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Sensitivity Analysis in Economic Simulations – A Systematic Approach

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Executive Summary

This paper gives a systematic introduction to sensitivity analysis in economic simulations. Sensitivity analysis studies how the variation in the numerical output of a model can be quantitatively apportioned to different sources of variation in basic input parameters. It thus provides a check of robustness of the numerical results of a model. We present a formalisation of the deterministic and the stochastic approach to sensitivity analysis. The former assumes that the basic economic parameter stems from a known interval (in higher dimensions: a compact set) and quantifies the spread of the corresponding equilibrium output variables. The latter treats the parameter as a stochastic variable with known distribution and calculates mean and variance of output variables accordingly. Deterministic sensitivity analysis can be numerically implemented by peacemeal formulae. Stochastic sensitivity analysis is implemented by a Monte-Carlo or a Gauss-Quadrature algorithm. We develop an improved version of Gauss-Quadrature based on orthogonal polynomials. In an application of both algorithms to a CGE model we demonstrate its advantages: It significantly reduces the computational burden of sensitivity analysis and tends to be more stable numerically.

Das Wichtigste in Kürze

Dieses Discussion Paper ist eine systematische Anleitung zur Sensitivitätsanalyse bei ökonomischen Simulationen. Eine Sensitivitätsanalyse untersucht, wie die Variation in den numerischen Ergebnissen eines Modells der Variation verschiedener Grundparameter zugeordnet werden kann. Auf diese Weise ermöglicht sie eine Überprüfung der Robustheit numerischer Modellergebnisse. Wir stellen eine Formalisierung des deterministischen und des stochastischen Ansatzes zur Sensitivitätsanalyse vor. Bei ersterem geht man davon aus, dass der ökonomische Grundparameter aus einem bekannten Intervall (in höheren Dimensionen: aus einer kompakten Menge) stammt und man quantifiziert die Ausdehnung der zugehörigen Modellvariablen im Gleichgewicht. Bei letzterem behandelt man den Grundparameter als stochastische Variable mit bekannter Verteilung und berechnet entsprechend Erwartungswert und Varianz der Modellvariablen. Deterministische Sensitivitätsanalyse kann numerische durch Stückformeln (engl. "piecemeal formulae") implementiert werden. Stochastische Sensitivitätsanalyse wird durch das Monte-Carlo Verfahren oder Gauss-Quadratur implementiert. Wir entwickeln eine verbesserte Version des Gauss-Quadratur-Verfahrens, die auf orthogonalen Polynomen beruht. In einer Anwendung der beiden Verfahren auf ein allgemeines Gleichgewichtsmodell zeigen wir seine Vorteile: Es reduziert den numerischen Aufwand der Sensitivitätsanalyse erheblich und ist im allgemeinen numerisch stabiler.

Sensitivity analysis in economic simulations - a systematic approach

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Abstract

Sensitivity analysis studies how the variation in the numerical output of a model can be quantitatively apportioned to different sources of variation in basic input parameters. Thus, it serves to examine the robustness of numerical results with respect to input parameters, which is a prerequisite for deriving economic conclusions from them. In practice, modellers apply different methods, often chosen ad hoc, to do sensitivity analysis. This paper pursues a systematic approach. It formalizes deterministic and stochastic methods used for sensitivity analysis. Moreover, it presents the numerical algorithms to apply the methods, in particular, an improved version of a Gauss-Quadrature algorithm, applicable to one as well as multidimensional sensitivity analysis. The advantages and disadvantages of different methods and algorithms are discussed as well as their applicability.

Keywords: Sensitivity Analysis, Computational Methods
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1 Introduction

In economics as well as other model based sciences, a modeller has to do a sensitivity analysis to show the validity of results of his numerical model simulations. A sensitivity analysis is the study of how the variation in the output of a model (numerical or otherwise) can be apportioned, qualitatively or quantitatively, to different sources of variation in input parameters. It thus allows for an assessment of the robustness of numerical results, as it translates the range (confidence intervals) of fundamental (input) parameters into the model into ranges (confidence intervals) of economic (output) variables. The econometrician Edward Leamer makes it quite clear: "A fragile inference is not worth taking seriously. All scientific disciplines routinely subject their inferences to studies of fragility. Why should economics be different? ... What we need are organized sensitivity analyses." (Leamer, 1985)

In the context of CGE models, we ask whether the choice of basic parameters of the model, e.g. elasticities or time preference parameters, lead to stable equilibrium values of economic variables, e.g. GDP or labor participation. Usually, we refer to the equilibrium of the benchmark scenario. Quite importantly, a sensitivity analysis depends on the existence of equilibria for a sufficient range of parameters: If the model is not solvable for parameter values close to the ones we have chosen as benchmark values, model results are instable and thus worthless.

Basically, there are two methodological approaches to sensitivity analysis: a deterministic and a stochastic approach. Deterministic sensitivity analysis assumes that the tuple of basic parameters is an element of a given subset of all possible parameter choices. It seeks to determine upper and lower bounds on the corresponding subset of economic outcomes of the model. Stochastic sensitivity analysis treats the vector of parameters as a stochastic variable with a given distribution, rendering economic equilibria of the model into stochastic variables. It aims at calculating the first moments of these variables, with the variance indicating the robustness of the results. Both approaches are presented and discussed in section 2.

The choice a modeller has to make in a sensitivity analysis is, however, not only a methodological, but also a numerical one. Sensitiviy analysis can involve more or less calculations of equilibria, so that usually there is a trade-off between accuracy and calculation time. This holds already true for a comparison of the deterministic and the stochastic approach, and is particularly relevant for the case of a multidimensional sensitivity analysis. In section 3, we present and discuss different algorithms. In particular, an improved version of an algorithm based on Gauss-Quadrature is developed.

Section 4 presents a simple CGE model in Markusen's (2002) spirit. We conduct a stochastic sensitivity analysis with respect to demand elasticities with

a Monte-Carlo algorithm and Gauss-Quadrature. From the results we see that Monte Carlo is more burdensome computionally. Also, it appears less stable numerically than Gauss Quadrature.

2 Theory of Sensitivity Analysis

2.1 Mathematical Preliminaries

Before conducting a sensitivity analysis, indeed even before implementing a numerical model, the modeller has to understand whether his model is in fact solvable. Proving existence (and sometimes uniqueness) of economic equilibria is a challenge of its own. There exist, however, a set of mathematical theorems that - if applicable - guarantee the existence of solutions. For example, the existence of an equilibrium in a general equilibrium framework is proved by recurring on Kakutani's fix-point theorem (cf. Mas-Collel et al. 1995, chapter 17). In the sequel, we assume that models are (uniquely) solvable for at least some parameter values and discuss under which circumstances a sensitivity analysis is possible.

An equilibrium of a computable general equilibrium (CGE) model takes the mathematical form of a solution to a system of (non-linear) equations

$$F(x^*, a) = 0,$$

where $x^* \in \mathbf{R^n}$ is a vector of (equilibrium) state variables of the economy (such as capital or wage) and $a \in \mathbf{R^d}$ a vector of parameters of the economy (such as demand elasticity or time preference). G is a continously differentiable function

$$\mathbf{R^n} \times \mathbf{R^d} \rightarrow \mathbf{R},$$

that consists of first order conditions and (budget) constraints. This CGE model will be our standard example. Note, however, that the methods for sensitivity analysis presented can be generally be applied to economic models: The decisive distinction is the one between economic state variables x and basic parameters a.

Sensitivity analysis is concerned with the effect that (minor) changes of basic parameters Δa have on equilibrium state variables Δx^*. This notion will be formalized subsequently. At this point we ask under what circumstances an equilibrium x^{**} exists for a parameter value a' in a neighborhood of a, e.i. $a' \in B_\epsilon(a) = \{\tilde{a}\ s.t.\ |\tilde{a} - a| < \epsilon\}$. This is an important question: The existence of economic equilibria in a neighborhood of a is the theoretical prerequisite for sensitivity analysis. The implicit function theorem gives a definite answer.

Theorem 1 (Implicit function theorem) *If $det|\nabla_{x^*} F(x^*, a)| \neq 0$, then there exists an open neighbourhood $\mathcal{U}(a) \subset \mathbf{R^d}$ of a and a continously differentiable function $h : \mathcal{U}(a) \rightarrow \mathbf{R^n}$ that maps any vector of parameters on the corresponding equilibrium vector.*

Proof. see Rudin (1976), ch. 9., p. 223. ■

Thus we learn that there is some $\epsilon > 0$ so that the existence of equilibria in a ϵ-neighborhood of a is guaranteed whenever F is a regular function at a. Generally speaking, we would expect F to be regular as long as first order conditions and constraints are independent. While it may be difficult to prove the assertion in some cases, it can be checked without problem numerically.

We have formulated the implicit function theorem for the case of CGE models. Simular formulations can be given for partial equilibrium models that are characterized by first and second order conditions and, possibly, additional constraints. In the case of economic optimization problems, the role of the implicit function theorem is taken by the theorem of the maximum (cf. MasCollel et al. (1995), Mathematical Appendix M.K).

So far we have neglected the notion of uniqueness of equilibria. While in principle sensitivity analysis can be conducted in the presence of multiple equilibria as well, uniqueness facilitates the analysis considerably. It is usually ensured by adequate convexity assumptions (cf. MasCollel et al. (1995), ch. 17). In the more general case of multiple equilbria, caution is warranted. In this case,

$$h : \mathcal{U}(a) \rightarrow \mathcal{P}(\mathbf{R^n})$$

is a correspondence, mapping the vector of parameters into a set of solutions

$$h(a) = \{x_1^*, x_2^*, ..., x_m^*\}.$$

Ignoring multiplicity can seriously blur a sensitivity analysis whenever a numerical solver 'jumps' from a solution x_j^* to some other solution x_i^* along changes of underlying parameters a. In that case, sensitivity of an equilibrium with respect to the basic parameters can be seriously exaggerated: Instead of following the initial equilibrium x_j^* along a continous path for changes of a (cf. Judd (1998), p. 179), a numerical discontinuity occurs and the new equilibrium $\tilde{x}_i^*$ is more distant to x_j^* than the correct equilibrium $\tilde{x}_j^*$. The best provision against such fallacies is the calculation of all equilibria along the path of change; however, this can entail a considerable computational effort. At this point, we will not discuss the issue any further.

2.2 Deterministic Sensitivity Analysis

Sensitivity analysis is sometimes called robustness analysis. This term highlights its motivation: Assuming that we do not know the basic set of parameters exactly, how robust are the economic state variables in an equilibrium with respect to changes in the parameters? The **deterministic approach** to sensitivity analysis states is that there exists one true vector of economic parameters $a^* \in \mathbf{R^d}$, but that -instead of a^*- we only know its neighbourhood $\mathcal{A}$. Usually, we choose one vector of parameters $\hat{a} \in \mathcal{A}$ and call it the *benchmark scenario*. The point of sensitivity analysis then is to investigate whether equilibria vary considerably across $h(\mathcal{A})$ in comparison to the benchmark equilibrium $h(\hat{a})$.

Mathematically speaking, deterministic sensitivity analysis amounts to a geometric problem: Determine the relation of the volume of the image of $\mathcal{A}$ under h and the size of $h(\hat{a})$, weighted with a scaling factor w_k in each dimension[1]

$$\frac{vol(im(w * h))}{\|w * h(\hat{a})\|} = \frac{\int_{\mathcal{A}} \sqrt{\det(\partial_{ij}(w * h(a)))} da_1 ... da_d}{(w * h(\hat{a}), w * h(\hat{a}))},$$

where the vector w specifies the relative weight we want to attach to the different economic variables in equilibrium[2]. The findings of the model are robust whenever the relation is sufficiently small, where the assessment of sufficiency is left to the reader. The formal definition we have just given is a generalization of the more familiar notion of sensitivity analysis in one dimension: E.g. we might ask how big is maximal interval of values of GDP engendered by a model for a given interval of demand elasticities. We will discuss the issue in more detail in the next section where we present the *piecemeal approach* to sensitivity analysis.

2.3 Stochastic Sensitivity Analysis

The **stochastic approach** to sensitivity analysis takes a different view of the basic problem: It treats the vector of basic parameters as a stochastic variable a with values $a \in \mathcal{A} \subset \mathbf{R^d}$. The distribution G of a is given. While somewhat counterintiutive in the first place, the approach is in line with econometric estimations. These do not only produce mean values for parameters such as demand elasticities, but confidence intervals and higher moments for them. Under stochastic sensitivity analysis, h becomes a mapping onto a stochastical variable $x^* = h(a)$ of equilibria.

[1] By $'*'$ we denote component-wise multiplication of vectors.

[2] We do not have to attach a weight to an economic variable explicitly - instead we assess sensitivity by comparing the effect of changes in a on each dimension of the equilibrium state variable x. Implicitly, however, we will attach some weight to each dimension by accepting a certain outcome of the sensitivity analysis.

We then calculate the mean and the variation of the equilibrium vector x^*:

$$m = E[h(a)] = \int_{\mathcal{A}} h(a)dG, \tag{1}$$

$$v = Var[h(a)] = E[(h(a) - m)^2] = \int_{\mathcal{A}} (h(a) - m)^2 dG. \tag{2}$$

Attaching different weights to different economic variables, the stochastic sensitivity analysis assesses the size of

$$\sum_{k=1}^{n} w_k \frac{v_k}{m_k},$$

where index k is running over the dimension of x. In words: Given a distribution of basic parameters, we investigate the most likely equilibrium (the mean). We assess its robustness by assessing the relative size of the variance of equilibria with respect to the mean, possibly attaching different weights to different economic variables.

3 Practical Sensitivity Analysis

Having formalized the notion of sensitivity analysis in the preceeding section we now present the practical implementation of the (somewhat abstract) concepts. For the sake of clarity in the sequel we assume that all dimensions are weighted equally, thus dropping the vector w from all formulae.

3.1 The Piecemeal Approach

In a **piecemeal approach** to sensitivity analysis, we calculate

$$\Delta = \max_{a_i, a_j \in \{a_1, \ldots, a_M\}} |h(a_i) - h(a_j)|$$

for a set of representative parameters $a_i \in \mathcal{A}$. The relation of Δ to the weighted benchmark equilibrium $h(\hat{a})$ is used to assess the sensitivity of the model at equilibrium $h(\hat{a})$. The piecemeal approach is kindred to a deterministic sensitivity analysis. But instead of calculating the volume of $\mathcal{A}$ under h, it focusses on the maximal intervals of economic variables engendered by the set of parameters $a \in \mathcal{A}$.

If the set contains $\underline{a} = \mathbf{argmin}_{\mathbf{a} \in \mathcal{A}} \mathbf{h}(\mathbf{a})$ and $\overline{a} = \mathbf{argmax}_{\mathbf{a} \in \mathcal{A}} \mathbf{h}(\mathbf{a})$, then the following inequality holds (to facilitate the presentation and without loss of generality, we set $n = 1$):

$$\frac{vol(im(h))}{|h(\hat{a})|} \leq vol(\mathcal{A})\frac{h(\overline{a}) - h(\underline{a})}{|h(\hat{a})|} = vol(\mathcal{A})\frac{\Delta}{|h(\hat{a})|}.$$

A piecemeal approach can give a good idea of the sensitivity of the model if the set of parameters $a_i \in \mathcal{A}$ is sufficiently representative.

As an example, let both $n = 1$ and $d = 1$. Then

$$h : [\underline{a}, \overline{a}] \rightarrow [\underline{x}, \overline{x}]$$

for some scalar parameters $\underline{a}$, $\overline{a}$, $\underline{x}$ and $\overline{x}$. If h is monotonously increasing, then

$$\frac{vol(im(h))}{h(\hat{a})} \leq (\overline{a} - \underline{a})\frac{h(\overline{a}) - h(\underline{a})}{h(\hat{a})}.$$

In the one-dimensional case, the piecemeal approach and the more formal definition of deterministic sensitivity analysis given in the first section virtually coincide. Generally speaking, while our formal definition captures the intention of sensitivity analysis more accurately, the piecemeal approach is more practicable.

3.2 The Monte-Carlo Approach

The **Monte-Carlo** approach is the first of two practical implementations of stochastic sensitivity analysis. Both mean and variance of equilibrium x^*, as defined by equation 1 and 2, are approximated in the following way: Draw a (large) set of realisations $\{a_1, ..., a_M\}$ from the distribution $G(\mathcal{A})$ and calculate

$$m = E[h(a)] \approx \frac{1}{M}\sum_{i=1}^{M} h(a_i) = \tilde{m}, \tag{3}$$

$$v = Var[h(a)] \approx \frac{1}{M}\sum_{i=1}^{M} (h(a_i) - \tilde{m})^2 = \tilde{v}. \tag{4}$$

The sums of the right-hand side converge stochastically to the true values of m and v. Beyond mean and variance of the stochastic variable x^*, we can easily approximate its distribution $h \circ G$. A great disadvantage of the Monte-Carlo approach is that in order to assure convergence, the number of draws M has to be high and thus the approximation is numerically costly. This is a problem in particular when the space of parameters a is high dimensional - the curse of dimensionality drives up the number of necessary draws exponentially (cf. Judd).

3.3 The Gauss-Quadrature Approach

The second way of practically implementing stochastic sensitivity analysis is by **Gauss quadrature** - in fact a numerical method to approximate integrals (cf. Stoer). Remember that we intent to approximate mean and variance, that are defined by integrals of the distribution of basic parameters a. We want to do so using a rather small number L of function evaluations $h(.)$.

Essentially, the Gauss quadrature gives us nodes x_i and weights ω_i to approximate the (one dimensional) integral

$$\int_a^b f(x)\omega(x)dx \approx \sum_{i=1}^{L} \omega_i f(x_i). \tag{5}$$

In our specific case, we look for nodes a_i and weights g_i to approximate mean, equation 1, and variance, equation 2, of equilibria. Here and in the following, we assume that probability distribution G can be represented by a (continuous) probability densitiy function $g(.)$. While somewhat limiting the applicability of the procedure, we can safely say that all economicly sensible distributions should fulfill this assumption.

$$m = \int_{\mathcal{A}} h(a)dG = \int_{\mathcal{A}} h(a)g(a)da \approx \sum_{i=1}^{L} g_i h(a_i) = \tilde{m}, \tag{6}$$

$$v = \int_{\mathcal{A}} (h(a)-m)^2 dG = \int_{\mathcal{A}} (h(a)-m)^2 g(a)da \approx \sum_{i=1}^{L} g_i (h(a_i)-\tilde{m})^2 = \tilde{v}, \tag{7}$$

where again, the dimension of economic variables is set to $n = 1$ (we present the generalisation to several variables below).

In the following, we develop a version of Gauss quadrature new to computational economics, in that it builds on *orthogonal polynomials*. While somewhat complicating the straightforward Gauss quadrature algorithm commonly used in economics (cf. Arndt 1996 and DeVuyst and Preckel 1997) conceptually, our approach simplifies the computation of a sensitivity analysis in cases of standard probability distributions, increasing the approximation quality at the same time. This is possible because the optimal nodes x_i turn out to be zeros of orthogonal polynomials. They have to be linearly transformed to fit the respective interval but can otherwise be taken from an existing table. In contrast, in Arndt's (1996) algorithm, the nodes are the solution of a system of non-linear equations (similarly in DeVuyst and Preckel 1997).

We define orthogonality in this context by the scalar product

$$(f_1, f_2)_g = \int_{\mathcal{A}} f_1(a) f_2(a) dG = \int_{\mathcal{A}} f_1(a) f_2(a) g(a) da \tag{8}$$

defines a scalar product $(.,.)_g$. We refer to orthogonality with respect to this scalar product. The following lemma holds:

Lemma 2 (Gram-Schmidt, Weierstrass) *For any scalar product $(.,.)$ on the space of continuous functions $\mathcal{C}([\underline{a}, \overline{a}])$, there is a complete system of orthogonal polynomials $\{p_0, p_1, ...|(p_i, p_j) = 0, i \neq j\}$.*

Proof.

For any given scalar product, orthogonal polynomials can be constructed from monomials 1, x, x^2,... by the Gram-Schmidt procedure

$$p_0 \equiv 1 \qquad p_i(x) = x^i - \sum_{j=1}^{i-1} \frac{(p_j, x^i)}{(p_j, p_j)} p_j.$$

We thus obtain an infite sequence of orthogonal polynomials. As for completeness, we know that the polynomials $(p_0(x), p_1(x), ..., p_n(x))$ span the same linear subspace of the space of continuous functions as the monomials $(1, x, x^2, ..., x^n)$. Consequently, we can apply Weierstrass' approximation theorem which states that the space of polynomials is dense in the space of continous functions (cf. Rudin 1976, ch. 7, p. 159) and the completeness of the family of orthogonal polynomials ensues.

■

There are well known examples of orthogonal polynomials, the best known being Legendre, Tchebychev, Laguerre and Hermite polynomials.

Examples of families of orthogonal polynomials

Name	$g(x)$	$[a, b]$	Definition
Legendre	1	$[-1, 1]$	$P_k(x) = \frac{(-1)^k}{2^k k!} \frac{d^k}{dx^k}[(1-x^2)^k]$
Tschebyscheff	$(1-x^2)^{-\frac{1}{2}}$	$[-1, 1]$	$T_k(x) = \cos(k \cos^{-1}(x))$
Laguerre	$\exp(-x)$	$[0, \infty)$	$L_k(x) = \frac{\exp(x)}{k!} \frac{d^k}{dx^k}(x^k \exp(-x))$
Hermite	$\exp(-x^2)$	$(-\infty, \infty)$	$H_k(x) = (-1)^k \exp(x^2) \frac{d^k}{dx^k}(\exp(-x^2))$

The proof of lemma is constructive, so that for any density function $g(a)$ orthogonal polynomials can be constructed from monomials $1, x, x^2, ...$ For a general distribution probability distribution $G(a)$, their calculation can entail considerable

To proceed, we need one property of orthogonal polynomials.

Lemma 3 *The zeros* $\{a_1, a_2, ..., a_l\}$ *of* $p_l(a)$ *are real and distinct.*

Proof. Stoer and Bulirsch (1990), ch 3.6, p. 173. ■

It is because that they are real and distinct that the zeros of a orthogonal polynomial are a possible choice of nodes for the evaluation of the approximation formula 5. The following theorem shows that they are indeed a good choice.

Theorem 4 *Let* $\{a_1, a_2, ..., a_l\}$ *be the zeros of* $p_l(a)$ *and* $g_1, ..., g_l$ *be the solution of the system of linear equations*

$$\sum_{i=1}^{n} g_i p_k(a_i) = \begin{cases} (p_0, p_0) & : \quad k = 0 \\ 0 & : \quad k = 1, 2, ..., l-1 \end{cases}$$

Then $g_i > 0$ *for* $i = 1, 2, ..., l$ *and*

$$\int_{\underline{a}}^{\overline{a}} p(a) g(a) da = \sum_{i=1}^{l} g_i p(a_i)$$

for all $p \in \Pi_{2l-1} = \ll p_0, ..., p_{2l-1} \gg$.

Proof. Stoer and Bulirsch (1990), ch 3.6, p. 176. ■

In words: For a given density function $g(a)$ (i.e. probability distribution G), we calculate the zeros $a_1, ..., a_l$ of the corresponding orthogonal polynomial of degree l. Calculating the weights $g_1, ..., g_l$ from a suitable system of linear equations, we obtain a integration formula of type 5 that integrates polynomials up to degree 2l-1 exactly.

Thus, for our purpose of numerical integration, we have to calculate the zeros of orthogonal polynomials and weights corresponding to the probability distribution G with weight function $g(a)$. *However, we have to do so only once for a given* G. While in general the numerical determination of the zeros of orthogonal polynomials for a given distribution may be tricky, in the case of standard probability distributions we have no problem. A look at the table of orthogonal polynomials confirms that for uniform distributions, we can use Legendre polynomials, and Hermite polynomials for normal distributions. This facilitates our task considerably: We can either (easily) calculate the zeros numerically from the

defining formulae for Legendre or Hermite polynomials, or take these from published tables. The next section shows an application of the method presented here to a simple example.

In higher dimensions $n > 1$, integrals can be approximated by product rules, combining one-dimensional nodes and weights

$$\int_{\underline{a}^1}^{\overline{a}^1} ... \int_{\underline{a}^d}^{\overline{a}^d} f(a^1, ..., a^d) g^1(a^1) ... g^d(a^d) da^d ... da^1$$
$$\approx \sum_{i_1=1}^{n} ... \sum_{i_d=1}^{n} g^1_{i_1} ... g^d_{i_d} f(a^1_{i_1}, ..., a^d_{i_d}).$$

We can thus approximate joint distributions of economic parameters instead of analysis the robustness of results with respect to single parameters separately. Note that by specifying probability density functions $g^i(a^i)$ we implicitly assume that the probability distributions are independent. In this case, higher-dimensional Gauss-Quadrature is straightforward - we only have to combine the sums for each dimension. As we will see in the example in the next section, it is in this case that Gauss-Quadrature integration has a great advantage over Monte-Carlo simulations, as the evaluation of nodes increases exponentially with each dimension, making MC simulations simply too expensive.

4 A Simple Example

In this section we present a simple numerical example to demonstrate the use of our approach to sensitivity analysis. We have chosen a CGE framework, the example is implemented in GAMS (Brooke, Kendrick and Meeraus, 1988).
We consider a two-by-two closed economy in the spirit of Markusen (2002). It represents an economy with two commodity goods, `X` and `Y`, two factors (capital `K` and labor `L`), and one single representative agent. The goods are produced through constant returns to scale production activities which combine primary factor inputs. As usual, we use a balanced equilibrium data set given by the square accounting matrix below. The accounts labeled `X` and `Y` in this matrix refer to markets for final commodities; account `W` correspond to final consumption. The `RA` account corresponds to the representative agent. It defines both the endowment and expenditures for the model's single representative agent.

Accounting matrix with benchmark flows:

Markets	X	Y	W	RA	Row sum
PX	100		-100		0
PY		100	-100		0
PW			200	-200	0
PL	-40	-60		100	0
PK	-60	-40		100	0
Column sum	0	0	0	0	

The accounts of the matrix do not by themselves completely characterize a general equilibrium framework because they provide a variety of benchmark value shares. Hence, a model formulation additionally relies on assumptions about elasticities of substitution in the various sectors. In our simple model there are three elasticities of substitution: esubx denotes the elasticity of substitution between inputs to X production, esuby the elasticity of substitution between inputs to Y production and esubw the one between inputs to final consumption. In this section we present a sensitivity analysis with respect to these three elasticities which are examples of economic fundamentals and typical parameters for a sensitivity analysis.
Numerically, we formulate the model as a mixed complementarity problem (MCP) using the share form for the specification of functions forms (cost and expenditure functions). The algebraic formulation is implemented in GAMS. For our example, we choose $\texttt{esubx} = \texttt{esuby} = \texttt{esubw} = 0.5$ as starting point and introduce an exogenous labor tax on good X by setting $\texttt{TAX}_\texttt{L}\texttt{X} = 1$.
In the following we compare the two different implementation methods of stochastic sensitivity analysis that were presented in Section 3.2 and in Section 3.3: the Monte-Carlo (MC) analysis and the Gauss-Quadrature (GQ). We calculate both the case of a uniform and a normal distribution in one dimension, as well as the multidimensional case of a uniform distribution.

4.1 One-Dimensional Sensitivity Analysis

First, we assume that the elasticity of substitution esubx is **uniformly distributed** on the interval 0.25 and 0.75 and that the other two elasticities in question are constant and equal to 0.5. Due to the assumption of uniform distribution we choose Legendre polynomials for the Gauss-Quadrature.
For the sensitivity analysis according to the **Monte-Carlo** approach we randomly draw values for esubx from the interval [0.25,0.75]: $\texttt{esubx} = \texttt{UNIFORM}[0.25, 0.75]$.
We calculate the mean and variance of normalized equilibrium variables X, Y and

W according to the following formulas:

$$\begin{aligned} \texttt{mean(output)} &= \texttt{SUM(i, results(i, output))/card(i)}; \\ \texttt{variance(output)} &= \texttt{SUM(i, (sqr(results(i, output) - mean(output))))/card(i)}; \end{aligned}$$

card(i) is equal to the number of drawings of esubx from the interval [0.25,0.75]. results(i, output) denotes the (normalized) equilibrium values for the set output $=$ {X, Y, Z} given the i-th drawing of the elasticity. sqr denotes the square root.

For the sensitivity analysis using **Gauss-Quadrature with Legendre polynomials** we use a MATLAB routine to calculate the zeros of Legendre polynomials, legendrenodes, and the associated weights, legendreweights. Hereby, we use the recursive formula for polynomials given by the Gram-Schmidt procedure. We calculate the zeros for these polynomials and solve the equation system given in Theorem 4 with respect to the corresponding weights. Zeros and weights are saved in a gdx file. Alternatively, schedules with zeros and weights for Legendre polynomials can also be taken from existing tables which are published in the internet or in math books (see for instance Belousov 1962). Once this gdx file is created, it can be used to run sensitivity analysis for every CGE model where the parameters are assumed to be uniformly distributed.
In our GAMS routine, the variables legendrenodes and legendreweights are loaded from the gdx file[3] and transformed into the variables grid and weights. We choose maxdegree which is the degree of the maximal Legendre polynomial that we want to consider. It also refers to the rows with zeros and weights that are loaded from the gdx file. A higher value of maxdegree raises the quality of the Legendre approach but is computationally more expensive. Additionally we have to transform legendrenodes linearly from the interval $[-1, 1]$ into a new vector grid on interval $[a, b] = [0.25, 0.75]$:

$$\texttt{grid(i)} = (\texttt{b} - \texttt{a}) * (\texttt{zerosval}('\texttt{maxdegree}', \texttt{i}) + 1)/2 + \texttt{a}.$$

This is due to the fact that standard Legendre polynomials are defined on the interval $[-1, 1]$. zerosval('maxdegree', i) denotes the i-th zero point for a Legendre polynomial of degree maxdegree which is loaded from legendrenodes. We calibrate the equilibrium values for X, Y and W by a loop of the model and vary the elasticity esubx across grid. We calculate the means and variances of these economic variables by summing up results, weighted by weights which can be drawn from the gdx file directly:

$$\begin{aligned} \texttt{mean(output)} &= \texttt{SUM(i, weights(i) * results(i, output))}, \\ \texttt{variance(output)} &= \texttt{SUM(i, weights(i) * (sqr(results(i, output)) - mean(output)))}. \end{aligned}$$

[3]We make the files available on our webside.

This implements the formulas (2) and (3) from Section 3. Table 1 reports the results for the Monte-Carlo approach (MC) and the Gauss-Quadrature with Legendre polynomials (GQ). For Monte-Carlo we choose 100, 1,000 and 6,000 drawings of the random variable `esubx` from the interval $[0.25, 0.75]$. For each drawing the equilibrium values of our model have to be calculated. Hence, computing time is proportional to the number of drawings. For the Gauss-Quadrature we choose the degree of the maximal Legendre polynomial to be equal to 10, 20 and 40. These values correspond to the numbers of zeros of the polynomial function (i.e. to the length of grid) at which the equilibrium problem has to be solved. Thus, they are also equal to the number of model runs. Table 1 shows the means and variances of the variables `X`, `Y` and `W` depending on the chosen sensitivity approach and the number of runs. We can see directly, that the results for the two different approaches converge with increasing number of model evaluations. Gauss-Quadrature converges more quickly to both mean and variance than Monte Carlo. At the same time, it is numerically much more efficient. It reduces calculation time by a factor of at least 50 in comparison to Monte Carlo.

Table 1: Uniformly distributed variable `esubx`

Means of X,Y and W

Name (Runs)	MC (100)	MC (1000)	MC (6000)	GQ (10)	GQ (20)	GQ (40)
X	-8.990	-9.022	-9.030	-9.036	-9.026	-9.030
Y	7.582	7.575	7.572	7.576	7.568	7.570
W	-1.396	-1.418	-1.421	-1.426	-1.427	-1.421

Variances of X,Y, and W

Name (Runs)	MC (100)	MC (1000)	MC (6000)	GQ (10)	GQ (20)	GQ (40)
X	0.027	0.031	0.031	0.031	0.031	0.031
Y	0.004	0.005	0.004	0.004	0.004	0.005
W	0.014	0.016	0.016	0.018	0.015	0.016

4.2 Multidimensional Sensitivity Analysis

The sensitivity analysis can be conducted in an equivalent way for the elasticities esubx and esubw separately or for all elasticities simultaneously. In this section we assume that they are independently, uniform distributed. For the case of Monte Carlo, the three elasticities of substitution ar drawn randomly but independently from given intervals $[a_x, b_x]$, $[a_y, b_y]$ and $[a_w, b_w]$. For the **Gauss-Quadrature** the Legendre nodes are transformed three times from the gdx file. In both cases, the equilibrium values are calculated by three loops over the three grids. Mean and variance can be derived by weighting the results with the associated weights:

$$\begin{aligned} &\texttt{mean}(\texttt{output}) = \\ &\texttt{SUM}(\texttt{i}, \texttt{SUM}(\texttt{ii}, (\texttt{SUM}(\texttt{iii}, \texttt{weights}_\texttt{x}(\texttt{i}) * \texttt{weights}_\texttt{y}(\texttt{ii}) * \\ &\texttt{weights}_\texttt{w}(\texttt{iii}) * \texttt{results}(\texttt{i}, \texttt{ii}, \texttt{iii}, \texttt{output}))))) \end{aligned}$$

$$\begin{aligned} &\texttt{variance}(\texttt{output}) = \\ &\texttt{SUM}(\texttt{i}, \texttt{SUM}(\texttt{ii}, \texttt{SUM}(\texttt{iii}, \texttt{weights}_\texttt{x}(\texttt{i}) * \texttt{weights}_\texttt{y}(\texttt{ii}) * \\ &\texttt{weights}_\texttt{w}(\texttt{iii}) * \texttt{sqr}(\texttt{results}(\texttt{i}, \texttt{ii}, \texttt{iii}, \texttt{output}) - \texttt{mean}(\texttt{output}))))) \end{aligned} \quad .$$

$\texttt{results}(\texttt{i}, \texttt{ii}, \texttt{iii}, \texttt{output})$ denotes the equilibrium values for the set $\texttt{output} = \{\texttt{X}, \texttt{Y}, \texttt{Z}\}$ given the (i)-th zero of the chosen Legendre polynomial for esubx, the (ii)-th zero for the Legendre polynomial of esuby and the (iii)-th zero of the one for esubw. $\texttt{weights}_\texttt{x}$, $\texttt{weights}_\texttt{y}$ and $\texttt{weight}_\texttt{w}$ refer to the Legendre weights belonging to the elasticities esubx, esuby, and esubw. Please note that different values for $\texttt{maxdegree}_\texttt{x}$, $\texttt{maxdegree}_\texttt{y}$ and $\texttt{maxdegree}_\texttt{w}$ can be chosen corresponding to different degrees of polynomials.
In our example run, we set $[a_x, b_x] = [a_y, b_y] = [a_w, b_w] = [0.25, 0.75]$ and $\texttt{maxdegree} := \texttt{maxdegree}_\texttt{x} = \texttt{maxdegree}_\texttt{y} = \texttt{maxdegree}_\texttt{w}$. To test the Monte-Carlo (MC) approach we choose $10 \times 10 \times 10$, $20 \times 20 \times 20$, $30 \times 30 \times 30$ and $40 \times 40 \times 40$ independent drawings from $[0.25, 0.75] \times [0.25, 0.75] \times [0.25, 0.75]$ which correspond to $1,000$, $8,000$, $27,000$ and $64,000$ model runs. To apply the Gauss-Quadrature (GQ) approach we study the cases $\texttt{maxdegree} = 10$, $= 20$,$= 30$ and $= 40$ with $1,000$, $8,000$, $27,000$ and $64,000$ model evaluations. The results are reported in Table 2.

We notice the rapid convergence for the Gauss-Quadrature. Using this approach, the approximated mean does not change anymore after 8,000 runs. In contrast, there is no trend observable from the results of the Monte Carlo approach, even after 64,000 runs. Besides, the values for the means and variances differ considerably from the values derived by Gauss-Quadrature. It can be suggested that the number of model evaluations for Monte-Carlo is still far too low to

achieve acceptable results, despite the high computational burden.

Table 2: Uniformly distributed variable esubx, esuby and esubw

Means of X,Y and W

Name (Runs)	MC (1,000)	MC (8,000)	MC (27,000)	MC (64,000)	GQ (1,000)	GQ (8,000)	GQ (27,000)	GQ (64,000)
X	-9.852	-11.189	-10.901	-11.165	-9.002	-9.001	-9.001	-9.001
Y	8.435	9.574	9.323	9.680	7.571	7.573	7.573	7.573
W	-1.475	-1.702	-1.652	-1.628	-1.404	-1.406	-1.406	-1.406

Variances of X,Y, and W

Name (Runs)	MC (1,000)	MC (8,000)	MC (27,000)	MC (64,000)	GQ (1,000)	GQ (8,000)	GQ (27.000)	GQ (64.000)
X	5.719	4.570	4.570	3.462	5.435	5.438	5.441	5.440
Y	5.605	3.758	4.358	3.324	5.331	5.318	5.318	5.317
W	0.068	0.095	0.083	0.071	0.062	0.062	0.062	0.062

4.3 Sensitivity Analysis with Normal Distribution

Similarly, sensitivity analysis can be conducted for normally distributed parameters. In our model we consider the case of a log-normal distributed elasticity `esubx`. Note that we cannot assume a normal distribution for an the elasticity because it is not defined for negative numbers. Instead, we assume that $\ln(\texttt{esubx})$ is normal distributed with mean 0.5 and variance 0.4. For the **Monte-Carlo** simulation, random draws are generated by

$$\texttt{esubx} = \exp(\texttt{NORMAL}[0.5, 0.4]).$$

The **Gauss-Quadrature** method used here is based on Hermite polynomials corresponding to the normal distribution. We calculate the zeros and weights of the Hermite polynomials again in a separate MATLAB program and store them in a `gdx`-files as `hermitenodes` and `hermiteweights`. It is also possible to take these

values from published tables like e.g. from Greenwood and Miller (1948). Then, we proceed in the same way as in the case of uniform distribution.
The results are reported in Table 3. Again, the values imply convergence of the Gauss-Quadrature results. But also an additional disadvantage of the Monte Carlo approach becomes visible. Our Monte-Carlo simulation was only feasible for small numbers of runs up to 628. After that, the random generator of GAMS started to produce repeatedly values of `exubx` for which our programm failed to find equilibrium solutions. Dropping these values from our sample produces an unknown bias to the MC estimations. Unfortunately, this procedure is very common in practice. In contrast, the Gauss-Quadrature is numerically much more stable.

Table 3: Log-normal distributed variable esubx with $\sigma^2 = 0.4,$ **and** $\mu = 0.5$

Means of X,Y and W

Name (Runs)	MC (100)	MC (500)	MC (628)	GQ (10)	GQ (20)	GQ (40)
X	-9.804	-9.844	-9.842	-9.835	-9.842	-9.853
Y	7.559	7.572	7.572	7.571	7.578	7.592
W	-1.878	-1.901	-1.900	-1.883	-1.891	-1.910

Variances of X,Y and W

Name (Runs)	MC (100)	MC (500)	MC (628)	GQ (10)	GQ (20)	GQ (40)
X	0.069	0.075	0.070	0.178	0.168	0.168
Y	0.004	0.004	0.003	0.006	0.007	0.005
W	0.019	0.022	0.020	0.051	0.042	0.052

5 Conclusion

Due to its general importance in economic modelling, sensitivity analysis merits a systematic understanding by economic modellers. This paper hopes to contributes to such an understanding and to serve as a guide in applying the appropriate algorithm. Dimensionality of sensitivity analysis (the number of parameters that are varied simultaneously) makes an important difference. Moreover,we argue that generally stochastic sensitivity analysis gives more and better insights than deterministic sensitivity analysis. However, it is also more burdensome computationally. Comparing stochastic methods, the paper shows that Monte-Carlo methods are easily applicable, but computationally expensive. Gauss-Quadrature methods reduce the computational burden and thus are suitable for higher dimensional problems. In an application of both methods Gauss-Quadrature turns out to be more stable than Monte-Carlo, which can be explained by the fact that it avoids the extreme parts of the distribution.

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