Appendix: Algebraic Model Description

A. Summary of the Generic Model

This section provides an algebraic summary of equilibrium conditions for an intertemporal small open economy model designed to investigate the economic implications of an environmental tax reform. For the generic model (without unemployment) the following basic assumptions apply:

(i) Commodity prices and factor prices are fully flexible within competitive markets (see section B for the specification of unemployment).

(ii) Labor is intersectorally mobile but not mobile at the international level.

(iii) Capital is freely mobile across sectors and domestic boundaries.[1]

(iv) Labor force productivity (efficiency) increases at an exogenous growth rate.

(v) Capital stocks evolve through constant geometric depreciation and new investment.

(vi) The level of investment in a given period is determined by competitive and individually rational entrepreneurs (investors) who allocate investment across sectors in order to maximize the present value of firms. Investors have no money illusion and issues such as debt-versus equity-financing are not considered.

(viii) The public budget is balanced on an intertemporal basis.

(viii) In international trade the domestic economy is considered as sufficiently small. Therefore the effects of exports and imports on international prices can be ignored (horizontal foreign export demand and import supply functions).[2] Within this small open economy framework, the model adopts the Armington assumption to differentiate between domestically produced commodities and foreign produced commodities in exports and imports. International capital flows (borrowing and lending) are endogenous, subject to an intertemporal balance of payments constraint, i.e. no change in net indebtedness over the model horizon.

(ix) Aggregate consumption and savings are derived from utility maximization of a representative household. To approximate an infinite horizon equilibrium with a finite model we assume that the representative consumer purchases capital in the post-horizon period at a price which is consistent with steady-state equilibrium growth (terminal condition).

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[1] The model version used for our simulations incorporates capital adjustment costs: We assume that 90% of the initial capital operates as a fixed-coefficient Leontief technology. Substitution possibilities between various forms of energy, labor, capital and material are possible only for 10% of the initial capital stock and new investment. This (partial) putty-clay formulation has the virtue of reflecting empirical evidence between lower short-run and higher long-run elasticities of input demands and accomodates *premature retirement* of extant capital.

[2] Hence, we can omit export demand and import supply function within the algebraic model formulation.

The model is formulated as a system of nonlinear inequalities. These inequalities correspond to three classes of equilibrium conditions: zero profit, market clearance, and income balance. The fundamental unknowns of the system are three vectors: activity levels, prices, and income levels. In equilibrium, each of these variables is linked to one inequality condition: an activity level to a zero-profit condition, a commodity price to a market clearance condition, and a income variable to an income definition equation.

In the following algebraic exposition, the notation Π_i^X is used to denote the profit function of sector *i*, where *X* is the name assigned to this activity. Formally, all production sectors exhibit constant returns to scale (CRTS), hence differentiating Π_i^X with respect to input and output prices provides compensated demand and supply coefficients, which appear subsequently in the market-clearance conditions.[3] All prices are expressed as present values reflecting the assumed international interest rate and consumer's intertemporal preferences, i.e. the pure rate of time preference. In order to simplify the notation, time indices are omitted from those equations which are strictly intra-period.

Zero profit conditions

Production

In domestic production, nested, separable, constant elasticity of substitution (CES) cost functions are employed to specify the substitution possibilities between inputs of capital (K), labor (L), an energy composite (E) and a material composite (M).[4] The energy composite is made up of the outputs of the energy industries. The materials consists of the outputs of the other non-energy industries. At the top level, the materials composite is employed in fixed proportions with an aggregate of energy, capital and labor. A constant elasticity describes the substitution possibilities between the energy aggregate and the aggregate of labor and capital at the second level. Finally, at the third level capital and labor trade off with a unitary elasticity of substitution. On the output side, production is split between goods produced for the domestic market and goods produced for the world market according to a constant elasticity of transformation. The resulting intra-period zero-profit condition for the production of good *i* is:

[3] Decreasing returns are accommodated in the CRTS framework through introduction of a specific factor under the assumption of perfect competition.

[4] The cost functions are based on the nesting structure and nest elasticities from the ETA-MACRO model (Manne and Richels 1992).

$$\Pi_i^{Y} = \left(q_j^{X} \, (P_i(1 - t_i^{Y}))^{1+1/s^{DX}} + (1 - q_i^{X})(PX_i \, (1 - t_i^{Y}))^{1+1/s^{DX}} \right)^{\frac{1}{(1+1/s^{DX})}}$$

$$q_i^{KLE} \left[q_i^{E} \, P_i^{E\;1-s^{KLE}} + (1 - q_i^{E}) \left((w(1 + t_i^{L}))^{q_i^{L}} (r(1 + t_i^{K}))^{1-q_i^{L}} \right)^{1-s^{KLE}} \right]^{\frac{1}{1-s^{KLE}}} - (1 - q_i^{KLE}) \, P_i^{M} = 0$$

where:

q_i^{x} is the benchmark export value share in output of sector *i*,

P_i is the output price of good *i*,

t_i^{Y} is the net production (output) tax on good *i*,

s^{DX} is the elasticity of transformation between production for the domestic market and production for the export market,

PX_i is the export price of good *i* (expressed in domestic currency)[5],

q_i^{KLE} is the benchmark value share of capital, labor and energy inputs (KLE aggregate) of region in sector *i*,

q_i^{E} is the energy input value share of the KLE aggregate in sector *i*,

P_i^{M} stands for the composite price of the materials composite input into sector *i*,

P_i^{E} stands for the composite price of the energy composite input into sector *i*,

s^{KLE} is the elasticity of substitution between the energy aggregate and the aggregate of capital and labor,

w is the economy-wide gross wage rate (net of payroll taxes),

t_i^{L} is the payroll tax rate in sector *i*,

r is the uniform rate of return on mobile capital (net of capital taxes),

t_i^{K} is the capital tax rate in sector *i*,

q_i^{L} denotes the value shares of labor in the value added of sector *i*,

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[5] The prices for exports PX_i and imports PM_i are expressed in domestic currency. Export prices $\overline{PX}_i$ and import prices $\overline{PM}_i$ in international currency (e.g. in $US) are exogenous for the small open economy. The real exchange rate µ relates international prices to domestic prices, i. e. $PM_i \equiv \overline{PM}_i \, m$.

and

Y_i is the associated dual variable, which indicates the activity level of production in sector i.

Armington aggregation across imports and goods produced for the domestic market

Each of the individual inputs which make up the energy and the materials composite is a composite of a domestic and an imported variety which trade off with a constant elasticity of substitution. The corresponding zero-profit condition for the production of the Armington good i is given by:

$$\Pi_i^A = P_i^A - \left[\left(q_i^{IM} ((1+t_i^{IM})PM_i)^{1-s^{DM}} + (1-q_i^{IM})P_i^{1-s^{DM}} \right)^{\frac{1}{1-s^{DM}}} + t^{CO2} a_i^{CO2} \right] = 0$$

where:

P_i^A is the Armington price of the composite good i,

q_i^{IM} is the benchmark value share of imports in the Armington good i,

PM_i is the import price of good i (expressed in domestic currency),

t_i^{IM} is the (ad-valorem) tariff rate on imported goods,

s^{DM} is the Armington elasticity of substitution between domestic goods and imports,

t^{CO2} is tax rate per unit (ton) of CO_2,

$a_i^{CO_2}$ is the physical carbon emission coefficient for good i,

and

A_i is the associated dual variable, which indicates the activity level of Armington good production.

Material composite

The material composite is produced in fixed proportions (Leontief):

$$\Pi_i^M = P_i^M - \sum_{j \notin EG} q_{ji}^M \; P_i^A = 0$$

where:

q_{ji}^M is the benchmark value share of the non-energy Armington good j ($j \neq EG$) in the materials composite of sector i,

and

M_i is the associated dual variable, which indicates the activity level of production of the materials composite for sector i.

Energy composite

As to the formation of the energy aggregate, we employ several levels of nesting to represent differences in substitution possibilities between primary fossil fuel types as well as substitution between the primary fossil fuel composite and secondary energy, i.e. electricity. In the bottom nest, liquid fuels (refined oil (OIL), crude oil (CRU), gas (GAS)) trade off with a constant elasticity of substitution. At the same level, hard coal and soft coal are combined within a CES aggregate. The liquid fuel composite and the coal composite enter the next level with a constant elasticity of substitution. At the top level, the aggregate of non-electric energy combines with electricity at a constant elasticity of substitution[6]:

$$\Pi_i^E = P_i^E - \left\{ q_i^{ELE} P_{ELE}^{A^{1-s^{ELE}}} + (1-q_i^{ELE}) \left[q_i^{COA} P_{COA}^{1-s^{COA\_LQD}} + (1-q_{ir}^{COA}) P_{LIQ}^{1-s^{COA\_LQD}} \right]^{\frac{1-s^{ELE}}{1-s^{COA\_LQD}}} \right\}^{\frac{1}{1-s^{ELE}}} = 0$$

where:

P_i^E is the price of the energy composite good for sector i,

q_i^{ELE} is the benchmark value share of electricity in the energy aggregate of sector i,

s^{ELE} is the elasticity of substitution between electricity and non-electric (fossil) energy,

q_i^{COA} is the benchmark value share of coal inputs within the fossil fuel aggregate input of sector i,

s^{COA_LQD} is the elasticity of substitution between the coal composite and the liquid fuel composite,

and

E_i is the associated dual variable, which indicates the activity level of production of the energy composite for sector i.

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[6] For the sake of transparency we have a dropped the explicit representation of the lowest nest and employ instead an artificial liquid fuel composite (price P_i^{COA}) and coal aggregate (price P_i^{LIQ}).

Public output

Public goods and services are produced in fixed proportions (Leontief aggregation) of commodity inputs which are composed as an Armington aggregate of domestic and imported commodities:

$$\Pi^G = P^G - \sum_i q_i^G \; P_i^A = 0$$

where:

P^G is the composite price for government demand,

q_i^G is the benchmark value share of Armington input i in public goods provision,

and

G is the associated dual variable which indicates the activity level of public goods provision.

Consumer good production

Consumer goods are produced in fixed proportions of Armington goods (Z-Matrix transformation with fixed coefficients):

$$\Pi_z^Z = P_z - \sum_i q_{iz} \; P_i^A = 0$$

where:

P_z is the price for consumption good z (net of value-added taxes),

q_{iz} is the benchmark value share of producer good i into the formation of consumer good z,

and

Z_z is the associated dual variable which indicates the activity level of consumer good production.

Non-leisure consumption composite

Substitution patterns within the aggregate of (non-leisure) consumption goods are described by a Cobb-Douglas function. The zero-profit condition for the (non-leisure) consumption composite is given by:

$$\Pi^{CG} = P^{CG} - \prod_z \left(P_z (1 + t_z^{CG}) \right)^{q_z^{CG}} = 0$$

where:

P^{CG} is the price for the (non-leisure) consumption composite (gross of value-added taxes),

q_z^{CG} is the benchmark value share of consumer commodity z in aggregate (non-leisure) household consumption

t_z^{CG} is the value-added tax rate on inputs of consumption goods into aggregate consumption,

and

CG is the associated dual variable which indicates the activity level of household consumption of commodities (excluding leisure).

Full consumption

Intra-period household demand is given as a separable nested CES function which describes the trade-off between leisure and consumption. The zero-profit condition reads as:

$$\Pi^C = P^C - \left[q_F \left(w(1-t_w)\right)^{1-s^F} + (1-q_F)\, P^{CG^{1-s^F}} \right]^{\frac{1}{1-s^F}} = 0$$

where:

PC is the composite price index of aggregate leisure and goods consumption,

q^F is the benchmark value share of leisure in intra-period household consumption,

is the price of current leisure,

s^F is the elasticity of substitution between current leisure and commodity consumption (calibrated consistently to a given supply elasticity of labor with respect to the real wage),

t_w is the labor income tax rate (applicable to the gross wage),

and

C is the associated dual variable which indicates the activity level of aggregate household consumption (commodities and leisure).

Capital formation and investment

An efficient allocation of capital, i.e. investment over time assures the following intertemporal zero-profit conditions which relates the cost of a unit of investment, the return to capital and the purchase price of a unit of capital stock in period t[7]:

$$\Pi_t^K = p_t^K - r_t^K - (1 - d)\, p_{t+1}^K = 0$$

and

$$\Pi_t^I = \sum_i q_i^I \, P_{it}^A - p_{t+1}^K \geq 0$$

where:

[7] The optimality conditions for capital stock formation and investment are directly derived from the maximization of lifetime utility by the representative household taking into account its budget constraint, the equation of motion for the capital stock and the condition that output in each period is either invested or consumed. Note that in our algebraic exposition we assume an investment lag of one period.

p_t^K	is the value (purchase price) of one unit of capital stock in period *t*,
d	is the capital depreciation rate,
$\sum_i q_i^I \, P_{it}^A$	is the cost of a unit of investment in period *t*[8],
q_i^I	is the benchmark value share of Armington input *i* in the homogeneous investment ´ good

and

K_t	is the associated dual variable, which indicates the activity level of capital stock in period *t*,
I_t	is the associated dual variable, which indicates the activity level of aggregate investment in period *t* [9].

Market Clearance[10]

Domestic supply

Producer goods produced for the domestic market enter Armington production:

$$Y_i \frac{\partial \Pi_i^Y}{\partial (P_i (1-t_i^Y))} = A_i \frac{\partial \Pi_i^A}{\partial P_i}$$

Armington supply

Armington goods enter intermediate demand for the production of producer goods and consumer goods as well as government and investment demand:

$$A_i = \sum_i Y_i \frac{\partial \Pi_i^Y}{\partial P_i^A} + \sum_z Z_z \frac{\partial \Pi_z^Z}{\partial P_i^A} + I \frac{\partial \Pi^I}{\partial P_i^A} + G \frac{\partial \Pi^G}{\partial P_i^A}$$

Intermediate energy supply

The sector-specific intermediate energy composite enters production:

$$E_i = \sum_i Y_i \frac{\partial \Pi_i^Y}{\partial P_i^E}$$

Intermediate material supply

The sector-specific intermediate material composite enters production:

$$M_i = \sum_i Y_i \frac{\partial \Pi_i^Y}{\partial P_i^M}$$

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[8] The investment good is produced subject to a Leontief technology which combines Armington inputs in fixed proportions.

[9] As written, we have taken explicit account of the non-negativity constraint for investment.

[10] We exploit Shepard's Lemma to provide a compact representation of compensated demand and supply functions.

Consumer goods supply

Consumer goods enter final consumption demand:

$$Z_z = CG \frac{\partial \Pi^{CG}}{\partial (P_z(1+t_z^C))}$$

Non-leisure consumption

Non-leisure consumption enters the aggregate consumption (including leisure):

$$CG = C \frac{\partial \Pi^C}{\partial P_{CG}}$$

Government provision

Public good provision increases at the exogenous (steady-state) growth rate of the economy:

$$G_t = \overline{G}_{t-1}(1+gr)$$

where:

gr	is the exogenous growth rate,
$\overline{G}_0$	is the base year level of public good provision.

Imports

The supply-demand balance for imported goods is:

$$IM_i = A_i \frac{\partial \Pi_i^A}{\partial (PM_i(1+t_i^{IM}))}$$

where:

IM_i	is the level of imports of good i.

Exports

The supply-demand balance for exported goods is:

$$EX_i = Y_i \frac{\partial \Pi_i^Y}{\partial (PX_i(1-t_i^Y))}$$

where:

EX_i	is the level of exports of good i.

Foreign closure

As to the trade balance with respect to ROW a simple foreign closure rule is used: In the small open economy framework, CIF import prices and FOB export prices are exogenous and unaffected by the level of imports and exports. An intertemporal balance of payments constraint (trade closure) assures no change in net indebtedness over the model horizon:[11]

$$\sum_{t=1}^{T}\sum_{i} PM_{it}\, IM_{it} = \sum_{t=1}^{T}\sum_{i} PX_{it}\, EX_{it}$$

Labor

The intra-period supply-demand balance for labor is written:

$$\overline{E} - \frac{\partial \Pi^{C}}{\partial (w(1-t_w))} = \sum_i Y_i \frac{\partial \Pi_i^{Y}}{\partial (w(1+t_i^{L})}$$

where:

$\overline{E}$ is the total endowment with time which grows at the exogenous (steady-state) rate gr of the economy.[12]

Capital

The supply-demand balance for capital services is written:

$$rK = \sum_i Y_i \frac{\partial \Pi_i^{Y}}{\partial (r(1+t_i^{K}))}$$

where:

K is the aggregate capital stock for domestic production.

Current period's investment augments the capital stock in the next period.[13] Capital stocks are updated as an intermediate calculation between periods.[14] The stock-flow accounting relationship for capital (equation of motion for the capital stock) can be written as:

$$K_{t+1} = (1 - d) K_t + I_t$$

Income Balances

Household

The representative household chooses to allocate lifetime income across consumption in different time periods:

[11] N.B.: In this framework, international flows of capital goods (borrowing and lending) are endogenous.

[12] We represent growth in terms of Harrod-neutral technological progress in producing labor or leisure services per unit of actual time (efficiency growth).

[13] Capital accumulation (i. e. the level of savings and investment) is determined by firms managers who allocate investment across sectors to maximize the present value of firms.

[14] In our simulations, we assume that the capital stock is augmented by new investment with a three-year and depreciated at a constant geometric rate.

$$\max U(u(C_1, F_1), u(C_2, F_2), \ldots, u(C_T, F_T)) = \sum_{t=1}^{T} r^t u(C_t, F_t)$$

s.t.

$$\sum_{t=1}^{T} P_t C_t = \sum_{t=1}^{T} w_t (1 - t^w)(\overline{E}_t - F_t) + p_1^K K_1 + \sum_{t=1}^{T} r_t K_t$$

where:

r^t reflects the pure rate of time preference which determines the intertemporal allocation of consumption.

For the intra-period utility function we assume the functional form $u(C_t, F_t) = \frac{1}{C_t^a F_t^{1-a}}$ which is consistent with an intertemporal elasticity of substitution equal to 0.5.

Government

Government income consists of tax revenues from the representative household, which assures a balanced budget (fiscal closure). The intertemporal income balance of the government is given by (for the sake of brevity we omit production taxes and tariffs):

$$\sum_{t=1}^{T} \sum_{z} (P_z t_z^C CG \frac{\partial \Pi^{CG}}{\partial (P_z (1 + t_z^{CG}))})$$
$$+ \sum_{t=1}^{T} w_t t_w L_t + \sum_{t=1}^{T} w_t \sum_{i} t_i^L w_t Y_{it} \frac{\partial \Pi_{it}^Y}{\partial (w_t (1 + t_{it}^L))}$$
$$+ \sum_{t=1}^{T} \sum_{i} t_i^K r_t Y_{it} \frac{\partial \Pi_{it}^Y}{\partial (r_t (1 + t_i^K))}$$
$$+ \sum_{t=1}^{T} \sum_{i} t^{CO2} a_i^{CO2} A_{it}$$
$$= \sum_{t=1}^{T} P_t^G G_t$$

Terminal Constraint

The finite horizon poses some problems with respect to capital accumulation. Without any terminal constraint, the capital stock at the end of the model's horizon would have no value and this would have significant repercussions for investment rates in the periods leading up to the end of the model horizon. In order to correct for this effect we define a terminal constraint which forces terminal investment to increase in proportion to final consumption demand:

$$\frac{I_T}{I_{T-1}} = \frac{C_T}{C_{T-1}}.$$

B. Unemployment

We introduce unemployment through the specification of a "wage curve", which postulates a negative relationship between the real wage rate and the rate of unemployment ur:

$$\frac{w(1-t^W)}{P^{CG}} = g(ur) \text{ with } g' < 0,$$

where:

P is the consumer goods price index (in our case P^{CG})

and

ur is the unemployment rate.

This type of wage curve can be derived from trade union wage models as well as from efficiency wage models (see e. g. Beißinger, 1996 or Hutton and Ruocco, 1999). Figure 1 illustrates the wage curve in the traditional labor market diagram. The real wage rate w/P is measured on the vertical axis and the labor supplied and demanded are measured on the horizontal axis.

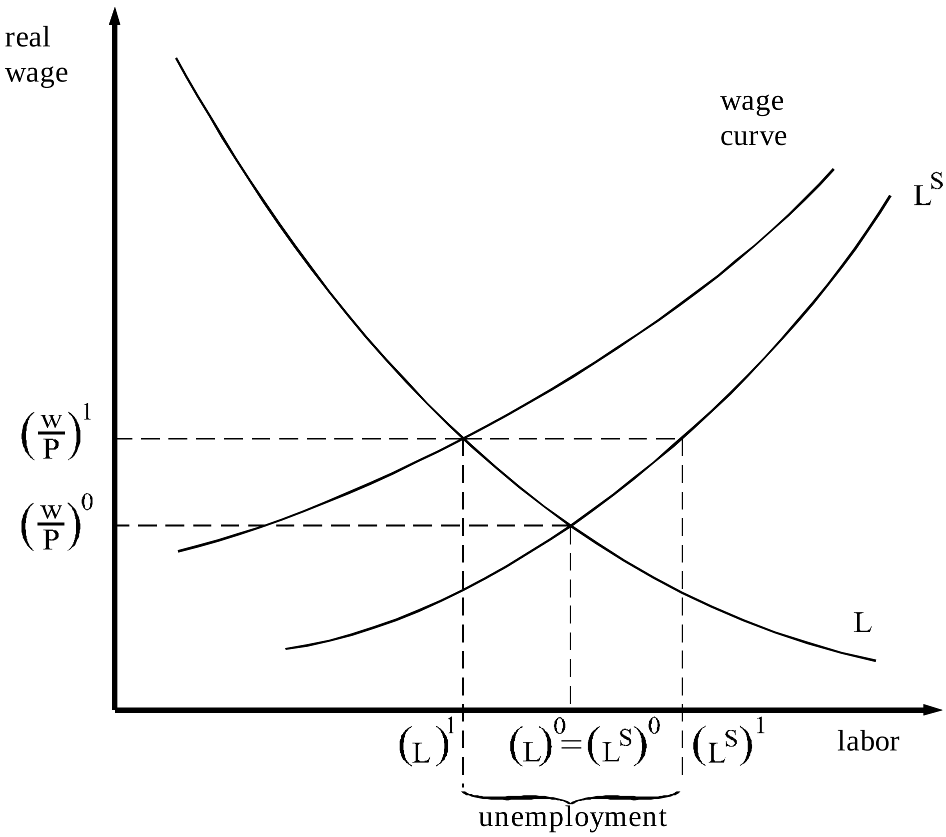

Figure 2: Wage curve and unemployment

Full employment occurs with the real wage rate of $(w/P)^0$ at the intersection of the (inverse) labor demand function L and the labor supply function L^S. The wage curve now replaces the labor supply curve. Consequently, the equilibrium wage rate $(w/P)^1$ lies above the market clearing wage rate. This causes unemployment at an amount of $(L^S)^1 - (L)^1$.

Our initial labor market clearance condition then becomes:

$$\left[\overline{E} - \frac{\partial \Pi^C}{\partial(\, w(1-t_w))}\right](1-ur) = \sum_i Y_i \frac{\partial \Pi_i^Y}{\partial\,(w(1+t_i^L\,)}$$

In our calculations, we assume that the unemployment benefit payments are constant in real terms and are not taxed. We can write the wage curve as a log-linear function:

$$\log\left(\frac{w}{P}\right) = \gamma_0 + \gamma_1 \log(ur)$$

where:

γ_0 is a positive scale parameter

and

$\gamma_1 < 0$ is the elasticity of the real wage with respect to the unemployment rate.

Real unemployment benefits are included in the parameter γ_0. Determination of welfare effects for the case of involuntary unemployment is also based on enforced (rather than voluntary) leisure consumption.