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On the Transition from Instantaneous to Time-Lagged Capital Accumilation

The Case of Leontief Type Production Functions

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Non-Technical Summary

Production processes can be considered as transforming inputs into outputs. In economic modelling it is usually assumed that this happens instantaneously. However, a "real world" production process takes time, meaning that the outputs are available with a certain time lag after assigning the inputs. The time lag may be substantial in the production of capital goods such as plants, buildings or larger network infrastructure.

We analyze the question how this time-lag influences the optimal investment over time at hand of an optimal control capital accumulation model. As known from the time-to-build literature, time-lagged optimal control problems may exhibit a qualitatively different system dynamics as compared to instantaneous capital accumulation models, namely cyclical and exponentially damped oscillating optimal investment paths. We confirm this system dynamics for the case of Leontief-type production functions and show under which conditions the optimal path is dominated by one major cycle.

As time-lagged optimal control problems are not analytically soluble, even in the linear approximation around the stationary state, state-of-the-art numerical optimization methods are used for the second major contribution of the analysis: the illustration of the transition from instantaneous to time-lagged production. We show the formation of the major cycle and illustrate that while for small time lags instantaneous production neoclassical economic theory is a good approximation, the validity of this approximation is challenged for large time lags. Calculating the major cycle already gives a good impression of what to expect from the optimal paths of investment.

On the Transition from Instantaneous to Time-Lagged Capital Accumulation

The Case of Leontief-Type Production Functions

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April 2005

Abstract: We formulate an optimal control capital accumulation model with a Leontief-type production function and an exogenously given time-lag between investment and the accumulation of the capital stock, to analyze the qualitative and quantitative influence of time-lags on the system dynamics. As known from the time-to-build literature, optimal investment paths for positive and finite time-lags are in general cyclical, in contrast to the monotonic optimal paths for instantaneous capital accumulation. We show that the transition between instantaneous and time-lagged capital accumulation is continuous, in the sense that the greater is the time-lag between investment and capital accumulation, the more likely and more pronounced becomes cyclical behavior of the optimal paths.

Keywords: cyclical optimal paths, numerical optimization, time-lagged optimal control, time-to-build

JEL-Classification: E32, C63, C61

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* We are grateful to Jürgen Eichberger, Malte Faber, Christoph Heinzel and John Proops for comments on an earlier draft. Ulf Moslener and Ralph Winkler gratefully acknowledge financial support by the Deutsche Forschungsgemeinschaft (German Research Foundation) under the postgraduate programme "Environmental and Resource Economics".

1 Introduction

All production takes time. That is, the transformation of inputs into outputs does not occur instantaneously. This ubiquitous experience has influenced economic theory in various ways and at different times. In his formulation of Austrian capital theory, von Böhm-Bawerk ([1889]1921) applied the average investment period, i. e. the average time span between the assignment of the non-produced inputs and the finished consumption goods in the production process, to avoid the problem of the ambiguity of an aggregate measure of capital. This time aspect of production was revived in the 1970s by the neo-Austrian capital theories (e. g. von Weizsäcker 1971, Hicks 1973 and Faber 1979). El-Hodiri et al. (1972) derived a generalized maximum principle for a growth model with heterogenous capital goods and exogenously given and constant time-lags between control and state variables. Benhabib and Rustichini (1991) interpreted the time structure of production as a special case of vintage-capital models, which they called *gestation lags*. The time aspect of production has also been discussed in the macroeconomic real business cycle theory. Following an idea first posed in Kalecki (1935), Kydland and Prescott (1982) empirically analyzed how far time consuming investment, which they called *time-to-build*, could explain real business cycles. While Kydland and Prescott (1982) argued that the time-to-build feature is essential to cyclical fluctuations in their model, this was doubted by Ioannides and Taub (1992). Rustichini (1989) and Asea and Zak (1999) showed in simple optimal control models with one capital good (but a different lag structure) that the time-to-build feature is the driving force for the cyclical system dynamics.

In contrast to the authors mentioned above, we explicitly analyze the *qualitative* and *quantitative* properties of the optimal paths in their dependence on the time-lag σ. Therefore, we formulate an optimal control capital accumulation model with a constant and exogenously given time-lag between investment and the accumulation of capital. For reasons of analytical tractability, we restrict our attention to a Leontief-type production function. Although the dynamics of our capital accumulation model is governed by a system of *functional differential equations*, which is not analytically soluble, we derive some qualitative properties of the optimal solution. As expected from the works of Rustichini (1989) and Asea and Zak (1999), the optimal investment paths for a finite investment period are shown to be cyclical, as opposed to the monotonic paths for instantaneous capital accumulation.

We present a systematic analysis of the impact of the length of the time-lag σ. We show analytically that there is a continuous transition from instantaneous to time-lagged capital accumulation, in the sense that the cyclical behavior becomes more pronounced with increasing time-lag σ. For time-lagged optimal control problems even the linear approximation around the stationary state is not analytically soluble, so numerical optimization is a relevant issue in order to analyze and understand the system dynamics of time-lagged problems. Using a method described in Winkler et al. (2004), we solve the time-lagged optimization problem numerically and discuss the results.

The paper is organized as follows. In section 2 we introduce the optimization model. Although the optimal control problem is not analytically soluble, we derive some analytical properties of the solution in section 3. In section 4 we apply advanced numerical optimization methods and discuss the optimal paths for an example. Section 5 concludes.

2 The Model

We analyze an optimal control capital accumulation model with an exogenously given time-lag between investment and capital accumulation. In general, time-lagged accumulation problems exhibit severe analytical difficulties, as even linear functional differential equations are in general not soluble. Therefore, we restrict our attention to a Leontief-type production function.[1] This specialization allows us to derive analytical properties of the optimal paths.

Suppose the following intertemporal welfare function W is to be maximized

$$W\{c(t)\} = \int_0^\infty V(c(t)) \exp[-\rho t]\, dt \ , \tag{1}$$

where ρ denotes the positive and constant discount rate and V the twice differentiable, monotonically increasing ($V' > 0$) and strictly concave ($V'' < 0$) instantaneous welfare function.

The only non-producible input factor, e. g. labor, is given in constant amount $\bar{l}$, which is distributed to three linear-limitational production processes. Without loss of generality, we assume that the first process produces one unit of the consumption good with one unit of labor. The second process combines λ units of labor together with κ units of capital to produce one unit of the consumption good. The third process creates one unit of investment from one unit of labor. Thus, we derive

$$\begin{aligned} c_1(t) &= l_1(t) \ , & (2)\\ c_2(t) &= \min\left[\frac{l_2(t)}{\lambda}, \frac{k(t)}{\kappa}\right] \ , & (3)\\ i(t) &= l_3(t) \ , & (4) \end{aligned}$$

where l_i denote the amount of labor employed in process i ($i = 1, 2, 3$). Assuming efficient production, i. e. $l_2(t)/\lambda = k(t)/\kappa$, and that the labor restriction holds with equality, i. e. $\sum_i l_i(t) = \bar{l}\ \forall\, t$, total production $P(t) = c_1(t) + c_2(t) + i(t)$ reads:[2]

$$P(k(t)) = \bar{l} + \frac{1-\lambda}{\kappa} k(t) \ . \tag{5}$$

Note that we can write total consumption $c(t) = c_1(t) + c_2(t)$ as total production minus investment:

$$c(t) = P(k(t)) - i(t) \ . \tag{6}$$

[1] The model introduced in the following is a slightly adapted continuous time version of the 3-process model discussed in Faber and Proops (1991).

[2] Although consumption goods and investment goods are different commodities, they can be summed up because they are all measured in units of labor.

Hence, the formal structure of our model is similar to the neoclassical growth models introduced by Cass (1965) and Koopmans (1965). The main difference is that we analyze a linear production function which does not satisfy the Inada conditions ($\lim_{k\to 0} P' = \infty$, $\lim_{k\to\infty} P' = 0$).

To model the time structure of production we assume that capital accumulation is time consuming: investment at time t increases the capital stock k delayed until time $t+\sigma$, where σ denotes the positive and constant time-lag between investment and capital accumulation. Furthermore, we assume that the capital stock deteriorates at the positive and constant rate γ:

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t) \ . \tag{7}$$

In addition, we assume that the capital stock k cannot be consumed, i. e. $i(t) \geq 0$. Hence, the optimal control problem reads:

$$\max_{i(t)} \int_0^\infty V(c(t)) \exp[-\rho t] \, dt \tag{8a}$$

subject to

$$c(t) = \bar{l} + \frac{1-\lambda}{\kappa} k(t) - i(t) \ , \tag{8b}$$

$$\dot{k}(t) = i(t-\sigma) - \gamma k(t) \ , \tag{8c}$$

$$i(t) \geq 0 \ , \tag{8d}$$

$$\bar{l} - \frac{\lambda}{\kappa} k(t) - i(t) = c(t) - \frac{1}{\kappa} k(t) \geq 0 \ , \tag{8e}$$

$$i(t) = \xi(t) = 0 \ , \quad t \in [-\sigma, 0) \ , \tag{8f}$$

$$k(0) = 0 \ . \tag{8g}$$

The restriction (8e) assures that $c_1 \geq 0$.[3] When it is binding, then all labor is used to employ and maintain the capital stock. This implies that the consumption good is exclusively produced by the capital intensive process (3). The equation of motion for the capital stock (8c) is the main difference from instantaneous capital accumulation models. Because of the positive time-lag σ, the ordinary differential equation becomes a *retarded differential-difference* equation, i.e. the variation in the capital stock depends not only on parameters evaluated at time t but also on parameters evaluated at the earlier time $t-\sigma$. Thus, the specification of an initial value for the capital stock k is no longer sufficient for a unique solution. In addition, we have to specify an initial path ξ for the investment i in the time interval $[-\sigma, 0)$. Hence, unlike the case of ordinary differential equations, the past does not condense into a single parameter – the initial value – but the time path has a crucial impact on the future dynamics. As a consequence, the complexity of the system dynamics increases greatly. For the sake of simplicity we assume that the initial path ξ is constant at 0.

[3] Note that restriction (8d) together with the initial condition (8g) assure that $c_2 \geq 0$.

3 Analysis of the Optimal Solution

Although the optimal control problem (8) is not analytically soluble, we can state some qualitative properties of the solution. We shall see that the optimal solution falls into one of two different classes. First, in the *trivial* case the accumulation of capital is not optimal. Then the optimal investment path is $i(t) = 0$ for all times t and the restriction (8d) is binding, while the restriction (8e) is not binding. The system will stay in the trivial stationary state $(i^\star = 0, k^\star = 0)$ forever. Second, in the *non-trivial* case investment is optimal and thus $i(t) > 0$ for all times t. As a consequence restriction (8d) is never binding. For small times t also the restriction (8e) is not binding. First, for times $t \in [0, \sigma)$ the capital stock is identical to 0 due to the initial path ξ (8f). After the time-lag σ investment turns into capital, and the capital stock increases. At some time t' the capital stock is big enough so that all available labor $\bar{l}$ is used to employ and maintain the capital stock. Hence the restriction (8e) is binding. The system will then converge to a stationary state, which is determined by the restriction (8e) and the time-lagged equation of motion (8c).

3.1 Necessary and Sufficient Conditions

We start the discussion of the properties of the optimal path by deducing the necessary and sufficient conditions. In contrast to Asea and Zak (1999), the lag structure applied in maximization problem (8) is not supported by the Maximum Principle of Pontrjagin et al. (1962). To determine the necessary conditions for an optimal solution we apply the generalized Maximum Principle derived in El-Hodiri et al. (1972). We obtain the following present-value Hamiltonian $\mathcal{H}$

$$\begin{aligned}\mathcal{H} \;=\;& V(c(t))\exp[-\rho t] + p_c(t)\left[\bar{l} + \frac{1-\lambda}{\kappa}k(t) - i(t) - c(t)\right] + p_k(t+\sigma)i(t) \\ & -p_k(t)\gamma k(t) + p_i(t)i(t) + p_l(t)\left[\bar{l} - \frac{\lambda}{\kappa}k(t) - i(t)\right] \;,\end{aligned}$$

where p_c, p_i and p_l denote the Kuhn-Tucker parameters of the corresponding restrictions and p_k the costate variable of the capital stock k, i.e. they are the shadow prices of the corresponding restrictions. The difference to instantaneous capital accumulation is covered by the term $p_k(t+\sigma)i(t)$. Although it might look odd at first sight to have p_k evaluated at a future time, while we have a *retarded* equation of motion (8c), the explanation is quite intuitive: p_k measures the net present value of all future welfare gains of one additional unit of capital. As investment takes the time period σ to turn into productive capital, the investment $i(t)$ gives rise to additional capital at $t+\sigma$, of which the net present value is given by $p_k(t+\sigma)$.

Assuming that $\mathcal{H}$ is continuously differentiable with respect to i, the necessary conditions for an optimal solution read:

$$\begin{aligned}
\frac{\partial\mathcal{H}}{\partial i(t)} &= -p_c(t)+p_k(t{+}\sigma)+p_i(t)-p_l(t)=0\ , && \text{(9a)}\\
\frac{\partial\mathcal{H}}{\partial c(t)} &= V'(c(t))\exp[-\rho t]-p_c(t)=0\ , && \text{(9b)}\\
\frac{\partial\mathcal{H}}{\partial k(t)} &= p_c(t)\frac{1-\lambda}{\kappa}-p_k(t)\gamma-p_l(t)\frac{\lambda}{\kappa}=-\dot{p}_k(t)\ , && \text{(9c)}\\
p_i(t) &\geq 0\ , \quad p_i(t)i(t)=0\ , && \text{(9d)}\\
p_l(t) &\geq 0\ , \quad p_l(t)\left[\bar{l}-\frac{\lambda}{\kappa}k(t)-i(t)\right]=0\ . && \text{(9e)}
\end{aligned}$$

As the Hamiltonian $\mathcal{H}$ is concave in k and i due to the assumed curvature properties of V, these necessary conditions are also sufficient if, in addition, the following transversality condition is satisfied:

$$\lim_{t\to\infty}[p_k(t)k(t)]=0\ . \tag{9f}$$

The economic interpretation of the necessary and sufficient conditions is straightforward. Equation (9b) states that along the optimal path the shadow price of the consumption good equals the net present value of marginal utility. Equation (9c) represents a linear first order differential equation for the shadow price of capital, which can be unambiguously solved together with the transversality condition (9f). As usual, the shadow price of capital $p_k(t)$ gives the present value gain in welfare of a marginal increase of capital at time t. Now we can interpret equation (9a). It says that along the optimal path and as long as investment is positive, i. e. $p_i(t)=0$, and restriction (8e) is not binding, i. e. $p_l(t)=0$, the present value of the costs for an investment in the capital good in terms of lost welfare has to equal the shadow price of capital p_k at time $t{+}\sigma$. As investment at time t accumulates the capital stock at time $t{+}\sigma$, the present value of the future welfare gains are captured by the future shadow price of capital $p_k(t{+}\sigma)$.

3.2 Stationary State

In the following we deduce a condition for the exogenous parameters to distinguish the trivial from the non-trivial case. Furthermore, we calculate the corresponding fixed points $(i^\star, k^\star)$, which are given by the conditions $\dot{i}(t)=\dot{k}(t)=0$.

Proposition 1 (Stationary State)
The unique fixed point $(i^\star, k^\star)$ of the optimal control problem (8) is given by:

- $(i^\star=0,\ k^\star=0)$, *if* $\dfrac{1-\lambda}{\kappa}\leq(\gamma+\rho)\exp[\rho\sigma]$, *and*
- $\left(i^\star=\dfrac{\gamma\kappa\bar{l}}{\lambda+\gamma\kappa},\ k^\star=\dfrac{\kappa\bar{l}}{\lambda+\gamma\kappa}\right)$, *if* $\dfrac{1-\lambda}{\kappa}>(\gamma+\rho)\exp[\rho\sigma]$.

The corresponding stationary state consumption levels are $c^\star = \bar{l}$ *for* $(i^\star = 0,\ k^\star = 0)$, *and* $c^\star = \frac{\bar{l}}{\lambda+\gamma\kappa}$ *for* $\left(i^\star = \frac{\gamma\kappa\bar{l}}{\lambda+\gamma\kappa},\ k^\star = \frac{\kappa\bar{l}}{\lambda+\gamma\kappa}\right)$.

Proof: Suppose investment in capital is not optimal. Then $i(t) = 0$, $p_i(t) \geq 0$ and $p_l(t) = 0\ \forall\, t$. Hence, (9a) reduces to:

$$p_c(t) \geq p_k(t+\sigma)\ . \tag{10}$$

Furthermore, if $i(t) = 0$, then also $k(t) = 0$ and $c(t) = \bar{l}\ \forall\, t$. Thus, we derive from (9b):

$$p_c(t) = V'\left(\bar{l}\right)\exp[-\rho t]. \tag{11}$$

The differential equation for the shadow price p_k can be solved together with the transversality condition (9f) to yield:

$$p_k(t) = \frac{1-\lambda}{(\gamma+\rho)\kappa}V'\left(\bar{l}\right)\exp[-\rho t]\ . \tag{12}$$

Inserting (11) and (12) in (10) and simplifying yields the following inequality:

$$\frac{1-\lambda}{\kappa} \leq (\gamma+\rho)\exp[\rho\sigma]\ . \tag{13}$$

Thus, investment is optimal if this inequality does not hold. In this case investment is positive and capital is accumulated until all labor is used to employ and maintain the capital stock, i. e. the restriction (8e) is binding. Hence, $c^\star = k^\star/\kappa$ and $i^\star = \gamma k^\star$. Inserting into (8b) yields the stated result for $i^\star$, $k^\star$and $c^\star$. □

In the following we shall concentrate our attention to the non-trivial case of positive investment, where $\frac{1-\lambda}{\kappa} > (\gamma+\rho)\exp[\rho\sigma]$ holds. Then the stationary state is independent of the time-lag σ as long as this inequality holds. As already mentioned, the system dynamics of the non-trivial case splits into three phases. In the first phase, in the following called the *initial phase*, investment is positive but the capital stock is still 0 due to the initial path $\xi = 0$ (8f) and thus, the consumption good is solely produced by production process (2). During the second phase, in the following called the *growth phase*, the capital stock is accumulated while the consumption good is produced by both production processes (2) and (3). Hence, the restriction (8e) is not binding. In the third phase, in the following called the *consolidation phase*, consumption is solely produced by the capital intensive production process (3), i. e. the restriction (8e) is binding.

3.3 Initial Phase and Growth Phase

During the initial phase and the growth phase the dynamics of the optimal solution is governed by the following system of differential equations, which can be derived by the necessary and sufficient conditions (setting $p_i(t) = p_l(t) = 0$) and the equation of motion (8c):[4]

$$\dot{c}(t) = \frac{V'(c(t))}{V''(c(t))}(\gamma + \rho) - \frac{V'(c(t+\sigma))}{V''(c(t))}\frac{1-\lambda}{\kappa}\exp[-\rho\sigma] \ , \tag{14a}$$

$$\dot{k}(t) = \bar{l} + \frac{1-\lambda}{\kappa}k(t-\sigma) - c(t-\sigma) - \gamma k(t) \ . \tag{14b}$$

Note that $\dot{c}$ also depends on *advanced* (at a later time) and $\dot{k}$ on *retarded* (at an earlier time) variables. Hence, (14) forms a system of *functional differential equations.*[5] The usual procedure, to linearize the system of differential equations around some point of interest and discuss the resulting system of *linear* differential equations, is not applicable here, because during the growth phase there is no point of attraction like the stationary state. On the contrary, during the growth phase we expect the system to change rapidly. Furthermore, in general even the linearized system is not analytically soluble. As a consequence, little more can be said about the optimal paths than that the system dynamics is in general cyclical (Winkler 2004).

Nevertheless, it is worth noting that if $\sigma = 0$ both $c(t)$ and $k(t)$ increase monotonically while for $\sigma > 0$ cyclical paths are also feasible. Furthermore, the dynamics of the capital stock during the initial phase, ranging from $t = 0$ to $t = \sigma$, is completely determined by the initial investment path ξ, the initial capital stock $k(0)$ and the equation of motion (8c). Thus, the time-lagged accumulation of capital introduces an additional moment of inertia to the system dynamics.

3.4 Consolidation Phase

The situation changes as soon as restriction (8e) is binding and the system dynamics enters the consolidation phase. Here we expect the system to converge towards the stationary state. Inserting restriction (8e) into equation (8b) we derive:

$$k(t) = \frac{\kappa}{\lambda}(\bar{l} - i(t)) \ . \tag{15}$$

Differentiating with respect to time yields:

$$\dot{k}(t) = -\frac{\kappa}{\lambda}\dot{i}(t) \ . \tag{16}$$

Hence, the first result for the system dynamics during the consolidation phase is that capital and investment develop in opposite directions as $\dot{k}$ and $\dot{i}$ are of opposite sign. Inserting this equation into the equation of motion for the capital stock (8c) yields the following inhomogeneous retarded linear differential equation:

$$\dot{i}(t) + \gamma i(t) + \frac{\lambda}{\kappa}i(t-\sigma) = \gamma\bar{l} \ . \tag{17}$$

[4] Here we present the differential equations for $c(t)$ and $k(t)$ instead of $i(t)$ and $k(t)$. Note that once the paths for $c(t)$ an $k(t)$ are known, the path for $i(t)$ can easily be calculated using (8b).

[5] For an introduction to retarded functional differential equations see Asea and Zak (1999: section 2) and Gandolfo (1996: chapter 27). A detailed exposition for linear functional differential equations (differential-difference equations) is given in Bellman and Cooke (1963), and Hale (1977).

As for ordinary linear differential equations, the solution is the superposition of a particular solution of the *inhomogeneous* equation plus the general solution for the *homogeneous* equation. It can easily be verified that the stationary state investment $i^\star = \frac{\gamma\kappa l}{\lambda+\gamma\kappa}$ solves (17). Hence, we can restrict our attention to the solution of the homogeneous equation.

Similar to the case of ordinary linear first-order differential equations, the elementary solutions $\hat{i}_n$ for $\hat{i}(t) = i(t) - i^\star$ are exponential functions (e.g. Gandolfo 1996: 550–551). Hence, we can write the general solution as an (infinite) series of elementary solutions:

$$\hat{i}(t) = \sum_n i_n \exp[x_n t] \ , \tag{18}$$

where i_n denote constants, which can (at least in principle) be unambiguously determined by the set of initial conditions and the transversality condition (9f), and the x_n are the roots of the characteristic equation:

$$x + \gamma + \frac{\lambda}{\gamma} \exp[-\sigma x] = 0 \ . \tag{19}$$

Let us denote the real characteristic roots by x_r and the complex roots by $x_j = a_j \pm i b_j$ with $a_j, b_j \in \mathbb{R}$ (we shall see that all complex roots appear in conjugate pairs). For $\sigma = 0$ the characteristic equation (19) has a unique negative real root $x_r = -(\gamma + \lambda/\gamma)$. For $\sigma > 0$ the equation exhibits 0, 1 or 2 negative real roots x_r and in addition an infinite number of complex roots x_j as the following proposition states.

Proposition 2 (Roots of the characteristic polynomial)
Given positive constants λ and γ, the characteristic equation (19) has

- *one unique negative real root $x_r = -(\gamma + \frac{\lambda}{\gamma})$, if $\sigma = 0$, and*
- *0, 1 or 2 negative real roots with $x_r < -(\gamma + \frac{\lambda}{\gamma})$ and an infinite number of complex roots x_j, of which only a finite number has positive or vanishing real part, if $\sigma > 0$.*

Proof: 1. The case $\sigma = 0$ is obvious from equation (19).

2. Real solutions for $\sigma > 0$: Set $F(x) = x$ and $G(x) = -(\gamma + \frac{\lambda}{\kappa} \exp[-\sigma x])$. Then the real roots are given by $F(x) = G(x)$ for $x \in \mathbb{R}$. There are no positive roots because of $G(0) = -(\gamma + \frac{\lambda}{\kappa})$ and $\lim_{x\to\infty} G(x) = -\gamma$. As $\lim_{x\to-\infty} G(x) = -\infty$, $G'(x) = \frac{\sigma\lambda}{\kappa} \exp[-\sigma x] > 0$ and $G''(x) = -\frac{\sigma^2\lambda}{\kappa} \exp[-\sigma x] < 0$, $G(x)$ may not intersect $F(x)$, touch $F(x)$ for one multiple root or intersect $F(x)$ twice in the negative half-plane. As $G(0) = -(\gamma + \frac{\lambda}{\kappa})$ and due to the curvature properties of $G(x)$ all roots are smaller than $-(\gamma + \frac{\lambda}{\gamma})$.

3. Complex solutions for $\sigma > 0$: Set $x = a + ib$ with a, $b \in \mathbb{R}$. Inserting into (19) and separating real and imaginary parts yields the following equations, which have to hold for the characteristic roots:

$$a + \gamma + \frac{\lambda}{\kappa} \exp[-\sigma a] \cos[\sigma b] = 0 \ , \tag{20a}$$

$$b - \frac{\lambda}{\kappa} \exp[-\sigma a] \sin[\sigma b] = 0 \ . \tag{20b}$$

Unfortunately, this system of equations is not analytically soluble. Nevertheless, we can state some general properties of the solution. First, note that if $a+ib$ solves (20) then $a-ib$ also does. Hence, complex characteristic roots always appear in conjugate pairs and therefore we restrict the further analysis to positive b. Second, due to equation (20b), $\sin(\sigma b)$ has to be positive. Hence, the imaginary parts b are restricted to the following intervals:

$$\frac{2j\pi}{\sigma} < b_j < \frac{2(j+1)\pi}{\sigma} \ , \quad j \in \mathbb{N}_0 \ . \tag{21}$$

For further investigations we rearrange the equations (20)

$$a = \frac{1}{\sigma} \ln\left[\frac{\lambda\sigma}{\kappa}\frac{\sin\beta}{\beta}\right] \ , \tag{22a}$$

$$\ln\left[\frac{\kappa}{\lambda\sigma}\right] - \gamma = \ln\left[\frac{\sin\beta}{\beta}\right] + \frac{\beta}{\tan\beta} \ , \tag{22b}$$

where $\beta = \sigma b$. Note that the right-hand-side (RHS) of equation (22b) is independent of the exogenous parameters. Thus, we can determine the imaginary parts b by the intersection of the constant of the left-hand-side (LHS), which depends on the exogenous parameters, with the graph of the right-hand-side of equation (22b). Due to the strict monotonicity of ln and tan, there is one unique intersection in each interval described by (21) for $n > 0$ and in addition an intersection for $n = 0$ if the LHS of (22b) < 1 (figure 1). Hence, the characteristic equation (19) has an infinite number of complex solutions.

The last thing to show is that there is only a finite number of complex roots with $a_j \geq 0$. From equation (22a) we know that $a_j < 0$ if $\frac{\lambda\sigma}{\kappa}\frac{\sin\beta}{\beta} < 1$. As $\frac{sin[\sigma b_j]}{\sigma b_j} \to 0$ for $n \to \infty$, there is one j' for any given set of exogenous parameters so that $a_j < 0$ if $j > j'$. □

The space of solutions decomposes into a stable manifold spanned by the eigenvectors corresponding to the eigenvalues with negative real part and an unstable manifold spanned by the eigenvectors corresponding to the eigenvalues with positive real part.[6] Note that due to the transversality condition (9f), the optimal solution is restricted to the stable hyperplane. Concluding, the optimal solutions for investment $i(t)$ and capital stock $k(t)$ in the consolidation phase can be written as:

$$i(t) = i^\star + \sum_r i_r \exp[x_r t] + \sum_j i_j \exp[a_j t] \sin[\phi_j + b_j t] \ , \tag{23a}$$

$$k(t) = k^\star - \sum_r \frac{\kappa}{\lambda} i_r \exp[x_r t] - \sum_j \frac{\kappa}{\lambda} i_j \exp[a_j t] \sin[\phi_j + b_j t] \ , \tag{23b}$$

[6] Depending on the exogenously given parameters the characteristic polynomial may have one characteristic roots with vanishing real part($a_{j'} = 0$). If such a solely complex root exists, the system dynamics may exhibit a so called *limit-cycle*, i. e. the system oscillates around the stationary state without converging towards or diverging from it (Asea and Zak 1999). From the proof of proposition 2 it is clear that this can only happen accidentally for special sets of exogenous parameters.

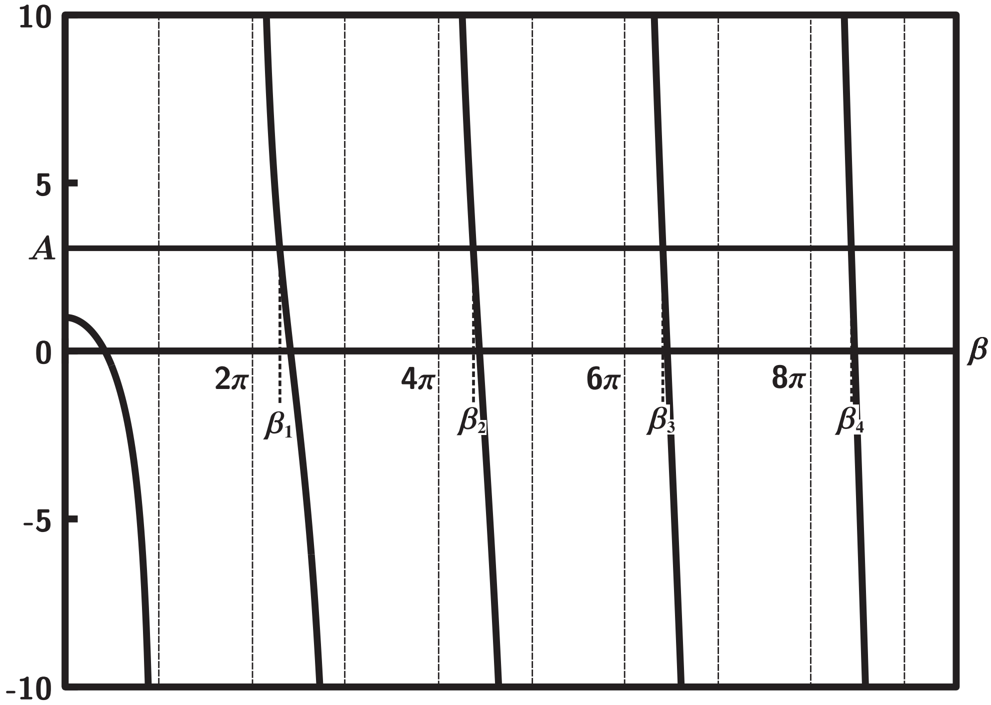

Figure 1: The imaginary parts of the characteristic roots are given by the intersection of the graph of the RHS of equation (22b) with the constant $A = \ln\left[\frac{\kappa}{\lambda\sigma}\right] - \gamma$, which is the LHS of equation (22b).

where i_r, i_j and ϕ_j are constants which have to be determined by the set of initial conditions and the transversality condition. Furthermore, if $a_j > 0$ then $i_j = 0$. Note that the optimal solution decomposes in a monotonic part (the first sum covering the real roots) and a cyclical part (the second sum covering the complex roots). The cyclical part itself is a composition of individual cycles, where the imaginary part b_j determines the period-length, the real part a_j the damping and the constant i_j the amplitude of the corresponding cycle.

3.5 Transition from Instantaneous to Time-Lagged Capital Accumulation

From our analysis so far it is obvious that the system dynamics exhibits a *qualitative* change for a transition from $\sigma = 0$ to $\sigma > 0$. The optimal paths for investment and capital converge towards the stationary state strictly monotonically and exponentially in the first case, and cyclically and exponentially damped in the latter case. However, so far it is not clear how this transition takes place *quantitatively*. Do the optimal paths exhibit more and more pronounced cyclical behavior with increasing time-lag σ, or do they experience a sharp change at the transition from $\sigma = 0$ to any $\sigma > 0$? Obviously, intuition would suggest a smooth and continuous transition. In fact, this time intuition holds (at least during the consolidation phase) as we shall show in the following analysis. Therefore, we first take a closer look at the frequencies, determined by x_j, and the amplitudes, determined by i_j, of the optimal solution. The following proposition states the result.

Proposition 3 (Properties of the optimal path)
Given the optimal control problem (8) together with the binding restriction (8e), the optimal paths for investment $i(t)$ and capital stock $k(t)$ exhibit the following properties:

- *There exists at most one* major cycle *with period-length $T_0 > 2\sigma$ corresponding to the characteristic root x_j with $j = 0$.*
- *There exists an infinite number of* minor cycles *with period-length $T_j < \sigma/j$ corresponding to the characteristic root x_j with $j \in \mathbb{N}$.*
- *The upper bound for the amplitude of the cycle corresponding to the characteristic root x_j is smaller the higher is $j \in \mathbb{N}_0$.*
- *The upper bound for the damping of the cycle corresponding to the characteristic root x_j is more negative the higher is $j \in \mathbb{N}_0$.*

Proof: 1. Period-lengths T_j: From (21) we know that there is a complex root with imaginary part b_j within each interval $(2j\pi/\sigma; 2(j+1)\pi/\sigma)$ for each $j \in \mathbb{N}$ and one complex root with imaginary part b_0 in the interval $(0; \pi/\sigma)$ if the LHS of equation (22b) is smaller than 1. Thus, the corresponding period lengths $T_j = 2\pi/b_j$ of the cycles are:

$$T_0 > 2\sigma \ , \quad j = 0 \ , \tag{24a}$$

$$T_j < \frac{\sigma}{j} \ , \quad j \in \mathbb{N} \ . \tag{24b}$$

2. Upper bound for the amplitudes: According to (23a), the absolute difference between the maximum and the minimum value of the investment path within one period-length is smaller than $|2i_j|$ as the cycles are also exponentially damped. As investment is non-negative according to restriction (8d), capital can decrease at most at the rate of deterioration γ. As the maximal possible amount for the capital stock $k(t)$ is given by $\bar{k} = \kappa\bar{l}$, the maximal decrease in capital within the time span T_j must not exceed $\gamma\kappa\bar{l}T_j$. Hence, the following relations for the constants i_j hold:

$$\left|2\frac{\kappa}{\lambda}i_j\right| < \kappa\gamma\bar{l}T_j = \frac{2\pi\kappa\gamma\bar{l}}{b_j} < \frac{\kappa\gamma\bar{l}\sigma}{j+1} \ , \quad j \in \mathbb{N}_0 \ . \tag{25}$$

Thus, an upper bound $\bar{i}_j$ for the constant i_j is given by:

$$|\bar{i}_j| = \frac{\kappa\gamma\lambda\bar{l}\sigma}{j+1} \ , \quad j \in \mathbb{N}_0 \ . \tag{26}$$

3. Upper bounds for real parts a_j (damping): According to (22a) the real parts a_j are given by:

$$a_j = \frac{1}{\sigma} \ln \left[\frac{\lambda\sigma}{\kappa} \frac{\sin\beta_j}{\beta_j} \right] \quad , \quad j \in \mathbb{N}_0 \ . \tag{27}$$

According to (21), $2\pi j$ is a lower bound for β_j. Setting 1 as an upper bound for $\sin\beta_j$, we derive as an upper bound $\bar{a}_j$ for the real part a_j:

$$\bar{a}_j = \frac{1}{\sigma} \ln \left[\frac{\lambda\sigma}{2\pi\kappa j} \right] \quad , \quad j \in \mathbb{N}_0 \ . \tag{28}$$

Hence, for given exogenous parameters λ, κ and σ the upper bound $\bar{a}_j$ is smaller the higher is j. □

From proposition 3 we expect that the optimal paths exhibit a dominant cycle, which corresponds to the characteristic root x_j with the smallest j that satisfies $a_j < 0$. If the smallest $j = 0$, then we observe a major cycle with a period-length bigger that 2σ. Otherwise we observe a minor cycle with period-length smaller than σ/j. In general, we observe a damped cyclical convergence towards the stationary state, but limit-cycles are possible for certain sets of exogenous parameters. In addition, we expect to observe the contributions of the minor cycles corresponding to the characteristic roots with higher j. Note that the higher is j, the smaller is the period-length T_j, the smaller is the upper bound for $|i_j|$ and thus the amplitude, and the higher is the damping due to the increasingly more negative real parts a_j. Hence, we expect the contribution of the characteristic root x_j to be smaller the higher is j.

Let us now take a closer look at the transition from $\sigma = 0$ to $\sigma > 0$. Therefore, we assume that all exogenous parameters are fixed except for σ, which we shall treat as a variable. Then we can analyze how the optimal paths change if we change σ. In particular, we are interested in the transition $\sigma \to 0$. The result is stated in the following proposition.

Proposition 4 (Continuous transition theorem)
Given the optimal control problem (8) together with the binding restriction (8e), the optimal paths for investment $i(t)$ and capital stock $k(t)$ exhibit ceteris paribus a continuous transition from monotonic to cyclical behavior for a transition from instantaneous ($\sigma = 0$) to time-lagged ($\sigma > 0$) capital accumulation in the following sense:

- *The period-lengths T_j of the cycles converge to 0 for $\sigma \to 0$.*
- *The upper bounds $\bar{i}_j$, and thus the amplitudes of the cycles, converge to 0 for $\sigma \to 0$.*

Proof: 1. Period-lengths T_j: According to (24b), it is obvious that the minor cycles ($j > 0$) have shorter period-lengths the smaller the time-lag σ. The situation is slightly more complicated for the major cycles ($j = 0$). Note that for $j = 0$ the LHS of equation (22b) tends to $+\infty$ for $\sigma \to 0$. Thus, there exists a σ' so that the LHS of equation (22b) equals one. As a consequence, there exists no major cycle for $\sigma < \sigma'$.

2. Upper bounds $|\bar{i}_j|$: From (26) follows directly that $\lim_{\sigma \to 0} |i_j| = 0$. □

According to proposition 4 we expect that the optimal paths exhibit increasingly more pronounced cyclical behavior if we increase the time-lag σ. However, note that for increasing σ, eventually the inequality (13) holds and the system dynamics will be of the trivial-solution-type.

Although the analysis carried out in this section contributed greatly to our understanding of the system dynamics of the linear-limitational optimal control problem (8), there are still unanswered questions. First, we can say hardly anything about the optimal paths during the growth phase. Second, although we were able to estimate upper bounds for the amplitudes of the cycles during the consolidation phase, we cannot tell if cycles play a significant role at all, as the amplitudes may be very small or even vanish.

For general production functions the situation is even worse. In general, time-lagged accumulation problems exhibit severe analytical difficulties, as even linear functional differential equations are in general not soluble. Hence, in general the standard method in optimal control theory to linearize the resulting system of differential equations around the stationary state does not lead lead to analytical solutions. As a consequence, numerical optimization methods play an important role to analyze and understand the behavior of time-lagged optimal control problems. In the following section we discuss a numerical example to illustrate our analytical results.

4 A Numerical Example

In this section we discuss an example of optimization problem (8) for a special set of exogenous parameters.[7] To analyze the transition from instantaneous to time-lagged capital accumulation we vary σ between 0 and 0.5. Table 1 shows numerically calculated values for the real characteristic roots x_r and the first three complex characteristic roots x_j together with their corresponding period-length T_j and upper bounds $|\bar{i}_j|$ for selected values of σ.

Note that there are real characteristic roots only for $\sigma = 0$ and $\sigma = 0.1$. Furthermore, for $\sigma = 0.1$ there exists no major cycle. As the first two minor cycles have very small period-lengths T_j and are strongly damped, due to highly negative real parts a_j, we expect the optimal paths to exhibit only slightly cyclical behavior. For $\sigma \geq 0.2$ there are no real characteristic roots. As a consequence, the optimal paths have to be cyclical. To what extent the major and minor cycles play a role in the system dynamics is impossible to say as we only know the period-lengths and upper bounds for their amplitudes. Nevertheless, we expect the major cycles with period-lengths T_j ranging from 1.27 to 2.01 to dominate the cyclical behavior as their upper bounds are much higher than the upper bounds for the minor cycles.

To test our expectations we solve the optimal control problem (8) numerically with the advanced optimal control software package MUSCOD-II developed by the Simulation

[7] The following functions and constants have been chosen for ease of graphical presenting of the results: $V(c(t)) = \ln c(t)$, $\bar{l} = 26\frac{2}{3}$, $\lambda = 0.8$, $\kappa = 0.3$, $\gamma = 0.15$, $\rho = 0.1$, $k_0 = 0$ and $\xi(t) = 0$.

σ	0	0.1	0.2	0.3	0.4	0.5
$x_{r=1}$	-2.82	-4.21	—	—	—	—
$x_{r=2}$	—	-20.15	—	—	—	—
β_0	—	—	$0.99\,(0.32\pi)$	$1.29\,(0.41\pi)$	$1.46\,(0.46\pi)$	$1.56\,(0.5\pi)$
a_0	—	—	-3.99	-1.73	-0.79	-0.32
b_0	—	—	4.96	4.31	3.63	3.13
T_0	—	—	$1.27\,(6.33\sigma)$	$1.46\,(4.86\sigma)$	$1.73\,(4.32\sigma)$	$2.01\,(4.02\sigma)$
$\bar{i}_0$	—	—	$2.03\,(10.14\sigma)$	$2.33\,(7.78\sigma)$	$2.76\,(6.9\sigma)$	$3.21\,(6.43\sigma)$
β_1	—	$7.44\,(2.37\pi)$	$7.53\,(2.4\pi)$	$7.58\,(2.41\pi)$	$7.62\,(2.42\pi)$	$7.66\,(2.43\pi)$
a_1	—	-34.17	-13.51	-7.62	-4.99	-3.54
b_1	—	74.4	37.64	25.26	19.04	15.29
T_1	—	$0.08\,(0.84\sigma)$	$0.17\,(0.83\sigma)$	$0.25\,(0.83\sigma)$	$0.33\,(0.82\sigma)$	$0.41\,(0.82\sigma)$
$\bar{i}_1$	—	$0.14\,(1.35\sigma)$	$0.27\,(1.34\sigma)$	$0.4\,(1.33\sigma)$	$0.53\,(1.32\sigma)$	$0.66\,(1.31\sigma)$
β_2	—	$13.87\,(4.41\pi)$	$13.92\,(4.43\pi)$	$13.94\,(4.44\pi)$	$13.96\,(4.45\pi)$	$13.98\,(4.45\pi)$
a_2	—	-39.88	-16.43	-9.59	-6.47	-4.72
b_2	—	138.67	69.58	46.48	34.91	27.96
T_2	—	$0.05\,(0.45\sigma)$	$0.09\,(0.45\sigma)$	$0.14\,(0.45\sigma)$	$0.18\,(0.45\sigma)$	$0.22\,(0.45\sigma)$
$\bar{i}_2$	—	$0.07\,(.72\sigma)$	$0.14\,(0.72\sigma)$	$0.22\,(0.72\sigma)$	$0.29\,(0.72\sigma)$	$0.36\,(0.72\sigma)$

Table 1: Numeric estimates for the real characteristic roots and the first three complex chararcteristic roots together with their corresponding period-lengths and upper bound for the constant i_j in absolute numbers and in units of π or σ respectively (terms in brackets).

and Optimization Group of the Interdisciplinary Center for Scientific Computing at the University of Heidelberg. For details about the numerical simulation see Winkler et al. (2004). As it is not possible to optimize numerically over an infinite time horizon τ, the time horizon has been set sufficiently high to ensure a close neighborhood of the optimal paths to the long-run stationary state ($\tau \approx 60$). For a more convenient exposition, the figures show the time paths up to $t = 15$ (figure 2) and $t = 8$ (figure 3) only.

Figure 2 shows numerical optimized paths of the time-lagged capital accumulation problem (8) for time-lags σ ranging from 0 to 0.5. As already mentioned, the optimal paths split into three phases: the initial phase, the growth phase and the consolidation phase. As the marginal productivity of capital changes discontinuous at the transition from one phase to another, the optimal paths are not necessarily differentiable at the phase borders. In fact, for $\sigma > 0$ we observe kinks in the optimal investment paths which correspond to these phase transitions (indicated by black triangles in figure 2). Consistent with proposition 2, the optimal paths converge monotonically towards the stationary state for instantaneous capital accumulation ($\sigma = 0$). We also observe monotonic optimal paths for $\sigma > 0$ during the initial phase and the growth phase, which was not necessarily expected from the system of functional differential equations (14). Whether this is a general feature of this model, or just an artifact of our choice of exogenous parameters, is impossible to say. In general, the system of functional differential equations (14) allows for cyclical system dynamics.

As expected from propositions 3 and 4, the system dynamics exhibits increasingly pronounced cyclical behavior for increasing time-lags σ. The fact that the optimal paths for $\sigma = 0.1$ show no visible non-monotonicity, suggests that the non-monotonicity is of a magnitude which cannot be traced in our graphical representation or perhaps even by the resolution of the numerical optimization procedure. This is not surprising as we know from table 1 that there is no major cycle for $\sigma = 0.1$ and the first minor cycle has a very small period-length and very high damping. For $\sigma \geq 0.2$ the optimal paths show clearly visible cyclical behavior, which becomes increasingly more pronounced the bigger the time-lag σ. Nevertheless, all optimal paths converge towards the stationary state. This is not necessarily the case as the system dynamics could exhibit a limit-cycle. For example, for our choice of parameters the major cycle turns into a limit-cycle for $\sigma \approx 0.65$.

Finally, we illustrate the continuous transition from instantaneous to time-lagged capital accumulation as stated in proposition 4. Figure 3 shows a 3-dimensional plot of the optimal paths, where the third axis denotes increasing time-lags σ. The exogenous parameters are identical to the calculations for figure 2. Again the time-lag $\sigma \in [0, 0.5]$, which has been split into a grid of 500 equidistant points. For each of these 500 σ the optimal control problem has been solved numerically and the resulting graphs have been composed to the 3-dimensional plots in figure 3. They show how the optimal paths evolve from monotonic to cyclical paths for increasing time-lag σ.

5 Conclusion

As known from the time-to-build literature, time-lagged optimal control problems exhibit in general a *qualitatively* different system dynamics compared to instantaneous capital accumulation models. While the optimal paths of the latter converge strictly monotonically towards the stationary state, the first show cyclical and exponentially damped optimal paths. In this paper we have drawn attention to the *quantitative* aspects of the system dynamics by a transition from instantaneous to time-lagged capital accumulation. To be able to derive analytical properties of the optimal solution, we have restricted our attention to a Leontief-type production function.

We have shown that there is a continuous transition from instantaneous to time-lagged capital accumulation in the sense that the greater is the time lag σ between investment and capital accumulation, the more the optimal paths display cyclical behavior and thus, the more they differ from the optimal paths of the instantaneous problem. Moreover, although the optimal solution exhibits in general an infinite number of cycles with different amplitudes, period-lengths and damping, the system dynamics is dominated by the contribution of the major cycle (if existing and otherwise by the first existing minor cycle) as amplitudes decrease and damping increases rapidly for cycles of higher order. As time-lagged optimal control problems are not analytically soluble, even in the linear approximation around the stationary state, numerical optimization is especially relevant to analyze and understand the system dynamics of time-lagged optimal control problems.

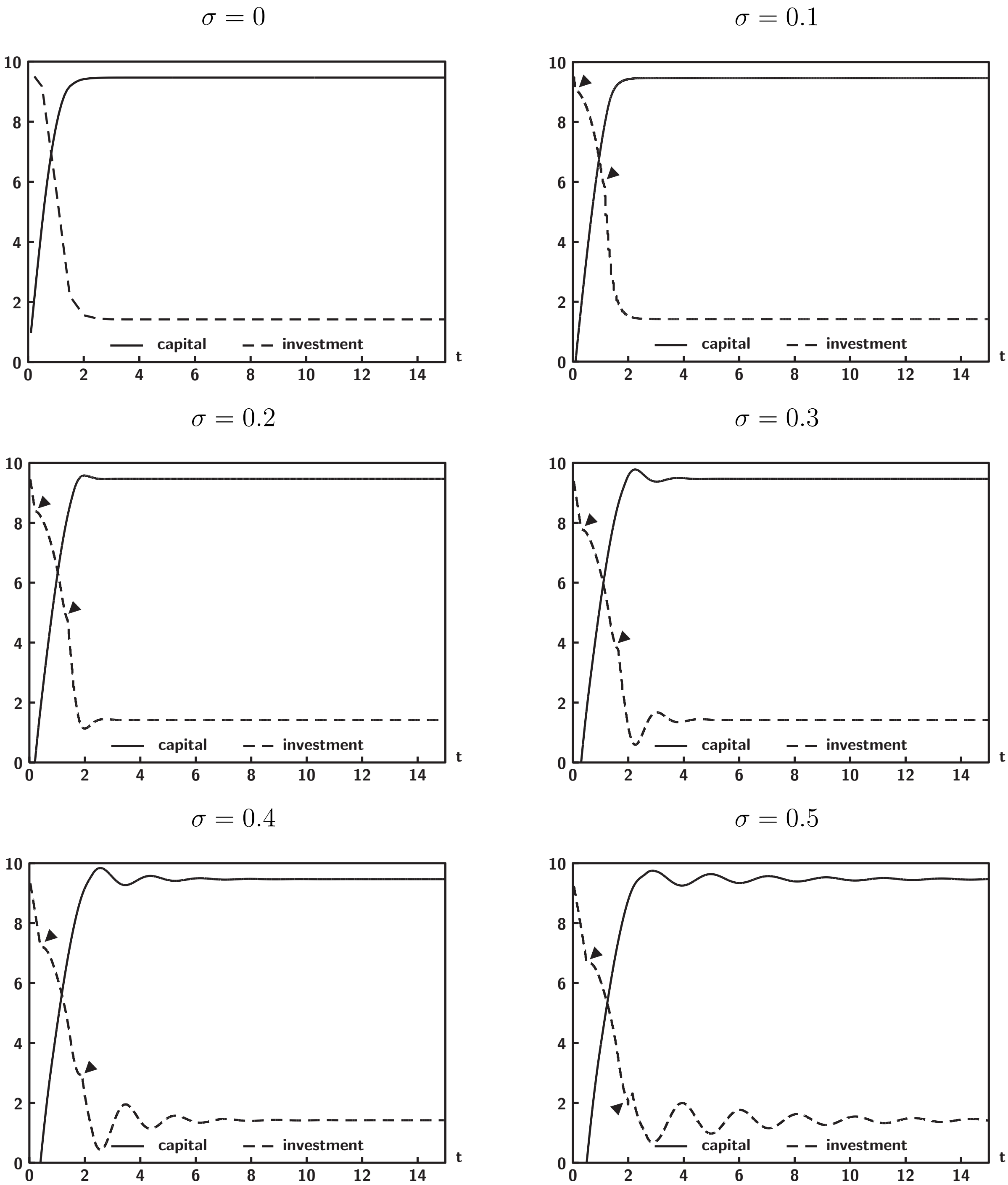

Figure 2: Optimal paths for capital and investment for time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation.

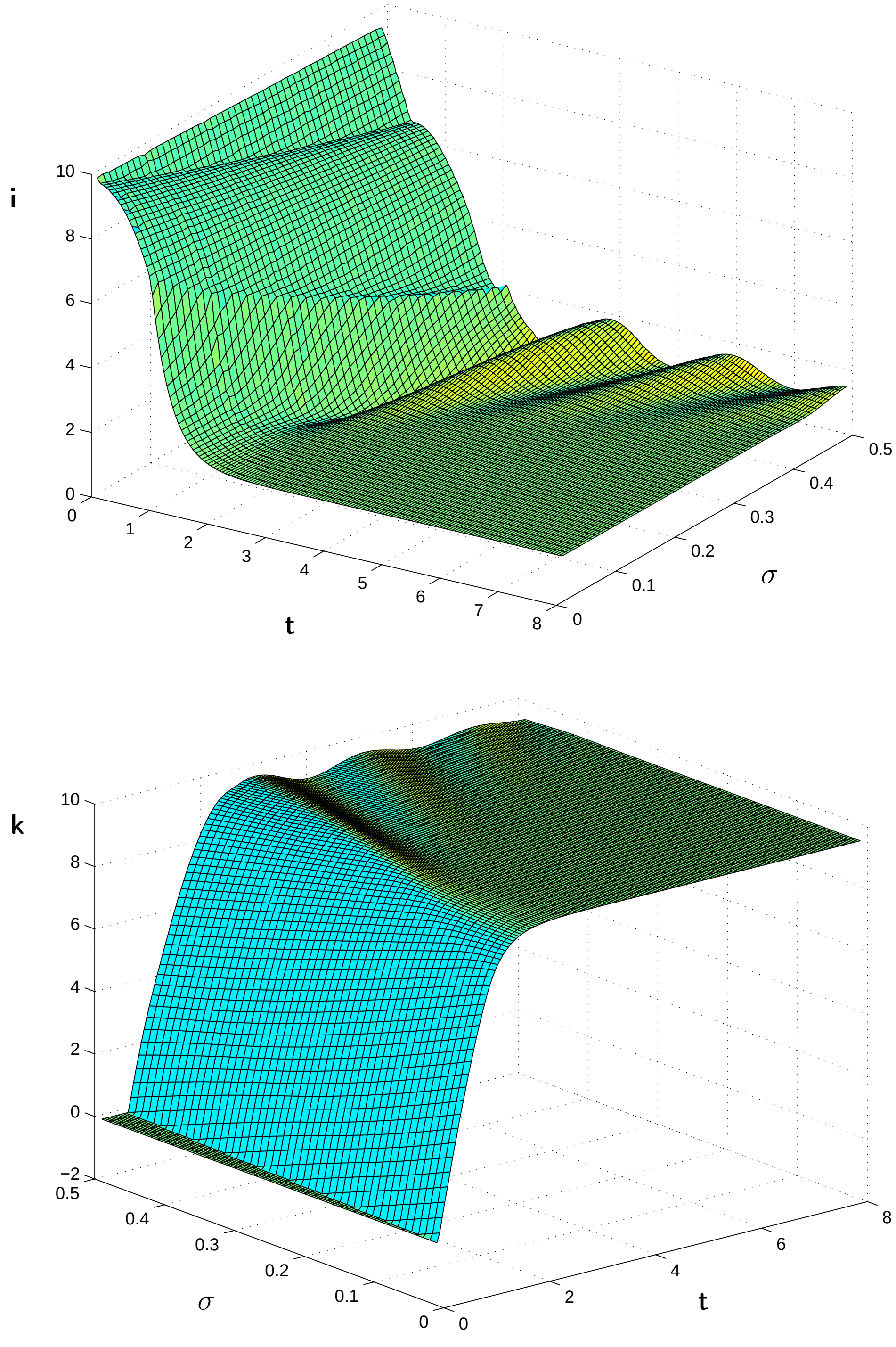

Figure 3: Optimal investment (top) and capital (bottom) paths for time-lags $\sigma \in [0, 0.5]$ between investment and capital accumulation. The third axis denotes increasing time-lags σ.

Thus, we have illustrated our analytical results by an example using state-of-the-art numerical optimization methods.

Our result is of direct interest to economic theory. It suggests that the standard assumption of instantaneous capital accumulation in neoclassical economic theory can be justified as a good approximation for small time-lags σ. However, for large time-lags, e. g. in plant construction or the pharmaceutical industry, the validity of this approximation is endangered. A priori it seems difficult to determine if a given time-lag should be considered to be small or large, as this depends on the whole set of exogenous parameters and the error one is willing to tolerate. However, as the optimal paths are dominated by the contribution of the first few cycles, the calculation of the period-length of the major cycle (or if non-existing the first existing minor cycle) and the corresponding real part and upper bound for its amplitude give a good impression of what to expect from the optimal paths.

So far we have solely analyzed a centralized economy. A priori it is not clear if the well known result of instantaneous capital accumulation that (under certain additional assumptions) a decentralized market solution is Pareto-optimal turns out to be true in the case of time-lagged capital accumulation. Intuition suggests that the households' knowledge about the time-lag might be crucial. To answer this question further investigations on this topic have to be carried out. Another promising area of research is the combination of growth models with time-lagged capital accumulation. This could give new insights for the analysis of real business cycles.

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