9	4;0
D	1;10	6;9	0;0

The unique Nash equilibrium in pure strategies, (A, X), is focal in the sense that it is the social optimum and leads to the highest possible utility for both players. However, a rational column player can also choose Y instead of X. Action Y is rationalizable and moreover, the application of the maximin or the minimax regret criterion would lead to action Y for the column player. In other words, the column player may prefer to get a utility of 9 with certainty instead of aiming for the utility of 10 and risking to get a utility of 1. Given this uncertainty about the column player's strategy, the row player may resort to the application of the maximin criterion. This leads to action C which ensures a utility of 4 for the row player. However, the row player can anticipate that action Z is strictly dominated for the column player. After the elimination of this action, C becomes strictly dominated for the row player. The iterated elimination of actions which are not best replies, i.e. the elimination of actions Z and C, leads to the following utility table:

	X	Y
A	10;10	0;9
B	5;1	5;9
D	1;10	6;9

Now the application of the maximin criterion leads to action B for the row player. That is, after anticipating that the column player will never play Z, the row player can ensure a utility of 5 instead a utility of 4.

These examples show how the proposed decision criterion provides recommendations under strategic uncertainty. Moreover, they show why players may not expect their opponents to play Nash equilibria and why the application of the maximin utility criterion alone may cause forgone profits. After discussing the two examples, I provide an intuition and a summary of the results for first-price auctions where bidders' valuations are identically and independently distributed according to a commonly known distribution function. Consider the simple example of a first-price auction with two bidders who can have either a valuation of zero with probability p or a valuation of 1 with probability $1-p$. For simplicity, assume an efficient tie-breaking rule. We have to compute the highest rationalizable bids of each type. The highest rationalizable bid of a bidder with valuation zero is zero. If a bidder with valuation 1 bids zero, she gets an expected utility of p. Hence, bidding too close to the own valuation (or even above) cannot be rational for a 1-type.



The highest rationalizable bid of a bidder with valuation 1 makes her indifferent between winning against the 0-type by bidding zero and winning with probability one. That is, it is obtained by the equation[2]

$$1 - \bar{b}_1 = p \;\Leftrightarrow\; \bar{b}_1 = 1 - p.$$

A bidder with valuation 1 who applies the proposed decision criterion has the subjective maximin belief that the other bidder with valuation 1 bids $\bar{b}_1$. Therefore, her best reply is to win against the 0-type of the other bidder with certainty by bidding zero.

For the general case with an arbitrary number of bidders and valuations, for every type there exists a unique highest rationalizable bid. A bidder applying the proposed decision criterion assumes that every other bidder places the highest rationalizable bid given her type. As a consequence, the bidder never expects to win against a bidder with an equal or higher type and therefore bids the highest rationalizable bid of a lower type in order to win against the lower type with certainty. If every bidder applies this decision criterion, then every bidder has the same beliefs about distributions and strategies. Every bidder calculates which highest rationalizable bid of a lower type maximizes her expected utility. It turns out that due to the symmetry of beliefs about valuation distributions and strategies, the higher the type of the bidder, the higher is the type whose highest rationalizable bid maximizes her expected utility. Therefore, the outcome is efficient, i.e. the bidder with the highest valuation wins the auction with probability one.

In an extension I analyze both, distributional and strategic uncertainty. In this case the strategy space of the adverse nature consists of all rationalizable strategies and all possible valuation distributions. For a restriction of the set of possible distributions I assume common knowledge of an exogenously given mean μ of bidders' valuations.[3] Although in reality bidders go at great lengths in order to learn about their competitors' valuations,

[2]For the case with two possible valuations the highest rationalizable bid of a bidder with the higher valuation coincides with the highest bid played in the unique Bayes-Nash equilibrium. With more than two valuations the highest rationalizable bid of a type is strictly higher than the highest bid played in the unique Bayes-Nash equilibrium.

[3]The assumption of common knowledge of an exogenously given mean under distributional uncertainty has been used before. See for example Montiero (2009).

such learning has its limits and bidders may be able to learn only the support and the mean of the valuation distribution.

Under strategic uncertainty with common knowledge of rationality and distributional uncertainty with common knowledge of an exogenously given mean, as before, for every type there exists a unique highest rationalizable bid. A bidder applying the proposed decision criterion assumes that every other bidder places the highest rationalizable bid given her type. Let θ_μ be the lowest valuation which is higher than the mean. The highest rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of such a bidder about the other bidders' valuation distributions is that the probability weight is distributed between her own valuation and θ_μ. As a consequence, a bidder with a valuation lower than θ_μ expects a utility of zero and is indifferent between any bid between zero and her valuation. Every bidder with valuation θ such that $\theta \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the subjective maximin belief of such a bidder about the other bidders' valuation distribution maximizes the probability weight on θ and makes the bidder indifferent between any highest rationalizable bid of lower types. As a consequence, the bidder mixes among all highest rationalizable bids of lower types. Therefore, the outcome is not efficient.

The remainder of the paper is organized as follows. I conclude the introduction with an overview over the related literature. The second section contains the formal description of the proposed decision criterion. In the third section I collect sufficient conditions for actions to be rationalizable which will be useful for the derivation of subjective maximin beliefs and outcomes under maximin strategies. Moreover, I provide sufficient conditions for the existence of such outcomes. In the fourth section I apply the decision criterion to first-price auctions under strategic uncertainty. The fifth section contains the formal description of the decision criterion under distributional and strategic uncertainty and its application to first-price auctions. The appendix contains the proofs not provided in previous sections.

Relation to the literature

This paper relates to two strands of literature - the literature on decision criteria under uncertainty and robustness and the literature on rationalizability. Two widely used decision criteria under uncertainty are the maximin utility and the minimax regret criterion. The axiomatization of the maximin expected utility criterion is provided in Gilboa and Schmeidler (1989), the axiomatization of the minimax regret criterion is provided in Stoye (2011). In Bergemann and Schlag (2008) both criteria are applied to a monopoly pricing problem where a seller faces uncertainty about the buyer's valuation distribution. Since the seller knows that the buyer will obtain the good if the price is equal or lower than her valuation, the seller does not face strategic uncertainty.

The maximin expected utility criterion has been applied to first-price auctions under distributional uncertainty. Lo (1998) derives Bayes-Nash equilibrium bidding strategies in a first-price auction under the maximin expected utility criterion where it is common knowledge that the true valuation distribution is an element of a given set of distributions. Salo and Weber (1995) assume that only the set of possible valuations is common knowledge and that ambiguity averse bidders use a convex transformation of the uniform distribution as a prior. They find, that the more ambiguity averse a bidder is, the higher is the bid. Chen et al. (2007) analyze first- and second-price auctions where bidders face one of two possible distributions which can be ordered with respect to first-order stochastic dominance. Thus, an ambiguity-averse bidder would assume the stochastically dominating distribution. In their experimental findings they reject the hypothesis that bidders are ambiguity-averse. These three papers use Bayes-Nash equilibria as a solution concept, that is, the issue of strategic uncertainty is not addressed.

Bose et al. (2006) derive the optimal auction in a setting where seller and bidders may face different degrees of ambiguity, that is, they may face different sets of possible valuation distributions. Carrasco et al. (2018) consider a seller facing a single buyer. The set of distributions the seller considers to be possible is determined by a given support and mean. In these two papers strategic uncertainty is not an issue since the seller chooses a strategy proof direct mechanism.

Renou and Schlag (2010) analyze strategic uncertainty using the minimax regret criterion. Besides Kasberger and Schlag (2017), I am the only one addressing distributional *and* strategic uncertainty. They use the minimax regret criterion and allow for the possibility that a bidder can impose bounds on the other bidders' bids or valuation distributions. For example, they consider the case where a bidder can impose a lower bound on the highest bid.

In their literature on robust mechanism design Dirk Bergemann and Stephen Morris consider the problem of a social planner facing uncertainty about the players' actions. In Bergemann and Morris (2005) a social planner can circumvent uncertainty about the players' strategies by choosing ex-post implementable mechanisms. Bergemann and Morris (2013) provide predictions in games independent of the specification of the information structure. In order to do so, they characterize the set of Bayes correlated equilibria. An application of this concept to first-price auctions is carried out in Bergemann et al. (2017). In Carroll (2016) two agents accept or reject a proposed deal where the valuation for each agent depends on an unknown state. The main result provides an upper bound of welfare loss among all information structures.

The concept of rationalizable strategies has been first introduced by Bernheim (1984) and Pearce (1984) for games with complete information. Battigalli and Siniscalchi (2003b) extend rationalizability to games of incomplete information. An application to first-price auctions has been carried out by Dekel and Wolinsky (2001). They apply rationalizable strategies to a first-price auction with discrete private valuations and discrete bids. They present a condition on the distribution of types under which the only rationalizable action is to bid the highest bid below valuation. Battigalli and Siniscalchi (2003a) assume that valuation distributions in a first-price auction are common knowledge but not the strategies of the bidders. They characterize the set of rationalizable actions under the assumption of *strategic sophistication*, which implies common knowledge of rationality and of the fact that bidders with positive bids win with positive probability. They find that for a bidder with a given valuation θ all bids in an interval $(0, b^{max}(\theta))$ are rationalizable where $b^{max}(\theta)$ is higher than the Bayes-Nash equilibrium bid. Using this result, one can immediately tell that under common knowledge of rationality a bidder applying the maximin expected utility criterion has the subjective maximin belief that every other bidder with valuation θ bids $b^{max}(\theta)$. I replicate this result in section 4 for first-price auctions with discrete valuations.

To the best of my knowledge I am the first one applying the maximin expected utility criterion to strategic uncertainty and the first one combining rationalizable strategies with a decision criterion under uncertainty.

2 Model

Underlying game of incomplete information The starting point of the model is a game of incomplete information which is denoted by $\left(\{1,\ldots,I\},\Theta,A,\{u_i\}_{i\in\{1,\ldots,I\}}\right)$ where $\{1,\ldots,I\}$ is the set of players and for every $i\in\{1,\ldots,I\}$, $A_i\subseteq\mathbb{R}$ is the set of possible actions and $\Theta_i\subseteq\mathbb{R}$ is the set of possible privately known types of player i. A and Θ are defined by $A=A_1\times\ldots\times A_I$ and $\Theta=\Theta_1\times\ldots\times\Theta_I$. A *pure strategy* of player i is a mapping

$$\beta_i:\Theta_i\to A_i$$

$$\theta_i\mapsto a_i.$$

The set S_i is the set of all pure strategies of player i. A *strategy* of player i is a mapping

$$\beta_i:\Theta_i\to\Delta A_i$$

$$\theta_i\mapsto a_i$$

where ΔA_i is the set of probability distributions on A_i. In the following $g_{\theta_i}^{\beta_i}$ will denote the density of the bid distribution $\beta_i(\theta_i)$ and $supp(\beta_i(\theta_i))$ its support.[4] Let

$$u_i : A \times \Theta_i \to \mathbb{R}$$

$$(a_1, \ldots, a_I, \theta_i) \mapsto u_i(a_1, \ldots, a_I, \theta_i)$$

denote the utility function of player i. That is, I consider a setting with private valuations.

For a given profile of strategies $(\beta_1, \ldots, \beta_n)$ and a given type distribution

$$F : \Theta \to [0,1]$$

the expected utility of a player i is given by

$$U_i(\theta_i, \beta_i(\theta_i), \beta_{-i}, F_{-i}) = \int_{\theta_{-i}} \int_{a_{-i}} u_i(a_1, \ldots, a_i, \ldots, a_I, \theta_i) \prod_{j \neq i} g_{\theta^j}^{\beta_j}(a_j)\, d\theta_{-j} dF_{-i}(\theta_{-i})\, d\theta_{-i} \quad (1)$$

where the function u_i stems from the underlying game of incomplete information and where F_{-i} is defined by $F_{-i}(\theta_{-i}) = F(\theta_{-i}, \theta_i)$.[5]

Action space of adverse nature In order to formalize the maximin expected utility criterion, a new player, denoted by n, is introduced, representing the adverse nature a player i applying the maximin expected utility criterion faces. Players i and n play a simultaneous zero-sum game where utilities are induced by the underlying game of incomplete information. The first step of a formal description of this game is the definition of the adverse nature's action space. It accounts for the residual uncertainty of player i. In sections 2-4 I study only strategic uncertainty and assume common knowledge of a type distribution given by

$$F : \Theta \to [0,1].$$

That is, the adverse nature's action space is the set of all other players' strategies which player i considers to be possible which is the set of rationalizable strategies.

[4]A pure strategy can be interpreted as distribution of bids which puts probability weight 1 on one bid. I abuse notation since in the case of a pure strategy, $\beta_i(\theta_i)$ denotes an element in A_i while in the case of a (mixed) strategy $\beta_i(\theta_i)$ denotes an element in ΔA_i. However, in the following it will be clear whether β_i is a pure or a mixed strategy.

[5]For a vector $(v_1, \ldots, v_I)$ I denote by v_{-i} the vector $(v_1, \ldots, v_{i-1}, v_{i+1}, \ldots, v_I)$.

Rationalizable strategies As argued in the introduction, in many economic settings players may face uncertainty about the other players' strategies. Even if a (Bayes-) Nash equilibrium exists, a player may consider also other strategies of her opponents to be possible. For example, multiple Nash equilibria can exist or the Nash equilibrium strategies are not aligned with preferences the other players may have, e.g. maximin or minimax regret preferences. In order to determine the set of strategies a player can expect from rational opponents, I assume common knowledge of rationality. That is, it is common knowledge that every player i maximizes her expected utility given her type, the commonly known type distribution F and a conjecture about the other players' strategies.

The assumption of common knowledge of rationality leads to the following reasoning. Every player i maximizes her expected utility given her type, the type distribution F and a conjecture about the other players' strategies. The strategy which player i assumes is played by some player $j \neq i$ has also to be compatible with common knowledge of rationality. Therefore, for every possible type of player j, the action prescribed by the strategy assumed by player i maximizes j's expected utility given her type, the type distribution F and a conjecture about the other players' strategies. Again, player j's conjecture has to be compatible with common knowledge of rationality. This reasoning continues ad infinitum. [6]

Given the type of a player, an action which is compatible with common knowledge of rationality is called *rationalizable*. Battigalli and Siniscalchi (2003b) have shown that it is equivalent to define a rationalizable action as follows.

Definition 1.

(i) Let $i \in \{1, \ldots, I\}$ be a player and $\theta_i \in \Theta_i$ be a type of player i. The set of rationalizable actions *for player i is defined as follows. Set $RS_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $RS_i^k(\theta_i)$ is already defined. Then the set $RS_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there exists a strategy profile β_{-i} of the other players such that it holds*

(i) $a_j \in supp(\beta_j(\theta_j))$ *for* $\theta_j \in \Theta_j \;\Rightarrow\; a_j \in RS_j^k(\theta_j)$ *for all* $j \neq i$

(ii) $a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i(\theta_i, a_i', \beta_{-i}, F_{-i})$

[6] As stated above, under strategic uncertainty a rational player acts optimally given a conjecture about the other players' strategies (and a conjecture about the other players' type distributions if also distributional uncertainty is present). Instead of "conjecture" other terms have been used in economic literature, e.g. belief, subjective prior, assumption, assessment ect. I use the term conjecture as proposed in Bernheim (1984).

and $RS_i(\theta_i)$ is given by

$$RS_i(\theta_i) = \bigcap_{k\geq 1} RS_i^k(\theta_i).$$

(ii) A strategy β_i of a player i is rationalizable *if for every $\theta_i \in \Theta_i$ every action a_i with $a_i \in supp(\beta_i(\theta_i))$ is rationalizable, i.e. an element of $RS_i(\theta_i)$.*

(iii) For a player i let RS_{-i} be the set of rationalizable strategies of the other $I-1$ players.

The intuition behind this definition is that an action for a player which is consistent with common knowledge of rationality, i.e. a rationalizable action, is an action which survives the iterated elimination of actions which are not best replies. An action is a best reply if it maximizes the player's expected utility given her type, the commonly known type distribution F and a conjecture about the other players' strategies which prescribe actions that have not been eliminated yet.

The definition of rationalizable strategies allows for a formal definition of the adverse nature's action space and therefore for a formal definition of the simultaneous game against the adverse nature.

Simultaneous game against adverse nature The following definition summarizes all components describing a game under strategic uncertainty.

Definition 2. *A* game under strategic uncertainty *consists of an underlying game of incomplete information, denoted by $(\{1,\ldots,I\},\Theta,A,\{u_i\}_i \in \{1,\ldots,I\})$, a subset of players $\{i_1,\ldots i_k\} \subseteq \{1,\ldots,I\}$ applying the maximin expected utility criterion, and a player n. For every $i \in \{i_1,\ldots,i_k\}$ player i chooses a strategy*

$$\beta_i : \Theta_i \to \Delta A_i.$$

A strategy of n is a mapping which for every player $i \in \{i_1,\ldots i_k\}$ and for every possible type of player i assigns a strategy of the other players:

$$\beta_n = (\beta^{n_{i_1}},\ldots,\beta^{n_{i_k}}) : \Theta_{i_1} \times \ldots \times \Theta_{i_k} \to RS_{-i_1} \times \cdots \times RS_{-i_k}.$$

$$(\theta_{i_1},\ldots,\theta_{i_k}) \to \left(\beta_{-i_1}^{n_{i_1},\theta_{i_1}},\ldots,\beta_{-i_k}^{n_{i_k},\theta_{i_k}}\right).$$

Here the superscript n_{i_j},θ_{i_j} for $j \in \{1,\ldots,k\}$ indicates that the other players' strategies $\beta_{-i_j}^{n_{i_j},\theta_{i_j}}$ are chosen by the adverse nature faced by player i_j and depend on the player's type θ_{i_j}. The utility of a player $i \in \{i_1,\ldots,i_k\}$ is given by

$$U_i\left(\theta_i,\beta_i(\theta_i),\beta_{-i}^{n_i,\theta_i},F_{-i}\right)$$

which is defined as in (1) and depends on the utility function of player i in the underlying game of incomplete information, denoted by u_i:

$$u_i : A \times \Theta_i \to \mathbb{R}.$$

$$(a_1, \ldots, a_I, \theta_i) \mapsto u_i (a_1, \ldots, a_I, \theta_i).$$

The utility of player nature is given by

$$-\sum_{j=1}^{k} U_i \left(\theta_i, \beta_i (\theta_i), \beta_{-i}^{n_i,\theta_i}, F_{-i}\right).$$

Throughout the remainder of the paper it will be assumed that a game under strategic uncertainty is given without explicitly stating all its ingredients.

Since the other players' strategies the adverse nature chooses for a player $i \in \{i_1, \ldots, i_k\}$, are not observed by a player $j \neq i$, $j \in \{i_1, \ldots, i_k\}$, the adverse nature faces an independent minimization problem for every player applying the expected maximin utility criterion.[7]

Note that after specifying the subset of players who apply the maximin expected utility criterion, a given game of incomplete information uniquely defines a game under strategic uncertainty.

Now it is possible to define a maximin strategy in a game under strategic uncertainty which can be seen as a recommendation for a player facing strategic uncertainty.

Definition 3. *In a game under strategic uncertainty for a player i a strategy*

$$\beta_i : \Theta_i \to \Delta A_i$$

is a maximin strategy *if there exists a Nash equilibrium in the simultaneous game between nature and player i such that β_i is player i's equilibrium strategy.*

The Nash equilibrium in the simultaneous game between nature and player i is called maximin equilibrium.

As described above, such a maximin strategy has two properties. First, if a player would not choose an action according to a maximin strategy, then there would exist a rationalizable strategy of the other players under which the player's expected utility is lower than under the action prescribed by a maximin strategy. Second, the strategy chosen by nature can be interpreted as the player's subjective belief about the state of the world

[7]Equivalently, one could introduce an additional adverse nature for every player applying the minimax expected utility criterion.

against which she maximizes her expected utility given her type. The second property is formalized in the following definition.

Definition 4. *In a game under strategic uncertainty let β^{n_i} be the adverse nature's maximin equilibrium strategy projected on the $i'th$ component. A* subjective maximin belief *of player i with valuation θ_i is defined as*

$$\beta^{n_i}(\theta_i) = \beta_{-i}^{n_i,\theta_i},$$

that is, the adverse nature's maximin equilibrium strategy evaluated at θ_i.

Note that the subjective maximin belief of player is not necessarily unique. However, every best reply of a player i to any subjective maximin belief induces the same expected utility for player i.

3 Outcomes under strategic uncertainty

So far, I have characterized the set of strategies of a player which are obtained if this particular player applies the maximin expected utility criterion. In addition to the derivation of maximin strategies for particular players, one can analyze what happens if *all* players adopt maximin strategies. Since under strategic uncertainty players do not observe each other's strategies, I do not use the term *equilibrium*, but the term *outcome.*

Definition 5. *In a game under strategic uncertainty an* outcome under maximin strategies *is a strategy profile $(\beta_1, \ldots, \beta_I, \beta_n)$ such that for every $i \in \{1, \ldots, I\}$ it holds that player i's strategy is a maximin strategy given the adverse nature's strategy β_n.*

In other words, for every player $i \in \{1, \ldots, I\}$ it holds that $(\beta_1, \ldots, \beta_I, \beta_n)$ constitutes a maximin equilibrium in the simultaneous game between *all* players and the adverse nature. The following Proposition follows from the definition of rationalizable strategies and of an outcome under maximin strategies.

Proposition 1. *In a game under strategic uncertainty let $(\beta_1, \ldots, \beta_I, \beta_n)$ be an outcome under maximin strategies. Then for every $i \in \{1, \ldots, I\}$ it holds that β_i is a rationalizable strategy for player i.*

One can prove this Proposition by showing per induction that for every $k \in \mathbb{N}$, for every $\theta_i \in \Theta_i$ and for every $a_i \in supp\,(\beta_i(\theta_i))$ it holds that a_i is an element in $RS_i^k(\theta_i)$. The formal proof is relegated to Appendix A.

The following conclusions can be derived from this proposition. First, this proposition shows that the maximin expected utility criterion is consistent with common knowledge of rationality. That is, every action resulting from the application of the maximin utility criterion is rationalizable. Second, it provides a sufficient condition for a strategy to be rationalizable which will be useful in subsequent proofs. Third, the same proof as for Proposition 1 can be used in order to show that an action which is a best reply to a rationalizable strategy is again rationalizable. The last statement is formalized in the following Corollary.

Corollary 1. *In a game under strategic uncertainty let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for every $j \in \{1, \ldots, I\} \backslash \{i\}$ let β_j be a rationalizable strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i}, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right),$$

then $a_i \in RS_i\left(\theta_i\right)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

Another sufficient condition for an action to be rationalizable is that it is played in a Bayes-Nash equilibrium. It follows from Corollary 1 that a best reply to strategies played in a Bayes-Nash equilibrium is rationalizable. This constitutes another sufficient condition for an action to be rationalizable. These two conditions are formalized in the following definition and proposition.

Definition 6. *In a game of incomplete information a strategy profile $(\beta_1, \ldots \beta_I)$ together with a profile of type distributions $\left(\hat{F}_1, \ldots, \hat{F}_I\right)$ is a* Bayes-Nash equilibrium with a common prior *if for every $i \in \{1, \ldots, I\}$, every $\theta_i \in \Theta_i$ and every $a_i \in A_i$ such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$ it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, \hat{F}_{-i}\right).$$

That is, every player maximizes her expected utility given the other players' strategies and the other players' commonly known type distributions.

Proposition 2. *Let the profile of strategies $(\beta_1, \ldots, \beta_I)$ together with the profile of type distributions $\left(\hat{F}_1, \ldots, \hat{F}_I\right)$ constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Then the following holds true:*

(i) For every $i \in \{1, \ldots, I\}$ the strategy β_i is rationalizable.

(ii) Let $i \in \{1, \ldots I\}$ be a player with valuation θ_i and let $a_i \in A_i$ be a best reply to β_{-i}

and some distribution of the other players' types $F_{-i}{}' \in \Delta_{\Theta_{-i}}$, i.e. it holds that

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i \left(\theta_i, a_i', \beta_{-i}, F_{-i}'\right),$$

then $a_i \in RS_i(\theta_i)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

The formal proof is relegated to Appendix B.

As mentioned above, for every player applying the maximin expected utility criterion, the adverse nature faces an independent optimization problem. Thus, if $\{i_1, \ldots, i_k\}$ is the set of players applying the maximin expected utility criterion, then the game against an adverse nature can be seen as k independent two-player zero-sum games. This allows for the application of all results for two-player zero-sum games including the existence result for Nash equilibria.

4 First-Price Auctions under Strategic uncertainty

In this section I apply the proposed decision criterion to first-price auctions. The first subsection specifies the model for first-price auctions. The second subsection gives a rather informal preview of the results. The third subsection provides a list of the necessary notation and definitions. The fourth and fifth subsection contain a detailed description and derivation of the results for first price auctions under strategic uncertainty with common knowledge of valuations and common knowledge of the valuation distribution, respectively.

4.1 Model

Underlying game of incomplete information As in the general model, the model description starts with the specification of the underlying game of incomplete information. There are I risk-neutral bidders competing in a first-price sealed-bid auction for one indivisible good. Before the auction starts, every bidder $i \in \{1, ..., I\}$ privately observes her valuation (type) $\theta_i \in \Theta = \{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\}$. A *pure strategy* of bidder i is a mapping

$$\beta_i : \Theta \to \mathscr{B}$$

$$\theta_i \mapsto \beta_i(\theta_i)$$

where $\mathscr{B}$ is a finite (arbitrarily fine) grid of bids on an interval $[0, B]$ with $\Theta \subseteq \mathscr{B}$.[8] A *strategy* of a bidder i is a mapping

$$\beta_i : \Theta \to \Delta\mathscr{B}$$

$$\theta_i \mapsto \beta_i\left(\theta_i\right)$$

where $\Delta\mathscr{B}$ is the set of bid distributions on $\mathscr{B}$. For every $b \in \mathscr{B}$ with $b > 0$ there exists a predecessor in $\mathscr{B}$ denoted by

$$b^{-} = \max_{b' \in \mathscr{B}} b' < b$$

and for every $b \in \mathscr{B}$ with $b < B$ there exists a successor in $\mathscr{B}$ denoted by

$$b^{+} = \min_{b' \in \mathscr{B}} b' > b.$$

In a first-price auction the bidders submit bids, the bidder with the highest bid wins the object and pays her bid. In addition, it holds an efficient tie-breaking rule.[9] Thus, the utility of bidder i with valuation θ_i and bid b_i given that the other bids are b_{-i} is denoted by

$$u_i\left(\theta_i, b_i, b_{-i}\right) = \begin{cases} \theta_i - b_i & \text{if } b_i > \max\limits_{j \neq i} b_j \\ \theta_i - b_i & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i > \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i < \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ \frac{1}{k}\left(\theta_i - b_i\right) & \text{if } b_i = \max\limits_{j \neq i} b_j \text{ and } \theta_i = \max\limits_{j \neq i}\{\theta_j \mid b_j = b_i\} \\ 0 & \text{if } b_i < \max\limits_{j \neq i} b_j \end{cases}$$

where θ_j denotes the valuation of bidder j with bid b_j for $j \in \{1, ..., n\}$ and $k = \#\{\max\{\theta_j | b_j = b_i\}\}$.

The bidders' valuations are identically and independently distributed according to a distribution function

$$F : \Theta \to [0, 1].$$

[8] A finite grid is used for the set of all possible bids instead of the interval $[0, B]$ because of the following reason. Consider two bidders 1 and 2 with the same valuation θ. If bidder 1 bids some amount $b < \theta$, one has to identify the smallest bid which is strictly higher than b since this would be the unique best reply of bidder 2. This allows a more formal analysis than using expressions like "bidding an arbitrarily small amount more than b". The grid is assumed to be finite in order to ensure that any subset of the bid grid is compact. Since the grid can be arbitrarily fine, I assume for simplicity that $\Theta \subseteq \mathscr{B}$.

[9] The core statements in the results do not depend on the choice of the tie-breaking rule, i.e. under a random tie-breaking rule for every bidder and every valuation the bid prescribed by the maximin strategy would change by at most one step on the bid grid.

It is assumed that all components of the underlying game of incomplete information as well as rationality are common knowledge among all bidders.

As mentioned above, the above defined game of incomplete information uniquely defines a game under strategic uncertainty (after specifying the players applying the maximin expected utility criterion). In the following I will call this game *first-price auction under strategic uncertainty*. Moreover, I also consider the case where the valuations of the bidders are common knowledge. In this case I use the term *first-price auction under strategic uncertainty and common knowledge of valuations*.

4.2 Preview of results

Common knowledge of valuations If in a first-price auction under strategic uncertainty and common knowledge of valuations there exists a unique bidder with the highest valuation, this bidder's maximin strategy is to bid the second-highest valuation and every other bidder's maximin strategy prescribes to be indifferent between any bid between zero and her valuation. If at least two bidders have the highest valuation, then every bidder's maximin strategy prescribes to be indifferent between zero and her valuation.

Common knowledge of the valuation distribution In a first-price auction under strategic uncertainty the bidders' strategies are equal in every outcome under maximin strategies and every outcome is efficient.

For every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest rationalizable bid given her type. As a consequence, a bidder applying the maximin expected utility criterion never expects to win against a bidder with an equal or higher type. The bidder calculates which highest rationalizable bid of a lower type maximizes her expected utility. It turns out that due to the symmetry of beliefs about distributions and strategies, the higher the type of the bidder, the higher is the type whose highest rationalizable bid maximizes her expected utility. Therefore, every outcome under maximin strategies is efficient.

4.3 Notation and definitions

For the formal analysis it is useful to have an overview over the notation and definitions which will be used in the remainder of this paper.

- For $\theta^k \in \Theta$ let $\overline{b}^{\theta^k}$ be the highest rationalizable bid of a bidder with valuation θ^k.

- For $\theta^k, \theta^l \in \Theta$, $f_{j,\theta^l}^{i,\theta^k}$ denotes the probability with which type θ^l of bidder j occurs in the subjective maximin belief of a bidder i with valuation θ^k.
- If $f_{j,\theta^l}^{i,\theta^k}$ does not depend on the identities of bidder i and j, I use the notation $f_{\theta^l}^{\theta^k}$.

Definition 7. *An* auction mechanism *is a double (x, p) of an allocation function x and a payment function p. The allocation function*

$$x : (\mathscr{B})^I \to [0,1]^I$$

$$x : (b_1, \ldots, b_I) \to (x_1, \ldots, x_I) \quad \textit{with } x_i \in [0,1], \ \sum x_i \leq 1$$

determines for each participant the probability of receiving the item and the payment function

$$p : (\mathscr{B})^I \to \left(\mathbb{R}^+\right)^I$$

$$p : (b_1, \ldots, b_I) \to (p_1, \ldots, p_I) \quad \textit{with } p_i \in \mathbb{R}^+$$

determines each participant's payment.

Definition 8. *In a first-price auction a bidder i with valuation θ_i and strategy β_i* overbids *a bidder j with valuation θ_j and strategy β_j if for every b, b' such that $b \in supp\,(\beta_i(\theta_i))$ and $b' \in supp(\beta_j\,(\theta_j))$ it holds that $b \geq b'$ if $\theta_i > \theta_j$ and $b > b'$ if $\theta_i \leq \theta_j$.*

Note that due to the efficient tie-breaking rule, a bidder who overbids every other bidder wins with probability 1 in any auction mechanism where the highest bid wins.

In order to evaluate outcomes in terms of social surplus, I introduce the following definition.

Definition 9. *Let $(\beta_1, \ldots, \beta_I, \beta_n)$ be an outcome under maximin strategies of an auction mechanism. The outcome $(\beta_1, \ldots, \beta_I, \beta_n)$ is* efficient *if for all bid vectors $(b_1, \ldots, b_I)$, such that for every $i \in \{1, \ldots I\}$ there exists a valuation θ_i with $b_i \in supp(\beta_i(\theta_i))$, it holds that*

$$x_i\,(b_1, \ldots, b_I) > 0 \ \Rightarrow \ \theta_i = \max_{j \neq i} \theta_j.$$

That is, the good is allocated with probability one to a group of bidders who have the highest valuation.

4.4 Common knowledge of valuations

Proposition 3. *Consider a first-price auction under strategic uncertainty and common knowledge of valuations. Then there exists an outcome under maximin strategies and the following holds true:*

(i) If $\theta_k > \max_{j \neq i} \theta_j$, i.e. there exists a unique bidder k with the highest valuation, then bidder k bids $\theta_{k'} = \max_{\theta_j \in \Theta \setminus \{\theta_k\}} \theta_j$, i.e. the bidder with the highest valuation bids the second-highest valuation and every bidder $i \neq k$ is indifferent between any bid between zero and her valuation.

(ii) If it holds that $\theta_k = \theta_l = \max_{j \in \{1,\dots,I\}} \theta_j$, i.e. there exist at least two bidders k and l with the highest valuation, then every bidder is indifferent between any bid between zero and her valuation.

The formal proof is relegated to Appendix C.

The intuition behind part (i) is that one can show that the second-highest valuation $\theta_{k'}$ is the highest rationalizable bid of bidder k with the highest valuation θ_k. If the adverse nature chooses for all other bidders the subjective maximin belief that bidder k bids $\theta_{k'}$, this induces a utility of zero for any other bidder. Hence, any strategy of the adverse nature has to induce an expected utility of at most zero for all bidders besides k. That is, the subjective maximin belief of a bidder $i \neq k$ with valuation θ_i is that at least one other bidder bids an amount which is equal or greater than θ_i. As a consequence, all bidders are indifferent between zero and their valuation. The adverse nature chooses the subjective maximin belief for bidder k such that the bidder with the second-highest valuation bids her valuation $\theta_{k'}$. Hence, it is a best reply for bidder k to bid $\theta_{k'}$. Similar arguments apply to part (ii).

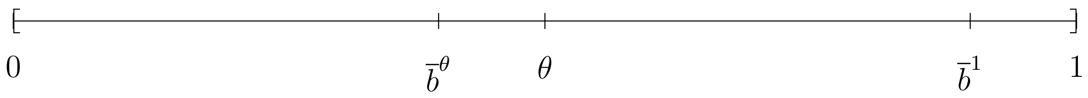
This Proposition describes the bidding behavior in a first-price auction under strategic uncertainty under the assumption that every bidder applies the proposed decision criterion. Since the bidders do not best reply to each other's strategies, the Proposition also provides the maximin strategy for every bidder. That is, the strategy prescribed in the outcome also serves as a recommendation for a bidder facing strategic uncertainty in a first-price auction.

Note that while the unique Nash equilibrium in this setting is rationalizable, there are much more rationalizable actions than played in the Nash equilibrium. In particular, in the case of two bidders who have the same valuation v all actions in the interval $[0, v]$ are rationalizable. This leaves room for more outcomes than the unique Nash-equilibrium which is weakly dominated.

4.5 Common knowledge of the valuation distribution

Now I consider the case where not the bidders' valuations but the distribution of the valuations is common knowledge. In this case in an outcome under maximin strategies for every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every other bidder will bid the highest rationalizable bid given her type. As a consequence, it is never a best reply for a bidder to overbid bidders with the same type. Hence, every bidder overbids only lower types and it depends on the commonly known valuation distribution which types are overbid. Since the strategy chosen by the adverse nature is the same for every bidder and every type, this results in an efficient outcome. This is illustrated by the following example.

Example 1. *Consider a first-price auction under strategic uncertainty with two bidders 1 and 2 and three possible valuations* $0, \theta$ *and* 1 *which are identically and independently distributed according to a commonly known distribution function* $F \in \Delta\{0, \theta, 1\}$*. For every type* $\theta^k \in \{0, \theta, 1\}$ *there exists a highest rationalizable bid* $\overline{b}^{\theta^k}$*. For every bidder and every type the adverse nature chooses a strategy of the other bidder to bid the highest rationalizable bid given her type. That is, every bidder with every type has the subjective maximin belief that the 0-type bids zero, the* θ*-type bids* $\overline{b}^{\theta}$ *and the 1-type bids* $\overline{b}^{1}$*.*



Hence, bidder 1 with type θ *never expects to win against bidder 2 with type* θ *and therefore bids 0. Bidder 1 with type 1 never expects to win against bidder 2 with type 1 and has to decide between bidding* 0 *and bidding* $\overline{b}^{\theta}$*. Since the same reasoning holds for bidder 2, in any case the outcome is efficient.*

The insights from this example are formalized in the following Proposition.

Proposition 4. *In a first-price auction under strategic uncertainty there exists an outcome under maximin strategies. Every outcome is efficient.*

The formal proof is relegated to Appendix D.

I will show the existence of an efficient outcome under maximin strategies by construction. Then I will show that every strategy of the adverse nature in an outcome under maximin strategies induces the same bidding strategies as in the constructed outcome and therefore every outcome has to be efficient. The proof by construction has the advantage that it determines the maximin strategies for every bidder, i.e. it provides a recommendation for a bidder facing strategic uncertainty.

For the construction of the efficient outcome it is crucial to calculate the highest rationalizable bid for every type. The following three steps serve as a preparation for this calculation.

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}^{\theta^k}$.

(II) Show that for every type zero is a rationalizable bid.

(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable.

The first step follows from the fact that $\mathscr{B}$ is compact and well-ordered with respect to $\leq$. For a proof sketch of step (II) consider a proof by induction with respect to the valuations in Θ. Assume it has been shown that for every bidder with valuation θ^j such that $j < k+1$ bidding zero is a rationalizable action. Assume that a bidder with valuation θ^{k+1} conjectures that all lower types bid zero. Due to step (I), for every type there exists a highest rationalizable bid. Assume further, that the bidder with valuation θ^{k+1} conjectures that all equal or higher types bid their highest rationalizable bid, then it is a best reply of this bidder to bid zero. As stated in Corollary 1, a best reply to a rationalizable strategy profile is rationalizable and therefore zero is a rationalizable action for a bidder with valuation θ^{k+1}.

For an intuition of step (III) consider the bid 0^+. Since bidding zero is a rationalizable action for every bidder and every type, it is straight-forward that for a sufficiently fine bid grid bidding 0^+ is a rationalizable action for every bidder and every type besides zero. Because if a bidder conjectures that all bidders bid zero, than she could win the auction with probability 1 by bidding 0^+. The same holds for $(0^+)^+$ and so on. This process reaches some bid b such that for type θ^2 it is more profitable to bid zero and win against the zero-type than to bid b^+ even if all other bidders with a type higher than zero bid b. Then b is the highest rationalizable bid for type θ^2 and all bids in the interval $[0, b]$ are rationalizable for a bidder with valuation θ^2. The analogous reasoning applies to every higher type. Since the bids in $\mathscr{B}$ are well-ordered with respect to $\leq$, one can show the result by double induction with respect to the types and the bids.

Given these steps, one can calculate the highest rationalizable bid for every type. The highest rationalizable bid $\overline{b}^{\theta^k}$ for a bidder i with valuation θ^k is induced by the belief about the other bidders' strategies such that

(i) All bidders with a lower type bid their highest rationalizable bid.

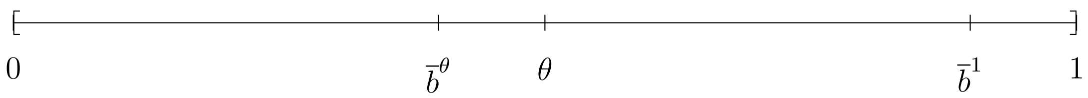
(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

That is, $\overline{b}^{\theta^k}$ is a best reply to the belief which maximizes the expected utility of bidding $\overline{b}^{\theta^k}$. The strategies in (i) are rationalizable by definition and it follows from step (III) that the strategies specified in (ii) are rationalizable. Hence, the highest rationalizable bid $\overline{b}^{\theta^k}$ of type θ^k makes this type indifferent between winning with probability 1 by bidding $\overline{b}^{\theta^k}$ and the most profitable overbidding of a lower type given that all lower types bid their highest rationalizable bid.

In an outcome under maximin strategies for every bidder and every type the adverse nature chooses as the strategy of the other bidders that every other bidder places the highest rationalizable bid given her type. As a consequence, a bidder never expects to win against an equal or higher type. The maximin strategy of a bidder is determined by the most profitable overbidding of a lower type given that every lower type places her highest rationalizable bid.

The following example continues with Example 1 and illustrates the calculation of the highest rationalizable bids and the maximin strategies.

Example 2. *Consider again the case with two bidders 1 and 2 and three possible valuations 0,θ and 1 which are identically and independently distributed according to a commonly known distribution function $F \in \Delta\{0, \theta, 1\}$.*



The highest rationalizable bid for type zero is zero. The highest rationalizable bid for type θ is given by the bid $\overline{b}^{\theta}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{\theta}$ and just overbidding type zero:

$$\theta - \overline{b}^{\theta} = F(0)(\theta - 0)$$

$$\Leftrightarrow \overline{b}^{\theta} = \theta(1 - F(0)) + F(0).$$

The highest rationalizable bid for type 1 is given by the bid $\overline{b}^{1}$ which makes her indifferent between winning with probability 1 by bidding $\overline{b}^{1}$ and the most profitable overbidding of a lower type. That is, type 1 has to be indifferent between bidding $\overline{b}^{1}$ and the maximum utility of bidding either $0 = \overline{b}^{0}$ or $\overline{b}^{\theta}$:

$$1 - \overline{b}^{1} = \max\left\{F(0)(1-0), F(\theta)\left(1 - \overline{b}^{\theta}\right)\right\}.$$

For a numerical example consider the parameters $\theta = \frac{1}{2}$, $F(0) = \frac{1}{3}$, $F(\theta) = \frac{2}{3}$ *and* $F(1) = 1$. *Then it holds that*

$$\overline{b}^{\theta} = \frac{1}{2}\left(1 - \frac{1}{3}\right) = \frac{1}{3}$$

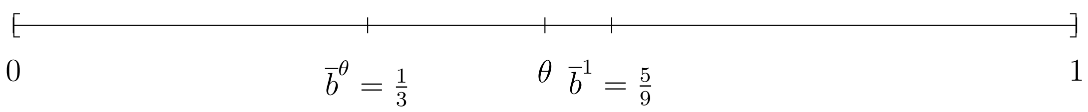
and

$$\max\left\{F(0), F(\theta)\left(1 - \overline{b}^{\theta}\right)\right\} = \max\left\{\frac{1}{3}, \frac{2}{3}\left(1 - \frac{1}{3}\right)\right\} = \frac{4}{9} = F(\theta)\left(1 - \overline{b}^{\theta}\right)$$

from which follows that

$$\overline{b}^{1} = 1 - \frac{4}{9} = \frac{5}{9}$$

which is illustrated below:



After computing the highest rationalizable bids, one can compute bidding strategies in an outcome under maximin strategies. Type zero bids zero. Note that the highest rationalizable bid of type θ is determined by the case that type θ wins with probability 1. Since type θ of bidder 1 has the subjective maximin belief that type θ of bidder 2 bids $\overline{b}^{\theta}$, bidder 1 would win only with probability $\frac{1}{2}$ by bidding $\overline{b}^{\theta}$. Hence, given the subjective maximin of bidder 1 with type θ, it is not a best reply to bid equal or higher than $\overline{b}^{\theta}$. Therefore, she does not expect to win against type θ of bidder 2 and bids zero. Similarly, type 1 of bidder 1 does not expect to win against type 1 of bidder 2 and has to decide whether to overbid type 0 or type θ of bidder 2, i.e. whether to bid 0 or $\overline{b}^{\theta}$. In any case the outcome is efficient.[10] *Bidding zero gives an expected utility of*

$$F(0) = \frac{1}{3}$$

and bidding $\overline{b}^{\theta} = \frac{1}{3}$ gives an expected utility of

$$F(\theta)\left(1 - \overline{b}^{\theta}\right) = \frac{2}{3}\left(1 - \frac{1}{3}\right) = \frac{4}{9}.$$

Hence, type 1 of bidder 1 will bid $\overline{b}^{\theta}$ (and analogously for type θ of bidder 2).

[10] Due to the efficient tie-breaking rule the outcome is efficient even if different types submit equal bids. However, efficiency does not depend on thy choice of the tie-breaking rule. Under a random tie-breaking rule type 1 would just decide between the bids 0^{+} and $\left(\overline{b}^{\theta}\right)^{+}$.

Applying the same procedure, one can compute the highest rationalizable bids for every number of types and every choice of parameters and then compute the bids under maximin strategies. The following two graphs show the highest rationalizable bids for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

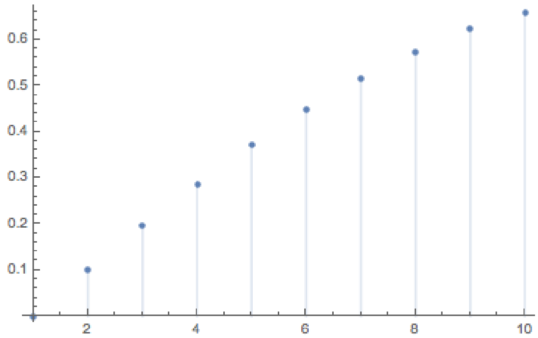


Figure 1: Highest rationalizable bids for $m = 10$

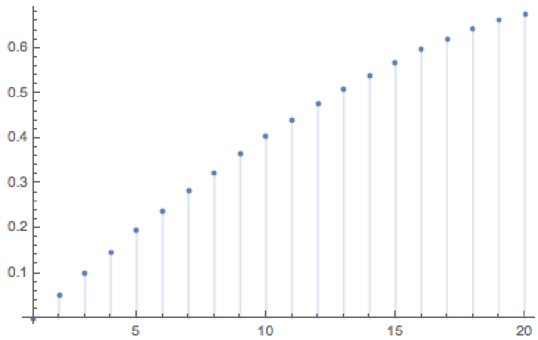


Figure 2: Highest rationalizable bids for $m = 20$

The following two graphs show the bids in an outcome under maximin strategies for m equidistant types with a uniform distribution for $m = 10$ and $m = 20$.

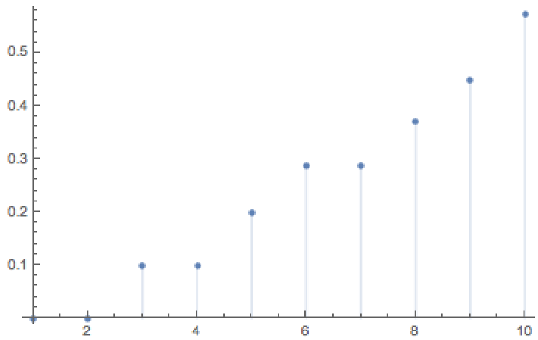


Figure 3: Bids in an outcome under maximin strategies for $m = 10$

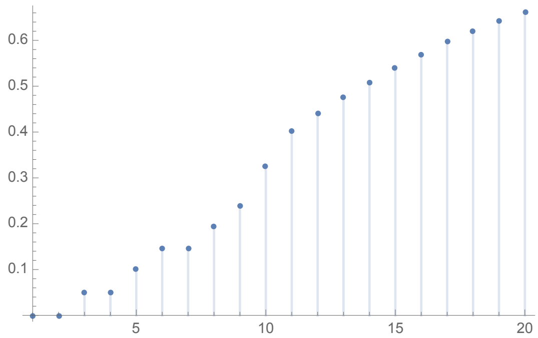


Figure 4: Bids in an outcome under maximin strategies for $m = 20$

Figures 3 and 4 show that the outcome under maximin strategies is efficient since the bidder with the highest valuation wins the auction with probability 1. However, it is possible that different types submit equal bids. Whenever a bidder is not indifferent between two bids, her bidding strategy is unique which is the case in figures 3 and 4.

The strategy of the adverse nature is not necessarily unique. Assume that it is a best reply of a bidder i with valuation $\theta^k \in \Theta$ to bid $\overline{b}^{\theta^l}$ for $l < k$. Then it is possible that the adverse nature decreases the bid of some bidder $j \neq i$ with type $\theta^{l'}$ for $l \neq l'$ without changing the best reply of bidder i and hence without changing her expected utility. Since all possible strategies of the adverse nature induce the same bidding strategies, the non-uniqueness of the adverse nature's strategy does not affect efficiency.

The recursive computation of the highest rationalizable bids for all types is formalized in the following Proposition.

Proposition 5. *In a first-price auction under strategic uncertainty the highest rationalizable bids can be defined by the following recursion. The highest rationalizable bid of a bidder with valuation zero is zero. Assume that for every type θ^j with $j < k$ the highest rationalizable bid $\overline{b}^{\theta^j}$ has been already defined. Then the highest rationalizable bid of a bidder with valuation θ^k is determined by the equality*

$$\theta^k - \overline{b}^{\theta^k} = \max_{\theta^j < \theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right). \tag{2}$$

The formal proof is relegated to Appendix D.[11]

Proposition 5 states that the highest rationalizable bid of a bidder with valuation θ^k makes this bidder indifferent between winning the auction with probability 1 by bidding

[11]For a simpler notation I assume that the highest rationalizable bids $\overline{b}^{\theta^j}$ for $1 \leq j \leq m$ lie on the bid grid. Otherwise, the highest rationalizable bid of a bidder with valuation θ^k would be defined by $\max\{b \in \mathscr{B} \mid b < b^{\theta^k}\}$ where the bid b^{θ^k} is defined by $\overline{b}^{\theta^k} = \theta^k - b^{\theta^k} = \max_{\theta^j < \theta^k} F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right)$.

$\overline{b}^{\theta^k}$ and the most profitable overbidding of some lower type given that all lower types bid their highest rationalizable bid.

After calculating the highest rationalizable bid for every type, one can specify the strategies played in an outcome under maximin strategies.

Proposition 6. *In a first-price auction under strategic uncertainty it holds for every outcome under maximin strategies and for every bidder i and every valuation θ^k that*

$$b \in supp\left(\beta_i\left(\theta^k\right)\right) \Rightarrow b \in \underset{\overline{b}^{\theta^j}}{\operatorname{argmax}}\left\{F^{I-1}\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right) \mid \theta^j < \theta^k\right\}.$$

This proposition states that every bidder chooses the most profitable overbidding of a lower type. The intuition for this result is as follows. I show in the proof of Proposition 4 that there exists an outcome under maximin strategies such that for every bidder the adverse nature chooses as the other bidders' strategy that every bidder places the highest rationalizable bid given her type. The strategy specified in Proposition 6 is a best reply to this strategy of the adverse nature. Moreover, I show in the proof of Proposition 4 that in every outcome under maximin strategies of a first-price auction under strategic uncertainty the bidders' strategies are equal. Therefore, in every outcome the bidders' strategies are as specified in Proposition 6. The formal proof is relegated to Appendix D. Note that as before, this Proposition also provides the maximin strategy, i.e. a recommendation, for a bidder facing strategic uncertainty in a first-price auction.

Proposition 6 shows that a bidder with a given type does not need to know the exact value or distribution of higher types but only of lower types. This stems from the fact that a bidder with a given type does not expect to win against bidders with the same or a higher type. This Proposition also provides an intuition for the fact that every outcome is efficient. The higher the valuation of a bidder, the higher is the type such that bidding the highest rationalizable bid of this type maximizes the bidder's expected utility.

Similarly as in the case where bidders' valuations are known, there are more rationalizable actions than actions played in the Bayes-Nash equilibrium as formalized in the following Proposition.

Proposition 7. *Consider a first-price auction under strategic uncertainty. Let $\overline{b}_*^{\theta^k} = \max supp\left(\beta^*\left(\theta^k\right)\right)$ in the unique Bayes-Nash equilibrium β^*.*[12] *If $m \geq 3$, it holds for all $k \neq 1$ that $\overline{b}_*^{\theta^k} < \overline{b}^{\theta^k}$.*

Proof. The formal proof is relegated to Appendix E. □

[12] It follows from Montiero (2009) that a unique Bayes-Nash equilibrium exists.

Proposition 5 provides the explanation for this result. Since the Bayes-Nash equilibrium is efficient, the highest bid in the Bayes-Nash equilibrium is induced if a bidder overbids all bidders with an equal or lower type. In contrast, the highest rationalizable bid is induced if a bidder overbids all other bidders.

5 Distributional and strategic uncertainty

In this section the formal model and the application to first-price auctions allow not only for strategic but also for distributional uncertainty. The first subsection contains the formal model, the second subsection collects all results for the general model. The third subsection specifies the formal model for first-price auctions and the fourth subsection provides the results.

5.1 Model

This subsection provides the formal model for a game under both, distributional and strategic uncertainty. The underlying game of incomplete information is the same as in section 2. As before, a player applying the maximin expected utility criterion plays a simultaneous game against an adverse nature.

Action space of adverse nature Under distributional uncertainty the adverse nature's action space does not only consist of rationalizable strategies but also of the set of distributions which the player considers to be possible. I allow for the possibility that a player does not know the exact type distribution but has more knowledge than just the other players' type spaces. This is formalized in the following definition.

Definition 10. *Let $\Delta\Theta_{-i}$ be the set of all probability distributions on Θ_{-i}. The set $\Delta_{\Theta_{-i}}$ is the smallest subset of $\Delta\Theta_{-i}$ such that player i knows that the true type distribution is an element in $\Delta_{\Theta_{-i}}$.*

Analogously as for strategic uncertainty, it holds that distributional uncertainty is present if the set $\Delta_{\Theta_{-i}}$ is not a singleton. Note that the assumption that a player knows that the true type distribution (or the true strategy) is an element of some set is w.l.o.g. since it covers any possible knowledge structure. For example, if a bidder i knows only the type spaces of the other bidders but nothing else about the type distribution, then $\Delta_{\Theta_{-i}}$ is equal to $\Delta\Theta_{-i}$, the set of all type distributions on Θ_{-i}. In contrast, if bidder i faces no distributional uncertainty and knows that the distribution of the other bidders' types is given by a function F_{-i}, then the set $\Delta_{\Theta_{-i}}$ is equal to $\{F_{-i}\}$.

Throughout the paper I use the axiomatization of the knowledge operator where the statement that a player knows something implies that it is true. Therefore, for every player i it holds that the true type distribution of the other player is indeed an element in $\Delta_{\Theta_{-i}}$.

As in section 2, the strategies which a player considers to be possible are the set of rationalizable strategies.

Rationalizable strategies As before, I assume common knowledge of rationality which implies that the adverse nature has to choose from the set of rationalizable strategies. If distributional uncertainty is added to strategic uncertainty, the definition of a rationalizable action changes. Under distributional and strategic uncertainty a player is rational if her action is a best reply given her type, a conjecture about the other players' strategies and a conjecture about the other players' type distributions which is an element in $\Delta_{\Theta_{-i}}$.

Definition 11.

(i) Let $i \in \{1, \ldots, I\}$ be a player and $\theta_i \in \Theta_i$ be a type of player i. The set of rationalizable actions *for player i is defined as follows. Set $RS_i^1(\theta_i) := A_i$. Assume that for $k \in \mathbb{N}$ the set $RS_i^k(\theta_i)$ is already defined. Then the set $RS_i^{k+1}(\theta_i)$ is defined as the set of all elements a_i in A_i for which there exists a type distribution $F_{-i} \in \Delta_{\Theta_{-i}}$ and a strategy profile β_{-i} of the other players such that it holds*

(i) $a_j \in supp(\beta_j(\theta_j))$ *for* $\theta_j \in \Theta \Rightarrow$ $a_j \in RS_j^k(\theta_j)$ *for all* $j \neq i$

(ii) $a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i(\theta_i, a_i', \beta_{-i}, F_{-i})$

and $RS_i(\theta_i)$ is given by

$$RS_i(\theta_i) = \bigcap_{k \geq 1} RS_i^k(\theta_i).$$

(ii) A strategy β_i of a player i is rationalizable *if for every $\theta_i \in \Theta_i$ every action a_i with $a_i \in supp(\beta_i(\theta_i)) > 0$ is rationalizable, i.e. an element of $RS_i(\theta_i)$.*

(iii) For a player i let RS_{-i} be the set of rationalizable strategies of the other $I-1$ players.

The definition of the possible distributions and strategies and of rationalizable strategies allows for a formal definition of the adverse nature's action space and therefore for a formal definition of the simultaneous game against an adverse nature.

Simultaneous game against adverse nature The following definition summarizes all components describing a game under distributional and strategic uncertainty.

Definition 12. *A* game under distributional and strategic uncertainty *consists of a game of incomplete information, denoted by* $(\{1,\ldots,I\},\Theta,A,\{u_i\}_i\in\{1,\ldots,I\})$*, a subset of players* $\{i_1,\ldots i_k\}\subseteq\{1,\ldots,I\}$ *applying the maximin expected utility criterion, and a player* n*. For every* $i\in\{i_1,\ldots,i_k\}$ *player* i *chooses a strategy*

$$\beta_i:\Theta_i\to\Delta A_i.$$

A strategy of n *is a mapping which for every player* $i\in\{i_1,\ldots i_k\}$ *and for every possible type of player* i *assigns a distribution of the other players' valuations in a convex set* $\Delta_{\Theta_{-i}}$ *and a strategy of the other players in* RS_{-i}*:*

$$\left(\beta^{n_{i_1}},\ldots,\beta^{n_{i_k}}\right):\Theta_{i_1}\times\ldots\times\Theta_{i_k}\to\left(RS_{-i_1}\times\Delta_{\Theta_{-i_1}}\right)\times\cdots\times\left(RS_{-i_k}\times\Delta_{\Theta_{-i_k}}\right).$$

$$\left(\theta_{i_1},\ldots,\theta_{i_k}\right)\to\left(\left(\beta_{-i_1}^{n_{i_1},\theta_{i_1}},F_{-i_1}^{n_{i_1},\theta_{i_1}}\right),\ldots,\left(\beta_{-i_k}^{n_{i_k},\theta_{i_k}},F_{-i_k}^{n_{i_k},\theta_{i_k}}\right)\right).$$

The utility of a player $i\in\{i_1,\ldots,i_k\}$ *is given by*

$$U_i\left(\theta_i,\beta_i\left(\theta_i\right),\beta_{-i}^{n_i,\theta_i},F_{-i}^{n_i,\theta_i}\right)$$

which is defined as in (1) and depends on the utility function of player i *in the underlying game of incomplete information, denoted by* u_i*:*

$$u_i:A\times\Theta_i\to\mathbb{R}.$$

$$\left(a_1,\ldots,a_I,\theta_i\right)\mapsto u_i\left(a_1,\ldots,a_I,\theta_i\right)$$

The utility of player nature is given by

$$-\sum_{j=1}^{k}U_i\left(\theta_i,\beta_i\left(\theta_i\right),\beta_{-i}^{n_i,\theta_i},F_{-i}^{n_i,\theta_i}\right).$$

The term uncertainty can include distributional uncertainty or strategic uncertainty or both. If only one type of uncertainty is present, I will refer to this case as *pure distributional* or *pure distributional uncertainty.* For instance, a game under pure strategic uncertainty as defined in section 2, is a special case of a game under distributional and strategic uncertainty. If the type distribution F is common knowledge as defined in section 2, then it holds for all players i that $\Delta_{\Theta_{-i}}=\{F_{-i}\}$.

Now it is possible to define a maximin strategy in a game under distributional and strategic uncertainty.

Definition 13. *In a game under distributional and strategic uncertainty for a player i a strategy*

$$\beta_i : \Theta_i \to \Delta A_i$$

is a maximin strategy *if there exists a maximin equilibrium in the simultaneous game between nature and player i such that β_i is player i's equilibrium strategy.*

The Nash equilibrium in the simultaneous game between nature and player i is called maximin equilibrium.

Analogously to section 2, one can define the subjective maximin belief of a bidder.

Definition 14. *In a game under uncertainty let β^{n_i} be the adverse nature's maximin equilibrium strategy projected on the i'th component. A* subjective maximin belief *of player i with valuation θ_i is defined as*

$$\beta^{n_i}(\theta_i) = \left(\beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right),$$

that is, the adverse nature's maximin equilibrium strategy evaluated at θ_i.

5.2 Outcomes under distributional and strategic uncertainty

The definition of an outcome in a game under distributional and strategic uncertainty is analogous to the definition in a game under distributional and strategic uncertainty.

Definition 15. *In a game under distributional and strategic uncertainty an* outcome under maximin strategies *is a strategy profile $(\beta_1, \ldots, \beta_I, \beta_n)$ such that for every $i \in \{1, \ldots, I\}$ it holds that player i's strategy is a maximin strategy given the adverse nature's strategy β_n.*

Analogously to Proposition 1 and Corollary 1 the following Proposition and Corollary hold true which state that a strategy played in an outcome under maximin strategies is rationalizable and that every action which is a best reply to a profile of rationalizable strategies is also rationalizable.

Proposition 8. *In a game under distributional and strategic uncertainty let $(\beta_1, \ldots, \beta_I, \beta_n)$ be an outcome under maximin strategies. Then for every $i \in \{1, \ldots, I, \beta_n\}$ it holds that β_i is a rationalizable strategy for player i.*

The proof is relegated to Appendix A.

Corollary 2. *In a game under distributional and strategic uncertainty let $i \in \{1, \ldots, I\}$ be a player with valuation θ_i and for every $j \in \{1, \ldots, I\} \backslash \{i\}$ let β_j be a rationalizable* *strategy for player j. Let $a_i \in A_i$ be a best reply to β_{-i}, i.e. it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}\right),$$

then $a_i \in RS_i\left(\theta_i\right)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

I will now provide another simple condition which is sufficient for an action to be rationalizable and therefore facilitates the derivation of maximin strategies. In order to do so, the following definition is needed.

Definition 16. *For a game under distributional and strategic uncertainty a profile of strategies $(\beta_1, \ldots, \beta_I) \in \Delta A_1 \times \cdots \times \Delta A_I$ together with a profile of subjective beliefs about the other players' type distributions $\left(F_{-1}^1, \ldots, F_{-I}^I\right) \in \Delta_{\Theta_{-1}} \times \cdots \times \Delta_{\Theta_{-I}}$ is called* subjective-belief equilibrium with given strategies *if every player acts optimally given her belief and the other players' strategies, i.e. for every $i \in \{1, \ldots, I\}$, every $\theta_i \in \Theta_i$ and every $a_i \in supp\left(\beta_i(\theta_i)\right)$ it holds that*

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \; U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}^i\right).$$

That is, in a subjective-belief equilibrium players best reply to each other's strategies but do not know each other's type distributions. Every player forms a subjective belief about the other players' type distributions and acts optimally given this subjective belief and the other players' strategies. An example for a subjective-belief equilibrium is a Bayes-Nash equilibrium with a common prior.

Example 3. *Let the strategy profile $(\beta_1, \ldots \beta_I)$ together with the profile of beliefs $\left(\hat{F}_{-1}, \ldots, \hat{F}_{-I}\right)$ be a Bayes-Nash equilibrium with a common prior. Then $(\beta_1, \ldots \beta_I)$ together with $\left(\hat{F}_{-1}, \ldots, \hat{F}_{-I}\right)$ constitutes a subjective-belief equilibrium.*

The following proposition states that a strategy which is played in a subjective-belief equilibrium is rationalizable.

Proposition 9. *In a game under distributional and strategic uncertainty an action $a_i \in A_i$ is rationalizable for a player i with valuation θ_i if there exists a subjective-belief equilibrium with strategies $(\beta_1, \ldots, \beta_I)$ such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$.*

Proof. Let $(\beta_1, \ldots, \beta_I)$ together with $\left(F_{-1}^1, \ldots, F_{-I}^I\right)$ be a subjective-belief equilibrium. Let i be a player with valuation θ_i and a_i be an action such that $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It is to show that $a_i \in RS_i(\theta_i)$. I show by induction with respect to k that for every $j \in \{1, \ldots, I\}$, for every $k \geq 1$ and for all $\theta_j \in \Theta_j$ it holds that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j^k\left(\theta_j\right).$$

Then it follows that $a_j \in RS_j\left(\theta_j\right)$ and one can conclude that $a_i \in RS_i\left(\theta_i\right)$ because $a_i \in supp\left(\beta_i\left(\theta_i\right)\right)$. It holds for all $j \in \{1, \ldots, I\}$ that

$$a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j^1\left(\theta_j\right) \quad \text{for all } \theta_j \in \Theta_j$$

since $RS_j^1\left(\theta_j\right) = A_j$ by definition. Assume it is already shown that for all $j \in \{1, \ldots, I\}$ it holds that

$$a_i \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_{\theta_j}^k \quad \text{for all } \theta_j \in \Theta_j.$$

Let j be some player with type θ_j and subjective belief $F_{-j}^j = \left(F_1^j, \ldots, F_{j-1}^j, F_{j+1}^j, \ldots, F_I^j\right)$. Then F_{-j}^j and β_{-j} fulfill the properties

(i) $a_l \in supp\left(\beta_l\left(\theta_l\right)\right) \Rightarrow a_l \in {RS_l}^k\left(\theta_l\right)$ for all $l \neq j$

(ii) $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in \underset{a_j' \in A_j}{\operatorname{argmax}} \; U_j\left(\theta_j, a_j', \beta_{-j}, F_{-j}^j\right)$.

The first property follows from the induction hypothesis and the second property follows from the definition of a subjective-belief equilibrium with given strategies. By definition of a rationalizable action, it follows that $\beta_j\left(\theta_j\right) \in RS_j^{k+1}$. Hence, it is shown that $a_j \in supp\left(\beta_j\left(\theta_j\right)\right) \Rightarrow a_j \in RS_j\left(\theta_j\right)$. □

The analogous result as in Proposition 2 holds also for games under distributional and strategic uncertainty. That is, every strategy played in a Bayes-Nash equilibrium is rationalizable and every action which is a best reply to a Bayes-Nash equilibrium is rationalizable.

Proposition 10. *Let the profile of strategies $\left(\beta_1, \ldots, \beta_I\right)$ together with the profile of type distributions $\left(F_1, \ldots, F_I\right)$ constitute a Bayes-Nash equilibrium with a common prior of a game of incomplete information. Then the following holds true:*

(i) For every $i \in \{1, \ldots, I\}$ the strategy β_i is rationalizable.

(ii) Let $i \in \{1, \ldots I\}$ be a player with valuation θ_i and let $a_i \in A_i$ be a best reply to β_{-i} and some distribution of the other players' types $F_{-i}' \in \Delta_{\Theta_{-i}}$, i.e. it holds that

$$a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} \, U_i\left(\theta_i, a_i', \beta_{-i}, F_{-i}'\right),$$

then $a_i \in RS_i\left(\theta_i\right)$, that is, a_i is a rationalizable action for player i with valuation θ_i.

It can be proved as a direct result of Proposition 9.

Proof. As stated in Example 3, every Bayes-Nash equilibrium is a subjective-belief equilibrium. Due to Proposition 9, every strategy played in a subjective-belief equilibrium is rationalizable. Hence, every strategy played in a Bayes-Nash equilibrium is rationalizable which proves the first part. Corollary 2 states that best replies to rationalizable strategies are rationalizable. Therefore, a best reply to a strategy which is played in a Bayes-Nash equilibrium is rationalizable which shows the second part. □

If $\{i_1, \ldots, i_k\}$ is the set of players applying the maximin expected utility criterion, then as under pure strategic uncertainty, the game against an adverse nature can be seen as k independent two-player zero-sum games. This allows for the application of all results for two-player zero-sum games including the existence result for Nash equilibria.

After presenting the formal model, I turn to the application to first-price auctions under distributional and strategic uncertainty.

5.3 First-price auctions under distributional and strategic uncertainty: Model

The underlying game of incomplete information is the same as for first-price auctions under strategic uncertainty in subsection 4.1. What differs is the set of distributions a bidder applying the maximin expected utility criterion considers to be possible. Before, the valuation distribution was common knowledge. Now I assume that the set of possible valuations, i.e. the support of the valuation distribution, and the mean of the valuation distribution is common knowledge.

Possible distributions It is common knowledge that every bidder's valuation is drawn from the set $\Theta = \{0 = \theta^1, \theta^2, \ldots, \theta^{m-1}, 1 = \theta^m\}$ according to a distribution with an exogenously given mean μ. Formally, let

$$\mathcal{F}_\mu^{I-1} = \left\{ F_1 \times \cdots \times F_{I-1} \in \Delta^{I-1}(\Theta) \;\middle|\; \sum_{i=1}^{m} \theta_i Pr\left(\theta^i\right) = \mu \right\},$$

the set of all distributions of independently drawn valuations for $I-1$ bidders with mean μ. Then it holds for every $i \in \{1, \ldots, I\}$ that

$$\Delta_{\Theta_{-i}} = \mathcal{F}_\mu^{I-1}.$$

As argued before, the above defined game of incomplete information uniquely defines a game under distributional strategic uncertainty (after specifying the player applying the maximin expected utility criterion). In the following I will call this game *first-price auction under distributional and strategic uncertainty.*

5.4 First-price auctions under distributional and strategic uncertainty: Results

Preview or results If in a first-price auction under distributional and strategic uncertainty there exist types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then every outcome is inefficient.

For every type there exists a unique highest rationalizable bid. For every bidder and every type the adverse nature chooses as the strategy of the other bidders that every bidder places the highest rationalizable bid given her type.

Let θ_μ be the lowest valuation which is higher than the mean. The highest rationalizable bid of a bidder with a valuation lower than θ_μ is her valuation. The subjective maximin belief of a bidder with valuation lower than θ_μ about the other bidders' valuation distributions is that the probability weight is distributed between her own valuation and θ_μ. As a consequence a bidder with a valuation lower than μ expects a utility of zero and is indifferent between any bid between zero and her valuation.

A bidder applying the maximin expected utility criterion with a valuation θ^k such that $\theta^k \geq \theta_\mu$ never expects to win against a bidder with the same valuation. Hence, the subjective maximin belief of the bidder about the other bidders' valuation distribution maximizes the probability weight on θ^k and makes the bidder indifferent between any highest rationalizable bid of lower types. As a consequence, the bidder mixes among all highest rationalizable bids of lower types. Therefore, if types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$ exist, then with positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

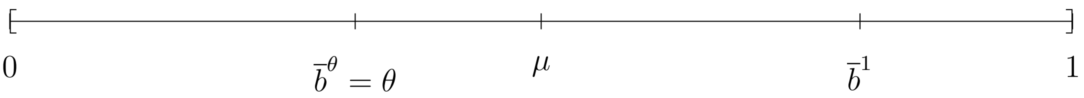
Example 4. *Consider a first-price auction under distributional and strategic uncertainty with two bidders 1 and 2 and three possible valuations* $0, \theta$ *and* 1 *which are identically and independently distributed with a commonly known mean* μ*. Assume that it holds* $\theta < \mu$*. The first step is to calculate the highest rationalizable bid for every valuation.*

The highest rationalizable bid of a bidder with valuation zero is zero. Assume that bidder 1 and bidder 2 have the subjective belief that the other bidder's valuation distribution distributes the probability weight between types θ *and 1, i.e. there is zero probability weight* *on type 0. Given that bidder 1 and bidder 2 have this subjective belief, the following strategies constitute a Bayes-Nash equilibrium:*

(i) Type θ of bidder 1 and bidder 2 bids θ.

(ii) Type 1 of bidder 1 and bidder 2 plays a mixed strategy on the interval $[\theta, b^1]$ for $\theta < b^1 < 1$.

Thus, it is part of a subjective-belief equilibrium that a bidder with valuation θ bids θ. It follows from Proposition 9 that bidding θ is a rationalizable action for a bidder with valuation θ. Since bidding above valuation cannot be rationalizable, the highest rationalizable action of a bidder with valuation θ is to bid θ.



Let $\overline{b}^1$ denote the highest rationalizable bid of a bidder with valuation 1. In order to compute $\overline{b}^1$, consider the conjecture of a bidder with valuation 1 that the strategy of the other bidder is such that

(iii) Type zero bids zero,

(iv) Type θ bids θ,

(vi) Type 1 bids $\left(\overline{b}^1\right)^-$.

It has been already shown that (iv) is rationalizable and similarly as in the case of pure strategic uncertainty, one can show that (vi) is rationalizable. It follows from Corollary 2 that a best reply to the strategy described in $(iii) - (vi)$ is rationalizable. Thus, this is the rationalizable strategy which maximizes the expected utility of bidding $\overline{b}^1$ and therefore induces the highest rationalizable bid of a bidder with valuation 1, i.e. bidding $\overline{b}^1$ is a best reply to this strategy.

A rationalizable bid is a best reply to a strategy of the other bidders and to a distribution of their valuations. Hence, in addition to the strategy inducing $\overline{b}^1$, one has to derive the valuation distribution inducing $\overline{b}^1$. Let $(f_0^1, f_\theta^1, f_1^1)$ denote the corresponding probability mass function. It must hold that

$$1 - \overline{b}^1 \geq f_0^1$$

$$1 - \overline{b}^1 \geq \left(f_0^1 + f_\theta^1\right)(1 - \theta)$$

which is equivalent to

$$1-\overline{b}^1 \geq \max\left\{f_0^1, \left(f_0^1+f_\theta^1\right)(1-\theta)\right\}.$$

Since $\overline{b}^1$ is the highest bid for which this condition is fulfilled, it holds that

$$\overline{b}^1 = 1-\min\max\left\{f_0^1, \left(f_0^1+f_\theta^1\right)(1-\theta)\right\}$$

which is equivalent to

$$\overline{b}^1 = 1-f_0^1 = 1-\left(f_0^1+f_\theta^1\right)(1-\theta). \tag{3}$$

Since probabilities have to add up to zero and the mean has to be preserved, the vector $(f_0^1, f_\theta^1, f_1^1)$ is the unique solution to the following system of linear equations

$$f_0^1+f_\theta^1+f_1^1=1$$

$$f_0^1 0+f_\theta^1\theta+f_1^1 1=\mu$$

$$f_1^0=\left(f_1^0+f_1^\theta\right)(1-\theta).$$

After obtaining the solution

$$f_0^1=\frac{1-\mu}{1+\theta}, \quad f_\theta^1=\frac{\theta(1-\mu)}{1-\theta^2}, \quad f_1^1=\frac{\mu-\theta^2}{1-\theta^2},$$

one can compute $\overline{b}^1$ using equation (3), i.e. it holds

$$\overline{b}^1=1-f_1^0=1-\left(f_1^0+f_1^\theta\right)(1-\theta)=\frac{\mu}{1+\theta}.$$

After deriving the highest rationalizable bids for every type, the second step is to derive the adverse nature's strategy. In the setting of strategic and distributional uncertainty the adverse nature's strategy determines for every bidder and every type a strategy and a valuation distribution of the other bidder. As in the case of pure strategic uncertainty, for every bidder and every type the adverse nature chooses as the strategy of the other bidder to place the highest rationalizable bid given her valuation.[13]

The subjective maximin belief of a bidder with valuation zero is irrelevant since such a bidder always earns a utility of zero. For a bidder with valuation θ the adverse nature

[13]As in the case of pure strategic uncertainty, the strategy of the adverse nature is not necessarily unique in a maximin equilibrium but in every equilibrium the strategies of the bidders coincide with the best reply to the strategy of the adverse nature as described.

chooses a distribution of the other bidder's valuations which puts zero probability weight on type zero. Since type θ bids θ, this induces an expected utility of zero for a bidder with valuation θ. A bidder with valuation 1 never expects to win against a bidder with valuation 1. Therefore, a bidder with valuation 1 has to decide between bidding zero and bidding θ. Hence, the adverse nature has to choose a valuation distribution $\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right)$ such that it holds

$$\min \max \left\{ \tilde{f}_1^0, \left(\tilde{f}_1^0 + \tilde{f}_1^\theta\right)(1-\theta) \right\}.$$

Since probabilities have to add up to one and the mean has to be preserved, the vector $\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right)$ is the unique solution of the same system of linear equations as the vector $\left(f_1^0, f_1^\theta, f_1^1\right)$. Therefore, it holds that

$$\left(\tilde{f}_1^0, \tilde{f}_1^\theta, \tilde{f}_1^1\right) = \left(f_1^0, f_1^\theta, f_1^1\right).$$

In the final step, for every bidder and every type one has to find the set of best replies to the adverse nature's strategy. Moreover, one has to identify the best replies such that the adverse nature does not have an incentive to deviate from her strategy derived in the second step. Since the expected utility of a bidder does not decrease if one of the other bidders places a lower bid, the adverse nature does not have an incentive to deviate from the strategy where for every bidder and every type she prescribes the highest rationalizable bid.[14] *Hence, it is sufficient to check whether the adverse nature has an incentive to deviate from the chosen distributions.*

A bidder with valuation zero bids zero. A bidder with valuation θ expects a utility of zero and is indifferent between any bid in the interval $[0, \theta]$. Hence, the adverse nature does not have an incentive to deviate. A bidder with valuation 1 is indifferent between bidding 0 and θ. In a maximin equilibrium in the game against the adverse nature, a bidder with valuation 1 mixes between 0 and θ in a way such that the adverse nature is indifferent among any valuation distribution which fulfills the constraints that probabilities add up to one and the mean μ is preserved. Therefore, the adverse nature does not have an incentive to deviate.

Note that the distribution of the other bidder's valuations which the adverse nature chooses for a type is the same distribution which induces the highest rationalizable bid for this type. That is, a bidder i with a given type assumes that her opponent j has the same assumption about i's valuation distribution as i's assumption about j's valuation

[14] An exception is that if one bidder bids above her valuation, it would be a best reply of the adverse nature to choose as the strategy of the other bidders that every other bidder bids zero. This would induce a strictly negative utility for the bidder bidding above her valuation. However, one can exclude this exception in a maximin equilibrium.

distribution. But bidder i assumes that j has a different belief about i's strategy than i's belief about j's strategy.

The insights from the example about bidders' strategies are generalized in the following Proposition.

Proposition 11. *Consider a first-price auction under distributional and strategic uncertainty. There exists an outcome under maximin strategies. If there exist types $\theta^k, \theta^{k'}, \theta^{k''} \in \Theta$ such that $0 < \theta^k \leq \mu < \theta^{k'} < \theta^{k''}$, then there does not exist an efficient outcome.*

The proof is relegated to Appendix F.

The inefficiency stems from the fact that every type above μ mixes between all highest rationalizable bids of all lower types. With positive probability type $\theta^{k''}$ bids zero and type $\theta^{k'}$ bids the highest rationalizable bid of type θ^k which is θ^k. Conclusively, the outcome is not efficient.

Similarly as under pure strategic uncertainty, I will show the existence of an outcome under maximin strategies by construction. The following three steps serve as a preparation for the calculation of the highest rationalizable bids.

(I) Show that for every type $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}^{\theta^k}$.

(II) Show that for every type zero is a rationalizable bid.

(III) Show that for every type $\theta^k \in \Theta$ every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable.

The explanation for steps (I)-(III) works analogously as for steps (I)-(III) in the case of pure strategic uncertainty. For the calculation of the highest rationalizable bids, first, consider valuations equal or below μ. Analogously as in the example, on can show that the highest rationalizable bid of a bidder with valuation θ^k such that $\theta^k \leq \mu$ is θ^k. This bid is induced by the subjective belief equilibrium where the probability weight is distributed between types θ^k and θ_μ and all bidders with valuation θ^k bid θ^k, where θ_μ is the smallest valuation strictly higher than μ.

The calculation of the highest rationalizable bids for higher types works by recursion. Assume that for a bidder i with valuation $\theta^k \geq \theta_\mu$ and that for all $j < k$ the highest rationalizable bids has been already computed. The highest rationalizable bid $\overline{b}^{\theta^k}$ of a bidder with valuation θ^k is a best reply to a conjecture about the other bidders' strategies and distributions.

The strategies which induce $\overline{b}^{\theta^k}$ are given by

(i) Every bidder with valuation θ^j such that $\theta^j < \theta^k$ bids her highest rationalizable bid.

(ii) Every bidder with valuation θ^k bids $\left(\overline{b}^{\theta^k}\right)^{-}$.

The valuation distribution of the other bidders which induces $\overline{b}^{\theta^k}$ has to minimize the incentive to bid another bid. In addition, probabilities have to add up to zero and the mean μ has to be preserved. Let $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ be a vector of probabilities such that according to the valuation distribution inducing $\overline{b}^{\theta^k}$, type θ^l of some bidder $j \neq i$ occurs with probability $f_{\theta^l}^{\theta^k}$.

Hence, the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ is the solution to the following minimization problem

$$\min\max\left\{\left(f_{\theta^1}^{\theta^k}\right)^{I-1}\theta^k, \left(f_{\theta^1}^{\theta^k} + f_{\theta^2}^{\theta^k}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^2}\right), ..., \left(f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^{k-1}}^{\theta^k}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^{k-1}}\right)\right\}$$

$$s.t.\;\; f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^m}^{\theta^k} = 1$$

$$f_{\theta^1}^{\theta^k}\theta^1 + \cdots + f_{\theta^m}^{\theta^k}\theta^m = \mu.$$

As proved in the Appendix, in the solution of this minimization problem all terms of the form

$$\left(\sum_{i=1}^{j} f_{\theta^i}^{\theta^k}\right)^{I-1}\left(\theta^k - \overline{b}^{\theta^j}\right) \quad \text{for } 1 \leq j < k$$

have to be equal.

The recursive calculation of the highest rationalizable bids and the distributions inducing them, is formalized in the following Proposition.

Proposition 12. *Consider a first-price auction under distributional and strategic uncertainty. For $\theta^k \leq \mu$ the highest rationalizable bid $\overline{b}^{\theta}$ is equal to θ^k.*

Assume that for all $j < k$, the highest rationalizable bid $\overline{b}^{\theta^j}$ has been already defined and it holds $\theta^k > \mu$. Then for the vector $f^{\theta^k} = \left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ it holds that $f_{\theta^j}^{\theta^k} = 0$ for $j > k$ and the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^k}^{\theta^k}\right)$ is the unique solution of the following system of k linear equations given by

$$\sum_{i=1}^{k} f_{\theta^i}^{\theta^k} = 1$$

$$\sum_{i=1}^{k} f_{\theta^i}^{\theta^k}\theta^i = \mu$$

$$\left(f_{\theta^1}^{\theta^k}\right)^{I-1} \theta^k = \left(\sum_{i=1}^{j} f_{\theta^i}^{\theta^k}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) \quad \textit{for } 1 < j < k.$$

The highest rationalizable bid $\overline{b}^{\theta^k}$ *is obtained by the equation*

$$\overline{b}^{\theta^k} = \theta^k - \left(f_{\theta^1}^{\theta^k}\right)^{I-1} \theta^k.$$

The proof is relegated to Appendix F. After calculating the highest rationalizable bid for every type, one can specify the strategies played in an outcome under maximin strategies.

Proposition 13. *In a first-price auction under distributional and strategic uncertainty it holds for every outcome under maximin strategies and for every bidder i and every valuation* θ^k *that*

(i) Every bidder with valuation θ^k *such that* $\theta^k \leq \mu$ *is indifferent between any bid in the interval* $[0, \theta^k]$.

(ii) Every bidder with valuation θ^k *such that* $\theta^k > \mu$ *mixes among the bids* $\{\overline{b}^{\theta^j} \mid j < k\}$, *that is, among the set of all highest rationalizable bids of lower types.*

The proof is relegated to Appendix F.

6 Conclusion

I conclude by first providing a short summary and afterwards discussing the assumptions made in this paper as well as possible extensions.

6.1 Summary

I propose a new decision criterion for players who face strategic uncertainty in games of incomplete information. The decision criterion works in two steps. First, I assume common knowledge of rationality and eliminate all strategies which do not survive the iterated elimination of strategies which are not best replies. Second, I apply the maximin expected utility criterion. With this decision criterion one can derive recommendations for a player facing strategic uncertainty and analyze outcomes under the assumption that every player follows this decision criterion. Moreover, I provide an extension of the model to distributional and strategic uncertainty.

I apply this decision criterion to first-price auctions under pure strategic uncertainty and under both, distributional and strategic uncertainty. In both cases every bidder has the subjective belief that every other bidder places the highest rationalizable bid given her type. Therefore, a bidder applying the proposed decision criterion resorts to win against lower types with certainty by bidding highest rationalizable bids of lower types. Besides providing recommendations for bidders facing strategic oder distributional uncertainty in first-price auctions, I characterize all outcomes under the assumption that every bidder applies this criterion. Under pure strategic uncertainty every outcome is efficient. Under distributional and strategic uncertainty every outcome is inefficient (under a mild condition on the number and distribution of possible valuations).

6.2 Discussion

Choice of decision criterion

The decision criterion under uncertainty used in this paper is the maximin expected utility criterion. The analogous analysis could be conducted with other criteria such as the minimax expected regret criterion.

Possible distributions and strategies

In this paper I restricted the set of possible strategies by assuming common knowledge of rationality and the set of possible distributions by assuming common knowledge of a mean. This restriction is crucial for the application of the maximin expected utility criterion. Otherwise, in first-price auctions there would exist a distribution or strategy inducing an expected utility of zero for a player independent of her action.

For example, Bergemann and Schlag (2008) apply the maximin expected utility criterion to a monopoly pricing problem where a seller faces uncertainty about the buyer's valuation distribution. Without a restriction of the set of possible distributions the adverse nature would choose a distribution which puts the whole probability weight on valuation zero. Thus, they assume that the seller knows that the buyer's true valuation distribution is in the neighborhood of a model distribution. Other papers applying the maximin expected utility criterion to distributional uncertainty also assume an exogenously given restriction of the set of possible distributions.

Due to a similar reasoning, a restriction of the set of possible strategies is required if the maximin expected utility criterion is applied to strategic uncertainty. For example, if one would apply the maximin expected utility criterion to a first-price auction where a bidder faces uncertainty about the other bidders' strategies, then without a restriction of the set of possible strategies the adverse nature would choose a strategy of the other bidders such that all bidders place arbitrarily high bids. First, such strategies do not seem plausible. Second, the maximin expected utility criterion does not provide a useful recommendation. In order to solve these issues, one could also exogenously restrict the set of possible strategies. For example, Kasberger and Schlag (2017) apply the minimax regret criterion to first-price auctions and assume common knowledge of an exogenously given restriction of the players' bidding strategies, for instance, in form of a lower bound of the highest bid.

However, the fact that rational agents interact strategically in a given economic setting already contains information about the possible strategies. Thus, it is possible to use an endogenous restriction of the set of possible strategies - which is given by the set of rationalizable strategies - in order to apply the maximin expected utility criterion. The model can be easily extended in a way which allows for additional (exogenously given) knowledge about possible strategies.

Under distributional uncertainty an exogenously given restriction of the set of possible distributions is still necessary. Besides fixing the mean, there exist other possibilities to restrict the set of possible distributions and strategies. For instance, one could investigate outcomes under distributional uncertainty under the assumption that further moment conditions of the type distribution are common knowledge.

Cognitive complexity

Formally, the derivation of the set of rationalizable actions for an agent with a given type requires an infinite intersection of sets. However, the proofs use a finite number of recursion steps. In the model under strategic uncertainty and in the model under distributional uncertainty the bid of a bidder with type θ^k is obtained after at most k recursion steps. One could argue that a sufficiently rational player can conduct the necessary calculations. But one could also argue that for some players these calculations may be too difficult. Therefore, similarly as in level-k models, one could define the concept of k-rationalizability. That is, a player i could know that her opponent can compute the set RS_j^k for all players j and for $k \in \mathbb{N}$, but cannot compute the sets $RS_j^{k'}$ for $k' > k$ (see Bernheim (1984)). Depending on the parameters, this knowledge can influence player i's maximin strategy.

Robustness

In addition to the maximin expected utility criterion, one could introduce an additional robustness criterion in the following sense: Does the maximin strategy of an agent change if the adverse nature deviates from her strategy to another strategy in an ϵ-neighborhood? If there is a change, does the strategy and the resulting expected utility change continuously?

As an example, consider a first-price auction under pure strategic uncertainty with a commonly known distribution function, two bidders and three valuations 0, θ and 1. Bidder 1 with valuation 1 has the subjective maximin belief that bidder 2 with valuation 1 bids $\overline{b}^1$. Hence, bidder 1 with valuation 1 bids either $\overline{b}_\theta$ or zero. However, all bids in the interval $[0, \overline{b}^1]$ are rationalizable for a bidder with valuation 1. Hence, (if the bid grid is sufficiently fince) an ϵ-neighborhood of $\overline{b}^1$ and its intersection with the set of rationalizable actions contains bids lower than $\overline{b}^1$. If bidder 1 with valuation 1 has the subjective belief that bidder 2 with valuation 1 bids lower than $\overline{b}^1$, e.g. $\left(\overline{b}^1\right)^-$, then $\overline{b}^1$ becomes a best reply for bidder 1 with valuation 1. This constitutes a discontinuity in her best reply.

As a second example, consider a first-price auction under pure strategic uncertainty with two bidders and a commonly known common valuation v. To bid v is the highest rationalizable action for both bidders. Therefore, bidder 1 has the subjective maximin belief that bidder 2 bids v. As a consequence, bidder 1 is indifferent between any bid in $[0, v]$. Assume that bidder 1 chooses the action v (or v^-). As any other bid, this leads to a utility of zero given the subjective maximin belief that bidder 2 bids v. An ϵ-neighborhood of v and its intersection with the set of rationalizable actions contains only bids below v, e.g. it contains the bids v,v^- and $(v^-)^-$. The best replies to these bids are in an ϵ-neighborhood of v (or v^-) and the induced utilities are in an ϵ-neighborhood of zero. Hence, bidding v (or v^-) fulfills the robustness property that an ϵ-deviation of the subjective maximin belief induces an ϵ-deviation of the best replies and expected utility.

Appendices

A Proof of Proposition 1 and 8

Proof. Proposition 1 is a special case of Proposition 8 such that for every player i it holds that $\Delta_{\Theta_{-i}} = F_{-i}$ where F is the commonly known valuation distribution as assumed in Proposition 1. Therefore, it is sufficient to prove Proposition 8. Every player maximizes her expected utility given a distribution of the other players' types and a rationalizable strategy of the other players chosen by nature. Let $(\beta_1, \ldots, \beta_I, \beta_n)$ be an outcome under maximin strategies. It is to show that for every player i and for every type θ_i an action a_i which is in the support of $\beta_i(\theta_i)$ is an element in $RS_i^k(\theta_i)$ for every $k \geq 1$. The proof works by induction. It is true that for every $i \in \{1, \ldots, I\}$ every action $a_i \in A_i$ is an element in $RS_i^1(\theta_i)$ since it holds by definition that $RS_i^1(\theta_i) = A_i$. Assume it is already shown for every $i \in \{1, \ldots, I\}$ and every $\theta_i \in \Theta_i$ that every action a_i with $a_i \in supp(\beta_i(\theta_i))$ is an element in $RS_i^k(\theta_i)$. Let i bid a bidder with valuation θ_i. Since n can choose only among rationalizable strategies, it holds for every $j \neq i$ that $\beta_j^{n_i,\theta_i}$ is a rationalizable strategy. By definition, this implies that for every $\theta_j \in \Theta_j$ and every action a_j with $a_j \in supp\left(\beta_j^{n_i,\theta_i}(\theta_j)\right)$ it holds that $a_j \in \bigcap_{k\geq 1} RS_j^k(\theta_j)$. It follows that

(i) $a_j \in supp\left(\beta_j^{n_i,\theta_i}(\theta_j)\right)$ for $\theta_j \in \Theta_j \Rightarrow a_j \in {RS_j}^k(\theta_j)$ for all $j \neq i$.

By definition of an outcome under maximin strategies, it holds for every action a_i with $a_i \in supp(\beta_i(\theta_i))$ that a_i is a best reply given the adverse nature's strategy, i.e. a best reply to the other bidders' valuation distribution and strategies chosen by the adverse nature. Therefore, it holds that

(ii) $a_i \in supp(\beta_i(\theta_i)) \Rightarrow a_i \in \underset{a_i' \in A_i}{\operatorname{argmax}} U_i\left(\theta_i, a_i', \beta_{-i}^{n_i,\theta_i}, F_{-i}^{n_i,\theta_i}\right).$

By definition of the set $RS_i^{k+1}(\theta_i)$, it follows from (i) and (ii) that for every a_i with $a_i \in supp(\beta_i(\theta_i))$ is an element in $RS_i^{k+1}(\theta_i)$ and it follows by induction that a_i is an element in $RS_i^k(\theta_i)$ for every $k \geq 1$. □

B Proof of Proposition 2

Since Proposition 2 is a special case of Proposition 10 such that for every player i it holds that $\Delta_{\Theta_{-i}} = F_{-i}$, where F is the commonly known valuation distribution as assumed in Proposition 2, the proof follows from the proof of Proposition 10.

C Proof of Proposition 3

Proof. (i) At first, I consider the case where there exists a unique bidder k who has the highest valuation and show that her highest rationalizable bid is the second-highest valuation, denoted by θ'_k. In order to do so, I will show by induction that for every bidder $i \neq k$ the bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. Let i be an arbitrary bidder which is not bidder k. Hence, bidder i's valuation is strictly lower than 1. The induction steps are descending and start with 1. Since 1 is the highest possible bid, bidder i wins with strictly positive probability if she bids 1 which cannot be rationalizable since she would earn a negative utility with positive probability. For the induction step assume that it has been shown that all bids equal or higher than b with $b \in (\theta_{k'}, 1]$ are not rationalizable for all bidders $i \neq k$. It is to show that for an arbitrary bidder $i \neq k$ the bid b^- is not rationalizable if $b^- > \theta_{k'}$. Since all bids strictly higher than b^- are not rationalizable for all bidders besides bidder k, it is also never a best reply for bidder k to bid strictly higher than b^-. Therefore, bidder i wins with strictly positive probability if she bids b^-. Since b^- is strictly higher than her valuation, this cannot be optimal. This completes the induction step from which follows that for all bidders $i \neq k$ the bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. It follows that for bidder k all bids in the interval $(\theta_{k'}, 1]$ are not rationalizable. In every Nash equilibrium the highest bidder bids the second-highest valuation $\theta_{k'}$. Since according to part (i) of Proposition 2 a strategy played in a Bayes-Nash equilibrium is rationalizable, the bid $\theta_{k'}$ is rationalizable for bidder k. It follows that $\theta_{k'}$ is the highest rationalizable bid of bidder k.

If the adverse nature chooses for all bidders $i \neq k$ as the action of bidder k to bid θ'_k , i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_{k'}$, every bidder $i \neq k$ with valuation θ_i expects a utility of zero independent of her action. Therefore, any other strategy of the adverse nature which is played in a maximin equilibrium, has to induce an expected utility of zero for every bidder $i \neq k$. That is, the subjective maximin belief of a bidder $i \neq k$ with valuation θ_i is that at least one other bidder bids equal or higher than θ_i. As a result, every bidder $i \neq k$ is indifferent between all bids in the interval $[0, \theta_i]$. It is left to show that a bidder $i \neq k$ does not bid above her valuation. Assume there exists a bidder i with valuation θ_i who bids $b > \theta_i$. Since for all bidders $j \neq k$ bidding zero is rationalizable, it is rationalizable for bidder k to bid zero. Given that bidder i bids b, the adverse nature chooses as the strategy of the other bidders to bid zero, i.e. for every $j \neq i$ it holds that $\beta_j^{n_i,\theta_i}(\theta_j) = 0$. As a result, bidder i wins with probability 1 and expects a negative utility which cannot be part of a maximin equilibrium. Hence, none of the bidders places bids strictly higher than her valuation in a maximin equilibrium.

In order to minimize the expected utility of bidder k, the adverse nature chooses as the strategy of the second-highest bidder, i.e. bidder k' with valuation $\theta_{k'}$, to bid her valuation i.e. $\beta_{k'}^{n_k,\theta_k}(\theta_{k'}) = \theta_{k'}$. This is the highest rationalizable bid which can be placed by a bidder who is not bidder k. As a consequence, bidder k bids $\theta_{k'}$.

(ii) Finally, I consider the case where at least two bidders have the highest valuation θ_k. Analogously as before, one can show by induction that for every bidder the bids in the interval $(\theta_k, 1]$ are not rationalizable. In every Nash equilibrium every highest bidder bids her valuation θ_k. Therefore, it holds due to Corollary 2 that the bid θ_k is rationalizable for every highest bidder. It follows that θ_k is the highest rationalizable bid and therefore is the action which the adverse nature chooses as the action of a highest bidder k for a bidder $i \neq k$, i.e. $\beta_k^{n_i,\theta_i}(\theta_k) = \theta_k$. This implies that every bidder does not expect to earn a positive utility and therefore is indifferent between any bid between zero and her valuation. Bids strictly higher than the own valuation can be excluded analogously as above. $\square$

D Proof of Propositions 4, 5 and 6

In order to prove Propositions 4 and 5, I will show the following Lemmas which formalize steps (I) -(III).

Lemma 1. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\bar{b}^{\theta^k}$. This bid does not depend on the identity of bidder i.*

Proof. For every bidder i and every valuation θ_i the set of rationalizable actions $RS_i(\theta_i)$ is a finite set in a metric space and therefore compact. Since every compact set contains its supremum, there exists a maximum element of the set $RS_i(\theta_i)$. Since this is a subset of $\mathscr{B}$ and by definition, $\mathscr{B}$ is well-ordered with respect to the relation $\leq$, the maximum element of $RS_i(\theta_i)$ has to be unique. Due to the symmetry of the bidders, the highest rationalizable bid does not depend on the identity of the bidder. $\square$

Lemma 2. *For every type $\theta^k \in \Theta$ zero is a rationalizable bid.*

Proof. The proof works by induction with respect to the types in Θ. The induction starts with $\theta^1 = 0$. Montiero (2009) shows that with a given commonly known distribution there exists a Bayes-Nash equilibrium in the first-price auction with discrete valuations where type zero bids zero. It follows from part (i) of Corollary 2 that zero is a rationalizable action for type zero.

For the induction step assume that it has been already shown for all types θ^j with $j \leq k$ that zero is a rationalizable action for type θ^j. Consider a bidder i with valuation θ^{k+1} who conjectures that all other bidders with type θ^j such that $j \leq k$ bid zero which is rationalizable by assumption. According to Lemma 1, for every bidder and every type there exists a highest rationalizable bid. Let the conjecture of bidder i with valuation θ^{k+1} be such that every other bidder with type θ^j such that $j > k$ bids her highest rationalizable bid. Since all types with valuation θ^j such that $j > k$ bid at least the highest rationalizable bid of type θ^{k+1} and all other types bid zero, it is a best reply of bidder i with valuation θ^{k+1} to bid zero. As stated in Corollary 1, a best reply to a rationalizable strategy profile is rationalizable and therefore zero is a rationalizable action for bidder i with type θ^{k+1}. This completes the induction step and hence one can conclude that for every bidder and every type zero is a rationalizable action. □

Lemma 3. *For every type $\theta^k \in \Theta$ every bid in $[0, \overline{b}^{\theta^k}]$ is rationalizable.*

Proof. The proof works by showing a formally stronger statement by induction with respect to the types in Θ. The statement is that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable for every type θ^j such that $j \geq k$.

The induction starts with $k = 1$, i.e. with $\theta^1 = 0$. The highest rationalizable bid for type θ^1 is zero and it follows from Lemma 2 that zero is a rationalizable bid for every type $\theta^j \geq \theta^1$.

For the induction step assume that it has been already shown that for all $l \leq k$ it holds that every bid in the interval $[0, \overline{b}^{\theta^l}]$ is rationalizable for every type θ^j such that $j \geq l$. By using induction with respect to the bids, I will show that the same statement holds for type θ^{k+1}. The induction starts with the bid zero. It follows from Lemma 2 that zero is rationalizable for every type. For the induction step assume that it has been already shown that every bid in the interval $[0, b]$ with $b < \overline{b}^{\theta^{k+1}}$ is rationalizable for every type θ^j with $j \geq k+1$. In order to show that b^+ is rationalizable for every type θ^j with $j \geq k+1$, consider a bidder i with valuation θ^j with $j \geq k+1$ and strategies of the other bidders such that for every other bidder it holds that

(i) Every type θ^h with $h < j$ bids her highest rationalizable bid.

(ii) Every type θ^h with $h \geq j$ bids b.

The strategies in (i) are rationalizable by definition and the strategies in (ii) are rationalizable by the assumption in the induction step (in the second induction with respect to the bids in the interval $[0, \overline{b}^{\theta^k}]$). Given this conjecture about the other bidders' strategies it is a best reply for bidder i with valuation θ^j to bid b^+. Any change in part (i) would imply that there exists a bidder with valuation θ^l such that $l < k+1$ who bids some bid $b^{\theta^l} < \overline{b}^{\theta^l}$ instead of $\overline{b}^{\theta^l}$ which does not increase the expected utility of bidding b^+. Any deviation from (ii) implies that there exists at least one bidder and a valuation θ^h with $h \geq j$ such that this bidder places either a lower or a higher bid than b. If the bid is lower, then the same reasoning as above applies. If a bidder with valuation θ^h deviates to a higher bid, then by bidding b^+ bidder i with valuation θ^j does not overbid type θ^l of the deviating bidder anymore which decreases bidder i's winning probability.

Conclusively, any conjecture deviating from the strategies in (i) and (ii) does not increase the expected utility of bidding b^+. Therefore, if bidding b^+ is not a best reply to the beliefs in (i) and (ii), then b is the highest rationalizable bid for type θ^j which is a contradiction to the assumption $b < \overline{b}^{\theta^k}$. This completes the induction step of the second induction. It follows that any bid in the interval $[0, \overline{b}^{\theta^{k+1}}]$ is rationalizable for every type θ^j with $j \geq k+1$. This completes the induction step of the first induction. Therefore, it has been shown that for every type $\theta^k \in \Theta$ it holds that every bid in the interval $[0, \overline{b}^{\theta^k}]$ is rationalizable for every type θ^j such that $j \geq k$. $\square$

After proving Lemmas 1-3, I continue with the proof of Proposition 5.

Proof of Proposition 5

Proof. Consider a bidder with valuation θ^k. As shown in the proof of Lemma 3, for every type the conjecture given by

(i) Every type θ^l with $l < k$ bids her highest rationalizable bid .

(ii) Every type θ^j with $j \geq k$ bids $\left(\overline{b}^{\theta^k}\right)^-$.

induces the highest rationalizable bid of a bidder with valuation θ^k , that is, the highest rationalizable bid $\overline{b}^{\theta^k}$ of a bidder with valuation θ^k is a best reply to the conjecture that all other bidders employ this strategy. Given this conjecture, the expected utility of a bidder with type $\theta^k \in \Theta$ who bids $\overline{b}^{\theta^k}$ is given by

$$\theta^k - \overline{b}^{\theta^k}.$$

This utility has to be higher than the utility induced by any other bid. A bid can be a best reply for a bidder if she just overbids some other bidder. Formally, a bid b can be best reply for a bidder with valuation θ^k only if there exists a bidder $j \neq i$ and a valuation $\theta^l < \theta^k$ such that bidder j with valuation θ^l bids b (or there exists a bidder j with valuation $\theta^l \geq \theta^k$ such that bidder j with valuation θ^l bids b^- or b). Hence, the only potential candidates for best replies besides $\overline{b}^{\theta^k}$ are bids $\overline{b}^{\theta^j}$ with $j < k$. Thus, equation (2) ensures that bidding $\overline{b}^{\theta^k}$ induces at least the same expected utility than any other bid which can be a best reply.

$\square$

Proof of Proposition 4

Proof. I show the existence of an efficient outcome under maximin strategies by construction and then show that bidders' strategies are equal in every outcome. Conclusively, there exists an outcome under maximin strategies and every outcome is efficient. The construction of an efficient outcome works as follows. According to Lemma 1 for every type there exists a unique highest rationalizable bid. For every type and every player the adverse nature chooses the other bidders' strategies such that every bidder places the highest rationalizable bid given her type, i.e. for pair of bidders i, j and for every pair of valuations θ_i, θ_j it holds that

$$\beta_j^{n_i,\theta_i}(\theta_j) = \overline{b}^{\theta_j}. \tag{4}$$

Let β_n denote this adverse nature's strategy. Independent of the bidders' strategies there does not exist another strategy of the adverse nature which induces a lower expected utility for any of the bidders.[15] Thus, the adverse nature does not have an incentive to deviate from this strategy. Every bidder plays a best reply given her type and the adverse nature's strategy. Due to the compactness of $\mathscr{B}$, such a best reply always exists. I will show that the outcome defined by these best replies is efficient.

For a bidder with valuation θ^k the best reply is given by the most profitable overbidding of a lower type, that is, by

$$\underset{\overline{b}^{\theta^j}<\theta^k}{\operatorname{argmax}}\; F\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right).$$

Let $\overline{b}^{\theta^l}$ be a best reply of a bidder with valuation θ^k. Then it holds for all $j \in \{1, \ldots, k-1\}$ that

$$F\left(\theta^l\right)\left(\theta^k - \overline{b}^{\theta^l}\right) \geq F\left(\theta^j\right)\left(\theta^k - \overline{b}^{\theta^j}\right) \tag{5}$$

$$\Leftrightarrow \theta^k\left(F\left(\theta^l\right) - F\left(\theta^j\right)\right) - F\left(\theta^l\right)\overline{b}^{\theta^l} + F\left(\theta^j\right)\overline{b}^{\theta^j} \geq 0. \tag{6}$$

Since $F\left(\theta^l\right) - F\left(\theta^j\right) \geq 0$ for $1 \leq j < l$, it follows from (6) that for all $\theta^{k'}$ such that $\theta^{k'} > \theta^k$ and for all $1 \leq j < l$ it holds that

$$\theta^{k'}\left(F\left(\theta^l\right) - F\left(\theta^j\right)\right) - F\left(\theta^l\right)\overline{b}^{\theta^l} + F\left(\theta^j\right)\overline{b}^{\theta^j} \geq 0$$

$$\Leftrightarrow F\left(\theta^l\right)\left(\theta^{k'} - \overline{b}^{\theta^l}\right) \geq F\left(\theta^j\right)\left(\theta^{k'} - \overline{b}^{\theta^j}\right). \tag{7}$$

[15] An exception is that if one bidder bids above her valuation, it would be a best reply of the adverse nature to choose as the strategy of the other bidders that every other bidder bids zero. This would induce a strictly negative utility for the bidder bidding above her valuation. However, one can exclude this exception in a maximin equilibrium.

First, consider the case where for every $j \in \{1, \ldots, k-1\}$ the inequality in (5) is strict. Then for a bidder with valuation θ^k there exists a unique best reply, denoted by $\overline{b}^{\theta^l}$. Hence, in order to show efficiency, it is sufficient to show that every best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

It holds for every $1 \leq j < l$ that the inequality in (7) is strict. It follows that none of the bids $\overline{b}^{\theta^j}$ for $1 \leq j < l$ can be a best reply for a bidder with valuation $\theta^{k'}$. Thus, a best reply of a bidder with valuation $\theta^{k'}$ with $\theta^{k'} > \theta^k$ is equal or greater than $\overline{b}^{\theta^l}$.

Second, consider the case where for at least one $j \in \{1, \ldots, k-1\}$ the expression in (5) holds with equality. Let $j_1, \ldots j_h$ be all indices for which it holds that the expression in (5) holds with equality where $\theta^{j_h} = \max_{j \in \{j_1, \ldots, j_h\}} \theta^j$. That is, $\overline{b}_{\theta^{j_h}}$ is the highest best reply of a bidder with valuation θ^k. Then for all $j \in \{j_1, \ldots, j_h\} \backslash \{j_h\}$ it must hold that $F\left(\theta^{j_h}\right) > F\left(\theta^j\right)$. Thus, for all $j < j_h$ it holds that

$$\theta^{k'}\left(F\left(\theta^{j_h}\right) - F\left(\theta^j\right)\right) - F\left(\theta^{j_h}\right)\overline{b}^{\theta^{j_h}} + F\left(\theta^j\right)\overline{b}^{\theta^j} > 0$$

$$\Leftrightarrow F\left(\theta^{j_h}\right)\left(\theta^{k'} - \overline{b}^{\theta^{j_h}}\right) > F\left(\theta^j\right)\left(\theta^{k'} - \overline{b}^{\theta^j}\right).$$

For all j such that $j < j_h$ and $j \notin \{j_1, \ldots, j_h\}$ as in the first case, it holds that the inquality in (5) is strict and therefore also the inequality in (7). Conclusively, the best reply of a bidder with valuation $\theta^{k'}$ is at least as high as the highest best reply of a bidder with valuation θ^k. Therefore, the outcome is efficient.

So far, I have shown by construction that an outcome under maximin strategies exists and that this outcome is efficient. More precisely, I have shown that any combination of best replies to the adverse nature's strategy β_n as defined in (4) is efficient. Formally, let B^{θ^j} be the set of best replies for a bidder with valuation θ^j given the adverse nature's strategy β_n, i.e. B^{θ^j} is defined by

$$B^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}}\, U\left(\theta^j, b', \beta_n, F\right).$$

Let $(b_1, \ldots, b_m)$ be a vector of bids such that $b_j \in B^{\theta^j}$ for all $j \in \{1, \ldots, m\}$. Then for every i, j with $j > i$, it holds that $b_j \geq b_i$.

It remains to show that every outcome is efficient. In order to show that every outcome is efficient, I will show that if $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ is an outcome under maximin strategies, then it holds for every bidder j and every valuation θ_j that

$$\hat{B}^{\theta_j} \subseteq B^{\theta_j}$$

where $\hat{B}^{\theta^j}$ is defined by

$$\hat{B}^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}} \, U\left(\theta^j, b', \hat{\beta}_n, F\right).$$

Assume there exists an outcome under maximin strategies, denoted by $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$, such that there exists a bidder i with valuation θ_i and a bid b such that $b \in supp\left(\hat{\beta}_i\left(\theta_i\right)\right)$ and $b \notin B^{\theta_i}$. This implies that there exists a bidder $j \neq i$ and a valuation θ^l such that $\hat{\beta}_j^{n_i,\theta_i}(\theta^l) \neq \beta^{n_i,\theta_i}(\theta^l)$ and

$$b \in supp\left(\beta_j^{n_i,\theta_i}\left(\theta^l\right)\right) \text{ or } b^- \in supp\left(\beta_j^{n_i,\theta_i}\left(\theta^l\right)\right)$$

(depending on whether $\theta_i > \theta^l$ or $\theta_i \leq \theta^l$). In other words, since the outcome is not efficient, there exists a bidder i who bids differently than in the efficient outcome by bidding b. This in turn implies that there exists another bidder j such that in the subjective maximin belief of bidder i with valuation θ_i bidder j's strategy differs from the strategy prescribed by β_n in a way which makes the bid b a best reply for bidder i.

Since in the subjective maximin belief of bidder i bidder j with valuation θ^l deviates from bidding her highest rationalizable bid and cannot bid higher than the highest rationalizable bid of type θ^l, it holds that $b < \overline{b}^{\theta^l}$. Therefore, the adverse nature could strictly decrease the winning probability of bidder i with valuation θ_i by deviating to the strategy which prescribes to bid $\overline{b}_{\theta^l}$ for bidder j with valuation θ^l. Thus, $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ cannot constitute an outcome under maximin strategies. Conclusively, every outcome under maximin strategies has to be efficient. □

Proof of Proposition 6

Proof. With the same reasoning as in the proof of Proposition 4, one can show that for every outcome $\left(\hat{\beta}_1, \ldots \hat{\beta}_I, \hat{\beta}_n\right)$ under maximin strategies it holds for every bidder i and every valuation θ_i that

$$B^{\theta_i} \subseteq \hat{B}^{\theta_i}$$

where $\hat{B}^{\theta_i}$ and B^{θ_i} are defined by

$$B^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}} \, U\left(\theta^j, b', \beta_n, F\right), \quad \hat{B}^{\theta^j} = \underset{b' \in \mathscr{B}}{\operatorname{argmax}} \, U\left(\theta^j, b', \hat{\beta}_n, F\right)$$

and β_n is defined as in (4). This is the strategy of the adverse nature which chooses as the subjective maximin belief for every bidder and every valuation that every other bidder places the highest rationalizable bid given her valuation. It follows that

$$\hat{B}^{\theta_i} = B^{\theta_i}$$

That is, in every outcome under maximin strategies, bidders play best replies to the adverse nature's strategy β_n. Therefore, in every outcome the bidders' strategies are as specified in Proposition 6.

$\square$

E Proof of Proposition 7

Proof. The proof works by induction with respect to the type. Since $\overline{b}^{\theta^1} = \overline{b}_*^{\theta^1}$, the induction starts with θ^2. The highest rationalizable bid for type θ^2 is obtained by the equation

$$\theta^2 - \overline{b}^{\theta^2} = F^{I-1}(0)\,\theta^2$$
$$\Leftrightarrow \overline{b}^{\theta^2} = \theta^2\left(1 - F^{I-1}(0)\right).$$

The highest bid which is placed by a bidder with valuation θ^2 in a Bayes-Nash equilibrium is obtained by the equation

$$F^{I-1}\left(\theta^2\right)\left(\theta^2 - \overline{b}_*^{\theta^2}\right) = F^{I-1}(0)\,\theta^2$$

$$\Leftrightarrow \overline{b}_*^{\theta^2} = \frac{\theta^2\left(F^{I-1}(\theta^2) - F^{I-1}(0)\right)}{F^{I-1}(\theta^2)}.$$

Since $m \geq 3$, it holds that $F^{I-1}(\theta^2) < 1$ from which follows that

$$F^{I-1}\left(\theta^2\right)F^{I-1}(0) < F^{I-1}(0)$$

$$\Leftrightarrow F^{I-1}\left(\theta^2\right) - F^{I-1}\left(\theta^2\right)F^{I-1}(0) > F^{I-1}\left(\theta^2\right) - F^{I-1}(0)$$
$$\Leftrightarrow 1 - F^{I-1}(0) > \frac{F^{I-1}(\theta^2) - F^{I-1}(0)}{F^{I-1}(\theta^2)}$$
$$\Leftrightarrow \overline{b}^{\theta^2} > \overline{b}_*^{\theta^2}.$$

For the induction step assume that it has been already shown that $\overline{b}^{\theta^j} > \overline{b}_*^{\theta^j}$ for all $j \leq k$. It has to be shown that

$$\overline{b}^{\theta^{k+1}} > \overline{b}_*^{\theta^{k+1}}.$$

As stated in Proposition 5, it holds that

$$\theta^{k+1} - \overline{b}^{\theta^{k+1}} = \max_{\theta^j < \theta^{k+1}} F^{I-1}\left(\theta^j\right)\left(\theta^{k+1} - \overline{b}^{\theta^j}\right).$$

Let

$$F^{I-1}\left(\theta^{l}\right)\left(\theta^{k+1}-\overline{b}^{\theta^{l}}\right)=\max_{\theta^{j}<\theta^{k+1}} F^{I-1}\left(\theta^{j}\right)\left(\theta^{k+1}-\overline{b}^{\theta^{j}}\right).$$

Since $\overline{b}_{*}^{\theta^{k+1}}$ is a best reply, it must induce an expected utility which is greater or equal than the expected utility induced by any other bid, given that every other bidder plays equilibrium strategies. Hence, it holds that

$$F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1}-\overline{b}_{*}^{\theta^{k+1}}\right) \geq F^{I-1}\left(\theta^{l}\right)\left(\theta^{k+1}-\overline{b}_{*}^{\theta^{l}}\right).$$

Due to the induction assumption, it holds that $\overline{b}_{*}^{\theta^{l}}<\overline{b}^{\theta^{l}}$ from which follows that

$$\begin{aligned}\theta^{k+1}-\overline{b}^{\theta^{k+1}} &= F^{I-1}\left(\theta^{l}\right)\left(\theta^{k+1}-\overline{b}^{\theta^{l}}\right)<F^{I-1}\left(\theta^{l}\right)\left(\theta^{k+1}-\overline{b}_{*}^{\theta^{l}}\right)\\ &\leq F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1}-\overline{b}_{*}^{\theta^{k+1}}\right)\end{aligned}$$

and therefore it holds that

$$\theta^{k+1}-\overline{b}^{\theta^{k+1}}<F^{I-1}\left(\theta^{k+1}\right)\left(\theta^{k+1}-\overline{b}_{*}^{\theta^{k+1}}\right).$$

$$\Leftrightarrow \overline{b}_{*}^{\theta^{k+1}}<\frac{\overline{b}^{\theta^{k+1}}-\theta^{k+1}\left(1-F^{I-1}\left(\theta^{k+1}\right)\right)}{F^{I-1}\left(\theta^{k+1}\right)}. \tag{8}$$

It holds that

$$\begin{aligned}\theta^{k+1}-\overline{b}^{\theta^{k+1}} &\geq 0\\ \Leftrightarrow \theta^{k+1}\left(1-F^{I-1}\left(\theta^{k+1}\right)\right)-\overline{b}^{\theta^{k+1}}\left(1-F^{I-1}\left(\theta^{k+1}\right)\right) &\geq 0\\ \Leftrightarrow \overline{b}^{\theta^{k+1}}-\theta^{k+1}\left(1-F^{I-1}\left(\theta^{k+1}\right)\right) &\leq F^{I-1}\left(\theta^{k+1}\right)\overline{b}^{\theta^{k+1}}\\ \Leftrightarrow \frac{\overline{b}^{\theta^{k+1}}-\theta^{k+1}\left(1-F^{I-1}\left(\theta^{k+1}\right)\right)}{F^{I-1}\left(\theta^{k+1}\right)} &\leq \overline{b}^{\theta^{k+1}}.\end{aligned}$$

Due to equation (8), it follows that

$$\overline{b}^{\theta^{k+1}}>\overline{b}_{*}^{\theta^{k+1}}.$$

This completes the induction step and the proof.

□

F Proof of Propositions 11, 12 and 13

First, I prove Proposition 12 which formalizes the recursive calculation of the highest rationalizable bids for every type. This calculation is crucial for the proofs of Propositions 11 and 13. In order to prove Proposition 12, I state the following three lemmas which formalize steps (I)-(III) in section 5. The proofs work analogously as for Lemmas 1, 2 and 3 in section 4 and are therefore omitted.

Lemma 4. *For every bidder i and every valuation $\theta^k \in \Theta$ there exists a unique highest rationalizable bid $\overline{b}_i^{\theta_i}$.*

Lemma 5. *For every type zero is a rationalizable bid.*

Lemma 6. *For every type $\theta^k \in \Theta$ it holds that every bid in $[0, \overline{b}^{\theta^k}]$ is rationalizable.*

Proof of Proposition 12

Proof. First, I examine the highest rationalizable bids of a bidder with valuation θ^k such that θ^k is lower or equal than μ. Consider a subjective belief equilibrium where every bidder has the subjective belief that the other bidders' valuation distribution distributes the probability weight between types θ^k and θ^μ where $\theta_\mu = \min\{\theta^k \in \Theta \mid \theta^k > \mu\}$. Formally, the distribution of the other bidders' valuation is defined by the vector $f^{\theta^k} = \left(f^{\theta^k}_{\theta^1}, \ldots, f^{\theta^k}_{\theta^m}\right)$ where for all $j \in \{1, \ldots, m\}$ it holds that $f^{\theta^k}_{\theta^j}$ denotes the probability with which type θ^j occurs. This vector is defined by

$$f^{\theta^k}_{\theta^k} = \frac{\theta_\mu - \mu}{\theta_\mu - \theta^k}, \quad f^{\theta^k}_{\theta_\mu} = \frac{\mu - \theta^k}{\theta_\mu - \theta^k} \text{ and } f^{\theta^k}_{\theta^j} = 0 \text{ for } \theta^j \neq \theta^k, \theta_\mu.$$

Given this subjective belief, in every subjective-belief equilibrium every bidder with valuation θ^k bids θ^k. It follows from Proposition 9 that bidding θ^k is a rationalizable action for a bidder with valuation θ^k. Since it is not rationalizable to bid above valuation, θ^k is the highest rationalizable bid of a bidder with valuation θ^k.

Now I examine the highest rationalizable bids of a bidder with valuation θ^k such that θ^k is strictly greater than μ. Analogously as in the proof Proposition 5, the highest rationalizable bid of a bidder with valuation θ^k is induced by the strategy of the other bidders' such that

(i) All bidders with a lower type bid their highest rationalizable bid.

(ii) All bidders with an equal or higher type bid $\left(\overline{b}^{\theta^k}\right)^-$.

The strategies in (i) are rationalizable by definition and the strategies in (ii) are rationalizable due to Lemma 6. It follows from Corollary 2 that a best reply to these strategies is rationalizable. The highest rationalizable bid of a bidder with valuation θ^k is a best reply to the strategies in (i) and (ii) and to a distribution of the other bidders' valuations. Let the vector $f^{\theta^k} = \left(f^{\theta^k}_{\theta^1}, \dots, f^{\theta^k}_{\theta^m}\right)$ be defined by $f_{\theta^j} = 0$ for $j > k$ and let $\left(f^{\theta^k}_{\theta^1}, \dots, f^{\theta^k}_{\theta^k}\right)$ be the unique solution of the system of k linear equations given by

$$\sum_{i=1}^{k} f^{\theta^k}_{\theta^i} = 1$$

$$\sum_{i=1}^{k} f^{\theta^k}_{\theta^i} \theta^i = \mu$$

$$\left(f^{\theta^k}_{\theta^1}\right)^{I-1} \theta^k = \left(\sum_{i=1}^{j} f^{\theta^k}_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^j}\right) \quad \text{for } 1 < j < k.$$

It is to show that this is the unique solution of minimization problem

$$\min \max_{l<k} \left\{ \left(\sum_{i=1}^{l} f_{\theta^i}\right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l}\right) \right\}$$

$$s.t. \;\; f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$f_{\theta^1} + \dots + f_{\theta^m} = 1$$

$$f_{\theta^1}\theta^1 + \dots + f_{\theta^m}\theta^m = \mu.$$

Assume, this is not true. Then let $\tilde{f}^{\theta^k} = \left(\tilde{f}^{\theta^k}_{\theta^1}, \dots, \tilde{f}^{\theta^k}_{\theta^m}\right)$ denote the solution vector of this minimization problem, which I will denote by M^{θ^k}. Let $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ be numbers such that

$$\left(\tilde{f}^{\theta^k}_{\theta^1}, \dots, \tilde{f}^{\theta^k}_{\theta^m}\right) = \left(f^{\theta^k}_{\theta^1} + \delta_{\theta^1}, \dots, f^{\theta^k}_{\theta^m} + \delta_{\theta^m}\right).$$

Since $\tilde{f}^{\theta^k} \neq f^{\theta^k}$, it holds that at least one δ_{θ^j} for $1 \leq j \leq m$ is unequal to zero. Therefore, as in the proof of Lemma **??** in section **??**, one can decompose the vector $(\delta_{\theta^1}, \dots, \delta_{\theta^m})$ into δ-sequences and if there does not exist a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, the process of decomposing into δ-sequences end with a δ-sequence of length 2, i.e. with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$.

Assume there exists a $1 \leq t \leq m$ with $\sum_{j=1}^{t} \delta_{\theta^j} > 0$. Since a bidder with valuation θ^k never expects to win against an equal type and the mean μ has to be preserved, it is not optimal for the adverse nature to put positive probability weight on types above θ^k. If there would be positive probability weight on types above θ^k, one could shift probability weight from types above θ^k and types below θ^k to type θ^k in a way which preserves the mean. Since this reduces the winning probability of a bidder with valuation θ^k, it cannot be optimal for the adverse nature to put positive probability weight on types above θ^k. Therefore, it holds that $\delta_{\theta^j} > 0$ for $j > k$. Since it must hold that $\sum_{j=1}^{m} \delta_{\theta^j} = 0$, it holds that $t < k$. Let

$$h \in \underset{l<k}{\operatorname{argmax}} \left\{ \left(\sum_{j=1}^{l} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l} \right) \right\}.$$

This implies that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right)$$

is the minimum value of the objective function of minimization problem M^{θ^k}. Since f^{θ^k} is an element of the feasible set of minimization problem M^{θ^k}, the vector f^{θ^k} cannot induce a lower value of the objective function than $\tilde{f}^{\theta^k}$. Therefore, it holds that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) \leq \left(\sum_{j=1}^{h} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right)$$

$$\Leftrightarrow \left(\sum_{j=1}^{h} f_{\theta^j} + \delta_{\theta^j} \right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^h} \right)} \leq \left(\sum_{j=1}^{h} f_{\theta^j} \right) \sqrt[I-1]{\left(\theta^k - \overline{b}^{\theta^h} \right)}$$

$$\sum_{j=1}^{h} \delta_{\theta^j} \leq 0. \tag{9}$$

By definition of the vector f^{θ^k}, it holds that

$$\left(\sum_{j=1}^{h} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) = \left(\sum_{j=1}^{t} f_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^t} \right).$$

By definition of h, it holds that

$$\left(\sum_{j=1}^{h} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^h} \right) \geq \left(\sum_{j=1}^{t} \tilde{f}_{\theta^j} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^t} \right)$$

from which follows that

$$\left(\sum_{j=1}^{h} \delta_{\theta^j} \right) \sqrt[I-1]{\theta^k - \overline{b}^{\theta^h}} \geq \left(\sum_{j=1}^{t} \delta_{\theta^j} \right) \sqrt[I-1]{\theta^k - \overline{b}^{\theta^t}}.$$

Since $\sum_{j=1}^{t} \delta_{\theta^j} > 0$, it follows that $\sum_{j=1}^{h} \delta_{\theta^j} > 0$ which is a contradiction to (9). Therefore, the process of decomposing into δ-sequences ends with some vector $\left(\delta_1^{final}, \delta_2^{final}\right)$ with $\delta_1^{final} < 0$ and $\delta_2^{final} > 0$ and there exists a θ^{final} such that

$$\sum_{j=1}^{m} \delta_{\theta^j} \theta^j > \sum_{j=1}^{m} \delta_{\theta^j} \theta^{final} = 0.$$

Since this is a contradiction to the fact that the vector $\tilde{f}^{\theta^k}$ fulfills the constraint

$$\sum_{j=1}^{m} \tilde{f}_j^{\theta^k} \theta^j = \mu,$$

one can conclude that the assumption that the solution of minimization problem M^{θ^k} does not coincide with the unique solution of the system of k linear equations as specified in Proposition 11, leads to a contradiction. Therefore, the solution of this system of linear equations is the unique distribution inducing the highest rationalizable bid of a bidder with valuation θ^k. □

Proof of Proposition 11 and Proposition 13

Proof. Given the distribution of the other bidders' valuations, the adverse nature chooses for a bidder with valuation $\theta^k \leq \mu$, the bidder expects the lowest possible utility of zero. Thus, the adverse nature does not have an incentive to deviate from this strategy. In order to choose for a bidder with valuation $\theta^k > \mu$ a distribution of the other bidders' valuations, the adverse nature has to solve the following minimization problem:

$$\min \max_{l<k} \left\{ \left(\sum_{i=1}^{l} f_{\theta^i}^{\theta^k} \right)^{I-1} \left(\theta^k - \overline{b}^{\theta^l} \right) \right\}$$

$$s.t. \quad f_{\theta^j} \geq 0 \text{ for all } 1 \leq j \leq m$$

$$f_{\theta^1}^{\theta^k} + \cdots + f_{\theta^m}^{\theta^k} = 1$$

$$f_{\theta^1}^{\theta^k} \theta^1 + \cdots + f_{\theta^m}^{\theta^k} \theta^m = \mu.$$

Since this minimization coincides with the minimization problem in the proof of Proposition 12 and the minimization problem has a unique solution, the distribution of the other bidders' valuations chosen by the adverse nature for a bidder with valuation $\theta^k > \mu$ coincides with the vector $\left(f_{\theta^1}^{\theta^k}, \ldots, f_{\theta^m}^{\theta^k}\right)$ as specified in Proposition 12. A bidder with valuation θ^k best replies to the adverse nature's strategy. If $\theta^k \leq \mu$, the bidder expects a utility of zero and is indifferent between any bid between zero and her valuation. If

$\theta^k > \mu$, the bidder is indifferent between any highest rationalizable bid of a lower type. Thus, mixing among all highest rationalizable bids of lower types is a best reply to the adverse nature's strategy. In a maximin equilibrium every player with valuation $\theta > \mu$ will mix in a way such that the adverse nature does not have an incentive to deviate from the proposed equilibrium. Thus, the strategies proposed in Propositions 11 and 13 indeed constitute a maximin equilibrium in the game with I players and an adverse nature. Analogously as in the proof of Proposition 4, one can show that bidders' strategies are equal in every outcome under maximin strategies. $\square$

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